§2.2 Computation and applications

CW complexes (p.5, p.519)

— a spicial class of spaces which are easier in computing homology

Def

A CW complex (or cell complex) is a space X constructed in the following way:

① Start with a discrete set X^0, the 0-cell of X.

② Inductively, form the n-skeleton X^n from X^{n-1} by attaching n-cells e^n_α via maps $\varphi_\alpha : S^{n-1} \to X^{n-1}$. That is,

$$X^n = X^{n-1} \sqcup_\alpha D^n_\alpha \Big/ x \sim \varphi_\alpha(x),\ x \in \partial D^n_\alpha \qquad (D^n_\alpha \cong D^n)$$

Note: the cell e^n_α is homeomorphic to $D^n_\alpha \setminus \partial D^n_\alpha$ under the quotient map.

③ $X = \bigcup_n X^n$ with the weak topology: A subset $A \subset X$ is open (or closed) iff $A \cap X^n$ is open (or closed) in X^n for each n.

For each cell e^n_α, the map

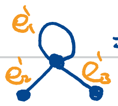
$$\Phi_\alpha : D^n = D^n_\alpha \hookrightarrow X^{n-1} \sqcup_\alpha D^n_\alpha \twoheadrightarrow X^n \hookrightarrow X$$

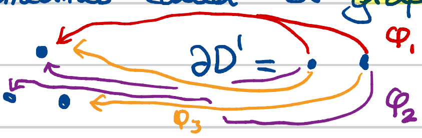
is called the characteristic map of e^n_α.

If $X = X^n$ for some n, then X is said to be finite-dimensional, and the smallest such number n is called the dimension of X.

Example 0.1

A 1-dim CW cx $X = X^1$ is sometimes called a graph in algebraic topology

e.g. $X =$ 

$= X^1$, $X^0 =$ 

Example 0.3

The sphere S^n has the structure of a CW cx with 2 cells, e^0 and e^n

$X^0 = \bullet = \cdots = X^{n-1}$, φ $\partial D^n = S^{n-1}$ 

Example 0.4

The real projective n-space $\mathbb{RP}^n$ is

$$\mathbb{RP}^n = \{\text{1-dim real vec. subsp in } \mathbb{R}^{n+1}\} = \mathbb{R}^{n+1}\setminus\{0\} \big/ \vec{v}\sim\lambda\vec{v},\ \lambda\neq 0,\ \vec{v}\in\mathbb{R}^{n+1}\setminus\{0\}$$

$$= S^n/\vec{v}\sim-\vec{v} \quad\cong\quad D^n/\vec{v}\sim-\vec{v},\ \vec{v}\in\partial D^n$$

$\Rightarrow$ $\mathbb{RP}^n = \mathbb{RP}^{n-1}\cup e^n$, $\varphi_n : \partial D^n = S^{n-1} \longrightarrow \mathbb{RP}^{n-1}$ is the quotient projection

$\overset{\text{inductively}}{\Rightarrow}$ $\mathbb{RP}^n = e^0 \cup_{\varphi_1} e^1 \cup_{\varphi_2} \cdots \cup_{\varphi_n} e^n$

Example 0.5

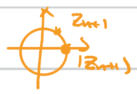
$$\mathbb{RP}^\infty = \bigcup_n \mathbb{RP}^n = e^0 \cup_{\varphi_1} e^1 \cup_{\varphi_2} \cdots$$

Example 0.6

The complex projective n-space $\mathbb{CP}^n$ is

$$\mathbb{CP}^n = \{\text{1-dim complex vec. subsps in } \mathbb{C}^{n+1}\} = \mathbb{C}^{n+1}\setminus\{0\}\big/\vec{v}\sim\lambda\vec{v},\ \lambda\neq 0,\ \vec{v}\in\mathbb{C}^{n+1}\setminus\{0\}$$

$$= {}^{\mathbb{C}^{n+1}\supset} S^{2n+1}\big/\vec{v}\sim\lambda\vec{v},\ |\lambda|=1$$



For $(z_1,\cdots,z_{n+1}) \in S^{2n+1} \subset \mathbb{C}^{n+1}$, $\lambda = \frac{|z_{n+1}|}{z_{n+1}}$

① if $z_{n+1}\neq 0$, then $(\lambda z_1,\cdots,\lambda z_n, |z_{n+1}|)$ is the uni equvector with the last Component >0 equivalent to $(z_1,\cdots,z_{n+1})$ in $\mathbb{CP}^n$.

$$\{[z_1:\cdots:z_{n+1}]\in\mathbb{CP}^n \mid z_{n+1}>0\} \cong \vec{w}\in D^{2n}\setminus\partial D^{2n} \subseteq \mathbb{C}^n$$

$[z_1:\cdots:z_{n+1}] \longmapsto (z_1,\cdots,z_n)$, $\quad [\vec{w} : \sqrt{1-|\vec{w}|^2}] \longleftarrow \vec{w}$

② if $z_{n+1}=0$: $\{[z_1:\cdots:z_n]\in\mathbb{CP}^n \mid z_{n+1}=0\} \cong \mathbb{CP}^{n-1}$ $\quad (z_1,\cdots,z_n,0)\sim(\lambda z_1,\cdots,\lambda z_n,0)$

So $\mathbb{CP}^n = \mathbb{CP}^{n-1}\cup_\varphi e^{2n}$, where $\varphi : \partial D^{2n}\cong S^{2n-1}\longrightarrow \mathbb{CP}^{n-1}$ is the quotient map

$\Rightarrow$ $\mathbb{CP}^n = e^0\cup e^2\cup e^4\cup\cdots\cup e^{2n}$

Similarly, one also has $\mathbb{CP}^\infty = \bigcup_n \mathbb{CP}^n = e^0\cup e^2\cup e^4\cup\cdots$

Def

A subcomplex of a CW complex X is a closed subspace $A\subset X$ that is a union of cells of X. A pair (X,A) of a CW cx X and a subcx A is called a CW pair

Remark

> exer: A CW pair is a good pair (the assumption of Thm 2.13)

"CW" refers to the following 2 properties of CW cxes:

① Closure-finiteness : The closure of each cell meets only finitely many other cells.

② Weak topology : A subset is closed iff the intersection with cl(each cell) is closed.

Cor 2.24

If a CW complex X is the union of subcomplexes A and B, then the inclusion $(B, A\cap B) \hookrightarrow (X, A)$ induces iso $H_n(B, A\cap B) \longrightarrow H_n(X, A)$ $\forall n$

pf

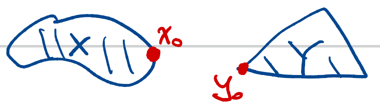
Since CW pairs are good pairs, Prop 2.22 implies that

$$\begin{array}{ccccc} H_n(B, A\cap B) & \xrightarrow{\cong} & H_n(B/A\cap B, A\cap B/A\cap B) & \xleftarrow{\cong} & \tilde{H}_n(B/A\cap B) \\ & & & & \downarrow \cong \\ H_n(X, A) & \xrightarrow{\cong} & H_n(X/A, A/A) & \xleftarrow{\cong} & \tilde{H}_n(X/A) \end{array}$$

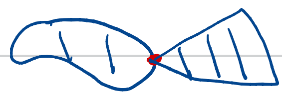
$\leftarrow$ $\because$ $B/A\cap B \xrightarrow{\cong} X/A$ is a homeomorphism

#

Wedge sum



$X \vee Y$:



Given spaces X and Y with $x_0 \in X$, $y_0 \in Y$, the wedge sum $X \vee Y$ is the quotient

$$X \vee Y := X \amalg Y / x_0 \sim y_0$$

If we are given a collection of spaces X_α and $x_\alpha \in X_\alpha$, then

$$\vee_\alpha X_\alpha := \coprod_{\alpha\in\Lambda} X_\alpha \Big/ x_\alpha \sim x_\beta,\ \alpha,\beta \in \Lambda$$

Example

Suppose X is the CW complex 

$X^0 = \cdot\ \cdot\ \cdot\ \cdot$ $4e^0$, $X^1 =$  $6e^1$, $X^2 =$  $3e^2$

$\Rightarrow$ $X^1/X^0 =$  $= \underbrace{S^1 \vee \cdots \vee S^1}_{6}$

$X^2/X^1 =$  $= S^2 \vee S^2 \vee S^2$

In general, $X^n/X^{n-1} = \vee_\alpha S^n_\alpha$, α: # of n-cells

Cor 2.25

For a wedge sum $\vee_\alpha X_\alpha$, the inclusions $i_\alpha : X_\alpha \hookrightarrow \vee_\alpha X_\alpha$ induces an iso

$$\oplus_\alpha i_\alpha : \bigoplus_\alpha \tilde{H}_n(X_\alpha) \xrightarrow{\cong} \tilde{H}_n(\vee_\alpha X_\alpha)$$

provided the pairs $(X_\alpha, \{x_\alpha\})$ are good.

pf

Apply Prop 2.22 to $(X, A) = (\amalg_\alpha X_\alpha, \amalg_\alpha \{x_\alpha\})$ #

Cellular homology

Lemma 2.34 If X is a CW complex, then

(a) $H_k(X^n, X^{n-1})$ is zero for $k \neq n$ and is free abelian for $k = n$, with a basis in 1-1 correspondence with the n-cells of X

(b) $H_k(X^n) = 0$ for $k > n$. In particular, if X is finite-dimensional, then $H_k(X) = 0$ for $k > \dim X$.

(c) The map $H_k(X^n) \longrightarrow H_k(X)$ induced by the inclusion $X^n \hookrightarrow X$ is an isomorphism for $k < n$ and surjective for $k = n$.

pf

(a) is true because (X^n, X^{n-1}) is a good pair and $X^n / X^{n-1} \cong S^n \vee \cdots \vee S^n$ #

(b) We have the long exact seq. for (X^n, X^{n-1}):

$$(*) \quad \cdots \to \underbrace{H_{k+1}(X^n, X^{n-1})}_{=0 \text{ if } k \neq n-1} \to H_k(X^{n-1}) \to H_k(X^n) \to \underbrace{H_k(X^n, X^{n-1})}_{=0 \text{ if } k \neq n} \to \cdots$$

if $k>n$,

$$\Rightarrow H_k(X^n) \cong H_k(X^{n-1}) \cong \cdots \cong H_k(X^0) = 0 \quad \because k>0$$ #

(c) finite-dim case ($X = X^N, \exists N$): By $(*)$, the map induced by inclusion $X^{n-1} \hookrightarrow X^n$

$$H_k(X^n) \longrightarrow H_k(X^{n+1}) \text{ is } \begin{cases} \text{1-1 if } k \neq n \\ \text{onto if } k \neq n+1 \end{cases}$$

So if $k < n$, then $H_k(X^n) \xrightarrow{\cong} H_k(X^{n+1}) \xrightarrow{\cong} \cdots \xrightarrow{\cong} H_k(X = X^N)$ is an iso

if $k = n$, then $H_k(X^n) \xrightarrow{\text{onto}} H_k(X^{n+1}) \xrightarrow{\cong} \cdots \xrightarrow{\cong} H_k(X = X^N)$ is onto.

infinite-dim case: exer. (p.138-139) #

Let X be a CW complex. By Lemma 2.34, $H_n(X^n, X^{n-1}) \cong \mathbb{Z}^{\sigma_n}$ where σ_n = # of n-cells

Define

$$\begin{tikzcd}
 & \mathbb{Z}^{\sigma_n} \arrow[d, phantom, "\cong" rotate=90] & & H_{n-1}(X^n) & \mathbb{Z}^{\sigma_{n-1}} \arrow[d, phantom, "\cong" rotate=90] & H_{n-2}(X^{n-2}) \\
d_n := j_{n-1} \circ \partial : & H_n(X^n, X^{n-1}) \arrow[rr, "\partial"] & & H_{n-1}(X^{n-1}) \arrow[u, dashed] \arrow[r, "j_{n-1}"] & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[ur, dashed] & \\
 & H_n(X^n) \arrow[u, dashed] & & H_{n-1}(X^{n-2}) \arrow[u, dashed] & &
\end{tikzcd}$$

Lemma

$$d_n \circ d_{n+1} = 0$$

pf $d_n \circ d_{n+1} = H_{n+1}(X^{n+1}, X^n) \xrightarrow{\partial} H_n(X^n) \overbrace{\boxed{\xrightarrow{j} H_n(X^n, X^{n-1}) \xrightarrow{\partial}}}^{=0} H_{n-1}(X^{n-1}) \xrightarrow{j} H_{n-1}(X^{n-1}, X^{n-2})$ #

The complex $(H_n(X^n, X^{n-1}), d_n)$ is called the cellular chain complex, and the corresponding homology group is called the cellular homology of X, temporarily denoted by $H_n^{CW}(X)$