Correction:

Hatcher's deformation retraction

= my strong deformation rection in a previous lecture

i.e. A deformation retraction of X onto A is

a homotopy $F: X \times I \to X$ s.t. ① $F(x,0)=x \quad \forall x \in X$

② $F(a,t)=a \quad \forall a \in A, t \in I$, ③ $f_1(X)=A$, $f_t = F(\cdot, t)$

Relative homology group

Let A be a subspace of a top. sp. X. $\Rightarrow$ $C_n(A)$ is a subgroup of $C_n(X)$.

Let $C_n(X,A) := C_n(X)/C_n(A)$

Since $\partial_n^X (C_n(A)) \subseteq C_{n-1}(A)$, ∂_n^X induces a homo $\partial_n^{X,A}: C_n(X,A) \longrightarrow C_{n-1}(X,A)$

$(\partial^X)^2 = 0 \Rightarrow (\partial^{X,A})^2 = 0 \quad \Rightarrow$ we have a chain complex

$$\cdots \longrightarrow C_n(X,A) \xrightarrow{\partial} C_{n-1}(X,A) \longrightarrow \cdots$$

The associated homology group $H_n(X,A) = \ker(\partial_n^{X,A}) / \operatorname{im}(\partial_{n+1}^{X,A})$ is called the relative homology group.

Note that we have a short exact seq. of complexes

$$0 \longrightarrow (C(A), \partial^A) \longrightarrow (C(X), \partial^X) \longrightarrow (\underbrace{C(X,A)}_{= C(X)/C(A)}, \partial^{X,A}) \longrightarrow 0$$

Therefore, by Thm 2.16, we have

Thm

Let $A \subset X$. Then we have the long exact seq.s

$$\cdots \longrightarrow H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(X) \longrightarrow \cdots$$

and

$$\cdots \longrightarrow \tilde{H}_n(A) \longrightarrow \tilde{H}_n(X) \longrightarrow \tilde{H}_n(X,A) \longrightarrow \tilde{H}_{n-1}(A) \longrightarrow \cdots$$

where $\tilde{H}_n(X,A) = H_n(X,A)$ if $A \neq \phi$ $\quad (\tilde{H}_n(X,\phi) = \tilde{H}_n(X))$

Example 2.17

For $(X,A) = (D^n, \partial D^n)$,

$$\cdots \to \underbrace{\tilde{H}_i(D^n)}_{=0} \longrightarrow \tilde{H}_i(D^n, \partial D^n) \to \underbrace{\tilde{H}_{i-1}(\partial D^n)}_{\cong S^{n-1}} \longrightarrow \underbrace{\tilde{H}_{i-1}(D^n)}_{=0} \to \cdots$$

$\Rightarrow \tilde{H}_i(D^n, \partial D^n) \cong \tilde{H}_{i-1}(S^{n-1}) \quad \forall i$

Example 2.18

Let $x_0 \in X$, $(X,A) = (X, \{x_0\})$

$\Rightarrow$

$$\cdots \longrightarrow \underbrace{\tilde{H}_i(\{x_0\})}_{=0} \longrightarrow \tilde{H}_i(X) \xrightarrow{\cong} \tilde{H}_i(X, \{x_0\}) \longrightarrow \underbrace{\tilde{H}_{i-1}(\{x_0\})}_{=0} \longrightarrow \cdots$$

$\Rightarrow H_i(X, \{x_0\}) \cong \tilde{H}_i(X) \quad \forall i$

We write $f:(X,A)\longrightarrow(Y,B)$ if $f(A)\subseteq B$. This kind of map induces a chain map

$$f_\#: C_n(X,A)=C_n(X)/C_n(A)\longrightarrow C_n(Y,B)=C_n(Y)/C_n(B)$$

because $f_\#(C_n(A))\subseteq C_n(B)$.

$\rightsquigarrow$ $f_*: H_n(X,A)\longrightarrow H_n(Y,B)$ $\forall n$

Cor If V deformation retracts onto A, then $\Rightarrow$ $H_n(V,A)\cong H_n(A,A)=0$

Prop 2.19

If two maps $f,g:(X,A)\longrightarrow(Y,B)$ are homotopic through maps $(X,A)\to(Y,B)$ (i.e. $\exists$ hpt $H: X\times I\to Y$ s.t. $H(A\times I)\subseteq B$), then $f_*=g_*: H_n(X,A)\to H_n(Y,B)$

pf

Note that the homotopy operator P in the proof of Thm 2.10 satisfies $P(C_n(A))\subseteq C_{n+1}(B)$

$\Rightarrow$ it induces a chain homotopy between $f_\#, g_\#: C(X,A)\to C(Y,B)$ $\Rightarrow$ ok. $\#$

Prop

A map $f:(X,A)\to(Y,B)$ of pairs induces a chain map $\leftarrow$ postpone to subsection "Naturality"

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(A) \arrow[r] \arrow[d, "f_*"] \arrow[dr, phantom, "\circlearrowleft"] & H_n(X) \arrow[r] \arrow[d, "f_*"] \arrow[dr, phantom, "\circlearrowleft"] & H_n(X,A) \arrow[r] \arrow[d, "f_*"] \arrow[dr, phantom, "\circlearrowleft"] & H_{n-1}(A) \arrow[r] \arrow[d, "f_*"] & \cdots \\
\cdots \arrow[r] & H_n(B) \arrow[r] & H_n(Y) \arrow[r] & H_n(Y,B) \arrow[r] & H_{n-1}(B) \arrow[r] & \cdots
\end{tikzcd}$$

pf: exer.

Remark

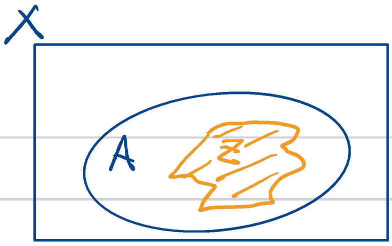
Let $B\subseteq A\subseteq X$ be spaces. Then we have a short exact seq. of complexes:

$$0\to C_\cdot(A,B)\to C_\cdot(X,B)\to C_\cdot(X,A)\to 0$$

So we obtain a long exact sequence

$$\cdots\to H_n(A,B)\to H_n(X,B)\to H_n(X,A)\to H_{n-1}(A,B)\to\cdots$$

need this for proving Thm 2.13



Excision

Thm 2.20 (Excision Thm)

Given subspaces $Z \subseteq A \subseteq X$ s.t. $\underset{\text{closure}}{cl(Z)} \subseteq \underset{\text{interior}}{int(A)}$, the inclusion $(X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$ induces isomorphisms

$$H_n(X \setminus Z, A \setminus Z) \xrightarrow{\cong} H_n(X, A) \quad \forall n$$

$B = X \setminus Z$

$Z = X \setminus B$

Equivalently, for subspaces $A, B \subseteq X$ with the property $int(A) \cup int(B) = X$, the inclusion $(B, A \cap B) \hookrightarrow (X, A)$ induces isomorphisms $H_n(B, A\cap B) \xrightarrow{\cong} H_n(X,A)$ $\forall n$

重心的

Barycentric subdivision

For a space X, let $\mathcal{U} = \{U_\lambda\}$, $U_\lambda \subseteq X$ (not necessarily open) s.t. $\bigcup_\lambda int(U_\lambda) = X$, and let

$$C_n^{\mathcal{U}}(X) := \{ \Sigma n_i \sigma_i \in C_n(X) \mid \sigma_i(\Delta^n) \subseteq U_\lambda \text{ for some } U_\lambda \in \mathcal{U} \} \leq C_n(X)$$

Since $\partial : C_n(X) \to C_{n-1}(X)$ takes $C_n^{\mathcal{U}}(X)$ to $C_{n-1}^{\mathcal{U}}(X)$, we obtain a subcomplex $(C_\bullet^{\mathcal{U}}(X), \partial)$

Its homology will be denoted by $H_n^{\mathcal{U}}(X)$.

Prop 2.21

The inclusion $\iota : C_n^{\mathcal{U}}(X) \hookrightarrow C_n(X)$ is a chain homotopy equivalence.

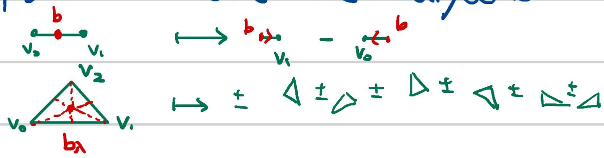
In particular, $H_n^{\mathcal{U}}(X) \cong H_n(X)$ $\forall n$.

sketch of pf (p.119 ~ 124)

① Consider (linear ver. of) $C_n(\Delta^N)$ (Note: Δ^N is a convex subset of a vec. sp.) and define the barycentric subdivision homo

$$S : C_n(\Delta^N) \longrightarrow C_n(\Delta^N)$$

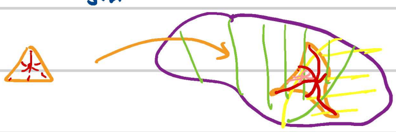
picture of S



by induction ($S(\lambda) = b_\lambda(S\partial\lambda)$ "add barycenter", $S = id$ on 0-simplexes)

② Define $T : C_n(\Delta^N) \to C_{n+1}(\Delta^N)$ inductively $\big(T(\lambda) = b_\lambda(\lambda - T\partial\lambda) \big)$ s.t.

$$\partial T + T\partial = id - S$$



So $S = id$ on homologies

③ For $\sigma \in C_n(X)$, $\sigma = \Delta^n \xrightarrow{id} \Delta^n \xrightarrow{\sigma} X$. apply S, T at $\Delta^n \xrightarrow{id} \Delta^n$. In this way, one can get S, T on $C_n(X)$

④ Apply S, T many times so that each small singular simplex in the formula maps into $int(U_\lambda)$ for some $U_\lambda \in \mathcal{U}$. #