Cor 2.14

$n>0$, $\tilde{H}_n(S^n) \cong \mathbb{Z}$, $\tilde{H}_i(S^n) = 0 \ \forall i \neq n$ (i.e. $H_i(S^n) = \begin{cases} \mathbb{Z} & \text{if } i = 0, n \\ 0 & i \neq 0, n \end{cases}$)

pf

For $n>0$, take $(X, A) = (D^n, \underset{\text{satisfies the nbd condition}}{S^{n-1}}) \Rightarrow X/A \cong S^n$.

Since D^n is contractible, $\tilde{H}_i(D^n) = 0 \ \forall i$

$\Rightarrow$ we have the exact seq.

$$\cdots \to \tilde{H}_k(S^{n-1}) \to \underset{\overset{\|}{0}}{\tilde{H}_k(D^n)} \to \tilde{H}_k(S^n) \to \tilde{H}_{k-1}(S^{n-1}) \to \underset{\overset{\|}{0}}{\tilde{H}_k(D^n)} \to \cdots$$

$\Rightarrow \tilde{H}_k(S^n) \cong \tilde{H}_{k-1}(S^{n-1}) \ \forall k$

$\Rightarrow \tilde{H}_n(S^n) \cong \cdots \cong \tilde{H}_0(S^0) \cong \mathbb{Z}$ and $\tilde{H}_i(S^n) = 0$ if $i \neq n$ #

Cor 2.15 see pf of Thm 1.9

$\underset{\overset{\|\wr}{S^{n-1}}}{\partial D^n}$ is not a retract of D^n. Hence every map $f: D^n \to D^n$ has a fixed point.

pf

Suppose $r: D^n \to \partial D^n$ is a retraction, i.e. $r \circ i = id_{\partial D^n}$, i = inclusion $S^{n-1} = \partial D^n \hookrightarrow D^n$

$\Rightarrow$ the induced map

$$\mathbb{Z} \cong \tilde{H}_{n-1}(\partial D^n) \xrightarrow{i_*} \underset{\overset{\|}{0}}{\tilde{H}_{n-1}(D^n)} \xrightarrow{r_*} \underset{\overset{\mathbb{Z}}{\wr\|}}{\tilde{H}_{n-1}(\partial D^n)} \quad = id_{\tilde{H}_{n-1}(\partial D^n)} \quad \to\leftarrow \quad \#$$

More homological alg: short exact seq of chain complexes

Let (A, ∂^A), (B, ∂^B), (C, ∂^C) be chain complexes, $i: (A, \partial^A) \to (B, \partial^B)$, $j: (B, \partial^B) \to (C, \partial^C)$ be chain maps. We say

$$0 \to (A, \partial^A) \xrightarrow{i} (B, \partial^B) \xrightarrow{j} (C, \partial^C) \to 0$$

is a short exact sequence of chain complexes if $0 \to A_n \xrightarrow{i_n} B_n \xrightarrow{j_n} C_n \to 0$ is a short exact seq for each n.

Thm 2.16

A short exact seq of complexes $0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$ induces a long exact seq of homolog groups:

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{j_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(B) \to \cdots$$

where the connecting homomorphism $\partial: H_n(C) \to H_{n-1}(A)$ is defined by diagram chasing:

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & A_{n+1} \arrow[r, "\partial^A_{n+1}"] \arrow[d, "i_{n+1}"] & A_n \ni x \arrow[r, "\partial^A_n"] \arrow[d, "i_n"] & A_{n-1} \arrow[r] \arrow[d, "i_{n-1}"] & \cdots \\
\cdots \arrow[r] & B_{n+1} \ni y \arrow[r, "\partial^B_{n+1}"] \arrow[d, "j_{n+1}"] & B_n \arrow[r, "\partial^B_n"] \arrow[d, "j_n"] & B_{n-1} \arrow[r] \arrow[d, "j_{n-1}"] & \cdots \\
\cdots \arrow[r] & C_{n+1} \ni z \arrow[r, "\partial^C_{n+1}"] \arrow[d] & C_n \arrow[r, "\partial^C_n"] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

$\partial(z)$

$\exists\, b \mapsto \partial^B_n(b) \in B_{n-1}$

$C_n \ni c \mapsto 0$

by exactness, $\exists!\, a \in A_{n-1}$

by commutativity

$[c] \in H_n(C)$

$\partial([c]) := [a] \in H_{n-1}(A)$

pf of Thm 2.16:

① ∂ is well-def:

- If $b' \in B_n$ is another choice s.t. $j_n(b') = c$, then $b' - b \in \ker(j_n) = \operatorname{im}(i_n)$
  $\Rightarrow \exists!\, x \in A_n$ s.t. $i_n(x) = b' - b \Rightarrow i_{n-1}(a + \partial^A_n(x)) = \partial^B_n(b) + \partial^B_n(b'-b) = \partial^B_n(b')$
  $\Rightarrow \partial([c]) = [a + \partial^A_n(x)] = [a]$
- If $c' \in C_n$ s.t. $[c'] = [c]$, then $c' = c + \partial^C_{n+1}(z)$ $\exists\, z \in C_{n+1}$
  $\exists\, y \in B_{n+1}$ s.t. $j_{n+1}(y) = z \Rightarrow j_n(b + \partial^B_{n+1}(y)) = c + \partial^C_{n+1}(z) = c'$
  Note: $\partial^B_n(b + \partial^B_{n+1}(y)) = \partial^B_n(b) + \underbrace{\partial^B_n \partial^B_{n+1}(y)}_{=0} = \partial^B_n(b) \Rightarrow \partial([c']) = [a] = \partial([c])$

② ∂ is a group homomorphism: clear from def

③ exactness:

- $\operatorname{im} i_* \subseteq \ker j_*$: because $j_* i_* = (ji)_* = 0$
- $\operatorname{im} j_* \subseteq \ker \partial$: for $[b] \in H_n(B)$, $\partial(j_*[b]) = [a] \in H_{n-1}(A)$, where $i_{n-1}(a) = \partial^B_n(b) = 0$ $\Rightarrow [a] = 0$
- $\operatorname{im} \partial \subseteq \ker i_*$: by def $i_*(\partial[c]) = [\partial^B_n(b)] = 0$ in $H_{n-1}(B)$
- $\ker j_* \subseteq \operatorname{im} i_*$: let $[b] \in H_n(B)$ s.t. $j_*[b] = 0$, i.e. $j(b) = \partial^C_{n+1}(z)$, $\exists\, z \in C_{n+1}$
  let $y \in B_{n+1}$ st. $j_{n+1}(y) = z \Rightarrow j(\partial^B_{n+1}(y)) = \partial^C_{n+1}(z) = j(b) \Rightarrow j(b - \partial^B_{n+1}(y)) = 0$
  $\Rightarrow \exists\, x \in A_n$ st. $i_n(x) = b - \partial^B_{n+1}(y) \Rightarrow i_*([x]) = [b - \partial^B_{n+1}(y)] = [b] \Rightarrow [b] \in \operatorname{im} i_*$
- $\ker \partial \subseteq \operatorname{im} j_*$: let $[c] \in H_n(C)$ s.t. $\partial([c]) = 0$, i.e., $a = \partial^A_n(x)$ $\exists\, x \in A_n$
  $\Rightarrow \partial^B_n(i_n(x)) = i_{n-1}(\partial^A_n(x)) = i_{n-1}(a) = \partial^B_n(b) \Rightarrow \partial^B_n(b - i_n(x)) = 0 \Rightarrow [b - i_n(x)] \in H_n(B)$
  and $j_*([b - i_n(x)]) = [j_n(b) - \overset{=0}{\boxed{j_n i_n}}(x)] = [j_n b] = [c] \Rightarrow [c] \in \operatorname{im}(j_*)$
- $\ker i_* \subseteq \operatorname{im} \partial$: let $[a] \in H_{n-1}(A)$ s.t. $i_*([a]) = 0 \Rightarrow i_{n-1}(a) = \partial^B_n(b)$ $\exists\, b \in B_n$
  Let $c = j_n(b) \Rightarrow \partial^C_n(c) = \partial^C_n j_n(b) = j_{n-1}\partial^B_n(b) = \underbrace{j_{n-1} i_{n-1}}_{=0}(a) = 0 \Rightarrow c$ descends to homology $[c] \in H_n(C)$

  and $\partial([c]) = [a] \Rightarrow [a] \in \operatorname{im} \partial$ #

Relative homology group

Let A be a subspace of a top. sp. X. $\Rightarrow$ $C_n(A)$ is a subgroup of $C_n(X)$.

Let $C_n(X,A) := C_n(X)/C_n(A)$

Since $\partial_n^X(C_n(A)) \subseteq C_{n-1}(A)$, ∂_n^X induces a homo $\partial_n^{X,A}: C_n(X,A) \longrightarrow C_{n-1}(X,A)$

$(\partial^X)^2=0 \Rightarrow (\partial^{X,A})^2=0$ $\Rightarrow$ we have a chain complex

$$\cdots \longrightarrow C_n(X,A) \xrightarrow{\partial} C_{n-1}(X,A) \longrightarrow \cdots$$

The associated homology group $H_n(X,A) = \ker(\partial_n^{X,A})/\mathrm{im}(\partial_{n+1}^{X,A})$ is called the relative homology group.

Note that we have a short exact seq. of complexes

$$0 \longrightarrow (C(A), \partial^A) \longrightarrow (C(X), \partial^X) \longrightarrow (C(X,A) = C(X)/C(A), \partial^{X,A}) \longrightarrow 0$$

Therefore, by Thm 2.16, we have

Thm

Let $A \subset X$. Then we have the long exact seq.s

$$\cdots \longrightarrow H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(X) \longrightarrow \cdots$$

and

$$\cdots \longrightarrow \tilde{H}_n(A) \longrightarrow \tilde{H}_n(X) \longrightarrow \tilde{H}_n(X,A) \longrightarrow \tilde{H}_{n-1}(A) \longrightarrow \cdots$$

where $\tilde{H}_n(X,A) = H_n(X,A)$ if $A \neq \emptyset$

Example 2.17

For $(X,A) = (D^n, \partial D^n)$,

$$\cdots \to \underbrace{\tilde{H}_i(D^n)}_{=0} \longrightarrow \tilde{H}_i(D^n, \partial D^n) \to \tilde{H}_{i-1}(\underbrace{\partial D^n}_{\cong S^{n-1}}) \longrightarrow \underbrace{\tilde{H}_{i-1}(D^n)}_{=0} \to \cdots$$

$\Rightarrow$ $\tilde{H}_i(D^n, \partial D^n) \cong \tilde{H}_i(S^{n-1})$ $\forall i$

Example 2.18

Let $x_0 \in X$, $(X,A) = (X, \{x_0\})$

$\Rightarrow$ $\cdots \longrightarrow \underbrace{\tilde{H}_i(\{x_0\})}_{=0} \longrightarrow \tilde{H}_i(X) \xrightarrow{\cong} \tilde{H}_i(X, \{x_0\}) \longrightarrow \underbrace{\tilde{H}_{i-1}(\{x_0\})}_{=0} \longrightarrow \cdots$

$\Rightarrow$ $H_i(X, \{x_0\}) \cong H_i(X)$ $\forall i$