Induced homomorphism

Let $f: X \to Y$ be a continuous map. $\forall\ \sigma: \Delta^n \to X$ singular n-simplex in X, we have an induced singular n-simplex $f \circ \sigma: \Delta^n \xrightarrow{\sigma} X \xrightarrow{f} Y$ in Y.

$\leadsto$
$$f_\# : C_n(X) \longrightarrow C_n(Y) : f_\#(\sum_i n_i \sigma_i) = \sum_i n_i \cdot (f \circ \sigma_i) \leftarrow \text{group homomorphism}$$

<u>Lemma</u>: $f_\# \circ \partial_n^X = \partial_n^Y \circ f_\#$ pf: computation

In short, we have a chain map $f_\# : (C_\bullet(X), \partial^X) \to (C_\bullet(Y), \partial^Y)$, i.e.,

a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1}(X) \arrow[r, "\partial_{n+1}^X"] \arrow[d, "f_\#"] \arrow[dr, phantom, "\circlearrowright"] & C_n(X) \arrow[r, "\partial_n^X"] \arrow[d, "f_\#"] \arrow[dr, phantom, "\circlearrowright"] & C_{n-1}(X) \arrow[r] \arrow[d, "f_\#"] & \cdots \\
\cdots \arrow[r] & C_{n+1}(Y) \arrow[r, "\partial_{n+1}^Y"] & C_n(Y) \arrow[r, "\partial_n^Y"] & C_{n-1}(Y) \arrow[r] & \cdots
\end{tikzcd}$$

$\Rightarrow$ $f_\#(\ker(\partial_n^X)) \subseteq \ker \partial_n^Y$

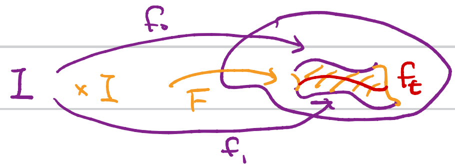
$f_\#(\mathrm{im}(\partial_{n+1}^X)) \subseteq \mathrm{im}(\partial_{n+1}^Y)$

$\Downarrow$

Therefore, we have a group homomorphism
$$f_* : H_n(X) = \ker(\partial_n^X) / \mathrm{im}(\partial_{n+1}^X) \longrightarrow H_n(Y) = \ker(\partial_n^Y) / \mathrm{im}(\partial_{n+1}^Y) \quad \forall\ n$$

<u>Prop</u>

If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are continuous maps, then
$$(g \circ f)_* = g_* \circ f_* \quad \text{and} \quad (id_X)_* = id_{H_n(X)}$$



<u>Homotopy equivalence</u> (Ch0, page 3)

Let $f_0, f_1 : X \to Y$ be continuous maps. We say f_0 and f_1 are homotopic if $\exists$ continuous $F: X \times I \longrightarrow Y$ s.t.
$$F(x, 0) = f_0(x), \qquad F(x, 1) = f_1(x) \qquad \forall\ x \in X.$$

Such F is called a homotopy. In this situation, we will denote $f_0 \simeq_F f_1$ or $f_0 \simeq f_1$.

A map $f: X \longrightarrow Y$ is called a homotopy equivalence if $\exists\ g: Y \to X$ s.t. $g \circ f \simeq id_X$, $f \circ g \simeq id_Y$

If $\exists$ a homotopy equi. $X \to Y$, we say X and Y are homotopy equivalent or have the same homotopy type.

Let $A \subseteq X$. A homotopy $F: X \times I \longrightarrow Y$ with the property $F|_{A \times I} = id_{A \times I}$ is called a homotopy relative to A. (e.g. A path homotopy is a homotopy $I \times I \to X$ rel to $\overset{I_{\cup}}{\{0,1\}}$)

$\longleftarrow$ topo ver of projection

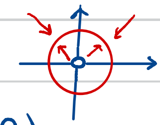
A retraction of X onto A is a map $r: X \to X$ s.t. $r(X) = A$ and $r|_A = id$ (or equivalently, $r^2 = r$)

A deformation retraction of X onto A is a homotopy from id_X to a retraction $r: X \to A$

$\uparrow$ a special case of homotopy equivalence contractible = homotopy equivalent to a point

A strong deformation retraction is a homotopy from id_X to r relative to A.

Example



Let $S^1 = \{(x,y)\in\mathbb{R}^2 \mid x^2+y^2=1\} \subseteq \mathbb{R}^2\setminus\{(0,0)\}$.

Write $(x,y)\in \mathbb{R}^2\setminus\{(0,0)\}$ by polar coordinate $(x,y)=(r_0\cos\theta, r_0\sin\theta)$

more precise: $i\circ r$

The map $r: \mathbb{R}^2\setminus\{(0,0)\} \longrightarrow S^1 \subseteq \mathbb{R}^2\setminus\{\vec 0\}$: $r(r_0\cos\theta, r_0\sin\theta) := (\cos\theta, \sin\theta)$ is a retraction

Let $F: (\mathbb{R}^2\setminus\{\vec 0\})\times I \to \mathbb{R}^2\setminus\{\vec 0\}$: $F((r_0\cos\theta, r_0\sin\theta), t) := ([(1-t)r_0+t]\cos\theta, [(1-t)r_0+t]\sin\theta)$

$\Rightarrow$ F is a strong deformation retraction

$\Rightarrow \begin{cases} ① \; S^1 \xrightarrow{i} \mathbb{R}^2\setminus\{\vec 0\} \xrightarrow{r} S^1 = id \\ ② \; \mathbb{R}^2\setminus\{\vec 0\} \xrightarrow{r} S^1 \xrightarrow{i} \mathbb{R}^2\setminus\{\vec 0\} \underset{F}{\simeq} id_{\mathbb{R}^2\setminus\{\vec 0\}} \end{cases}$

$\Rightarrow$ $i: S^1 \hookrightarrow \mathbb{R}^2\setminus\{\vec 0\}$, $r: \mathbb{R}^2\setminus\{\vec 0\} \to S^1$ are homotopy equivalences, and S^1 and $\mathbb{R}^2\setminus\{\vec 0\}$ have the same homotopy type

exer: ① "homotopic", "homotopy equivalence" are equivalence relations

② (Prop 1.18) If $\varphi: X\to Y$ is a homotopy equivalence, then $\varphi_*: \pi_1(X, x_0) \longrightarrow \pi_1(Y, \varphi(x_0))$ is an iso $\forall x_0\in X$

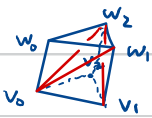
Homotopy invariance

Thm 2.10

If two maps $f,g: X\longrightarrow Y$ are homotopic, then they induce the same homomorphism

$$f_* = g_* : H_n(X) \longrightarrow H_n(Y) \quad \forall n$$

pf of Thm 2.10:



Divide $\Delta^n\times I$ into simplexes and define $P: C_n(X) \longrightarrow C_{n+1}(Y)$ by

$$P(\sigma) := \sum_{i=0}^{n} (-1)^i F\circ(\sigma\times id_I)|[v_0,\dots,v_i,w_i,\dots,w_n]$$

i.e.
$P(\sigma)(t_0,\dots,t_{n+1})$
$= \sum_{i=0}^{n} (-1)^i F\big(\sigma(t_0,\dots,t_{i-1},t_i+t_{i+1},\dots,t_{n+1}), t_{i+1}+\dots+t_{n+1}\big)$

where $F: X\times I \to Y$ is a homotopy from f to g, and

$(\sigma\times id_I)|[v_0,\dots,v_i,w_i,\dots,w_n] : \Delta^{n+1} \longrightarrow \Delta^n\times I \subseteq \mathbb{R}^{n+2} \longrightarrow X\times I$

$(t_0,\dots,t_{n+1}) \longmapsto (t_0,\dots,t_i,0,\dots,0;0) + (0,\dots,0,t_{i+1},\dots,t_{n+1}; t_{i+1}+\dots+t_{n+1})$

$\xmapsto{\sigma\times id_I} (\sigma(t_0,\dots,t_{i-1},t_i+t_{i+1},\dots,t_{n+1}), t_{i+1}+\dots+t_{n+1})$

Recall:
$\partial\sigma = \sum(-1)^i \sigma|[v_0,\dots,\hat{v}_i,\dots,v_n]$
$(t_0,\dots,t_{n-1})$
$\mapsto \sum(-1)^i\sigma(t_0,\dots,t_{i-1},0,t_i,\dots,t_n)$

$\Rightarrow$ ① $\partial P(\sigma)(t_0,\dots,t_n)$

$$= \sum_{i=0}^{n} (-1)^i P(\sigma)(t_0,\dots,t_{i-1},0,t_i,\dots,t_n)$$

$$= \sum_{i=0}^{n} (-1)^i \Big(\sum_{j=0}^{i-2} (-1)^j F(\sigma(t_0,\dots,t_j+t_{j+1},\dots,t_{i-1},0,\dots,t_n), t_{j+1}+\dots+t_n)$$
$$+ \underbrace{(-1)^{i-1} F(\sigma(t_0,\dots,t_{i-1},t_i,\dots,t_n), t_{i+1}+\dots+t_n)}_{\text{if } i\neq 0} + \cdots$$

if $i \neq n$

$$\cdots + \underline{(-1)^i F(\sigma(t_0, \cdots, t_i, \cdots, t_n), t_{i+1} + \cdots + t_n)}$$
$$+ \sum_{j=i+2}^{n-1} (-1)^{j+1} F(\sigma(t_0, \cdots, t_i, 0, t_{i+1}, \cdots, t_j + t_{j+1}, \cdots, t_n), t_{j+1} + \cdots + t_n) \Bigg)$$

② $P\partial(\sigma)(t_0, \cdots, t_n)$

$$= \sum_{j=0}^{n-1} (-1)^j F(\partial\sigma(t_0, \cdots, t_{j-1}, t_j + t_{j+1}, \cdots, t_n), t_{j+1} + \cdots + t_n)$$
$$= \sum_{j=0}^{n-1} (-1)^j \Bigg(\sum_{i=0}^{j} (-1)^i F(\sigma(t_0, \cdots, t_{i-1}, 0, t_i, \cdots, t_j + t_{j+1}, \cdots, t_n), t_{j+1} + \cdots + t_n)$$
$$+ \sum_{i=j+2}^{n} (-1)^{i+1} F(\sigma(t_0, \cdots, t_j + t_{j+1}, \cdots, t_{i-1}, 0, t_i, \cdots, t_n), t_{j+1} + \cdots + t_n) \Bigg)$$

So $(\partial P(\sigma) + P\partial(\sigma))(t_0, \cdots, t_n)$

$$= F(\sigma(t_0, \cdots, t_n), \underbrace{t_1 + \cdots + t_n}_{=1}) - F(\sigma(t_0, \cdots, t_n), 0)$$
$$= g(\sigma(t_0, \cdots, t_n)) - f(\sigma(t_0, \cdots, t_n))$$

In other words, P is a chain homotopy:

$$\partial P + P\partial = g_\# - f_\#$$

$\Rightarrow \quad f_* = g_* : H_n(X) \longrightarrow H_n(Y) \quad \forall n \quad \#$

Cor 2.11

If $f: X \to Y$ is a homotopy equivalence, then $f_*: H_n(X) \xrightarrow{\cong} H_n(Y)$ is an iso $\forall n$.

Examples

Exact seq and computation of H_n

Def

Recall: quotient topo X/A

A chain complex

$$\cdots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots$$

is exact if $\ker(\partial_n) = \operatorname{im}(\partial_{n+1}) \quad \forall n$

An exact seq of the form $0 \to A \to B \to C \to 0$ is called a short exact sequence.

Thm 2.13 ← prove later

If X is a space and A is a nonempty closed subsp that is a deformation retract of some nbd in X, then there is an exact seq

$$\cdots \to \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \xrightarrow{j_*} \tilde{H}_n(X/A) \xrightarrow{\partial} \tilde{H}_{n-1}(X) \xrightarrow{i_*} \cdots \to \tilde{H}_0(X/A) \to 0$$

where $i: A \hookrightarrow X$ is the inclusion, $j: X \to X/A$ is the quotient map

Cor 2.14

$n>0$, $\tilde{H}_n(S^n) \cong \mathbb{Z}$, $\tilde{H}_i(S^n) = 0 \ \forall i \neq n$ (i.e. $H_i(S^n) = \begin{cases} \mathbb{Z} & \text{if } i = 0, n \\ 0 & i \neq 0, n \end{cases}$)

pf

For $n>0$, take $(X, A) = (D^n, \underset{\swarrow}{S^{n-1}})$ ⟹ $X/A \cong S^n$. satisfies the nbd condition

Since D^n is contractible, $\tilde{H}_i(D^n) = 0 \ \forall i$

⟹ we have the exact seq.

$$\cdots \to \tilde{H}_k(S^{n-1}) \to \underbrace{\tilde{H}_k(D^n)}_{=0} \to \tilde{H}_k(S^n) \to \tilde{H}_{k-1}(S^{n-1}) \to \underbrace{\tilde{H}_k(D^n)}_{=0} \to \cdots$$

⟹ $\tilde{H}_k(S^n) \cong \tilde{H}_{k-1}(S^{n-1}) \ \forall k$

⟹ $\tilde{H}_n(S^n) \cong \cdots \cong \tilde{H}_0(S^0) \cong \mathbb{Z}$ and $\tilde{H}_i(S^n) = 0$ if $i \neq n$ ∎

Cor 2.15 see pf of Thm 1.9

$\underset{\cong S^{n-1}}{\partial D^n}$ is not a retract of D^n. Hence every map $f: D^n \to D^n$ has a fixed point.

pf

Suppose $r: D^n \to \partial D^n$ is a retraction, i.e. $r \circ i = id_{\partial D^n}$, i = inclusion $S^{n-1} = \partial D^n \hookrightarrow D^n$

⟹ the induced map

$$\mathbb{Z} \cong \underbrace{\tilde{H}_{n-1}(\partial D^n)}_{} \xrightarrow{i_*} \underbrace{\tilde{H}_{n-1}(D^n)}_{=0} \xrightarrow{r_*} \underbrace{\tilde{H}_{n-1}(\partial D^n)}_{\cong \mathbb{Z}} \quad = id_{\tilde{H}_{n-1}(\partial D^n)} \quad \rightarrow\leftarrow$$ ∎

<u>More homological alg: short exact seq of chain complexes</u>

Let (A, ∂^A), (B, ∂^B), (C, ∂^C) be chain complexes, $i: (A, \partial^A) \to (B, \partial^B)$, $j: (B, \partial^B) \to (C, \partial^C)$ be chain maps. We say

$$0 \to (A, \partial^A) \xrightarrow{i} (B, \partial^B) \xrightarrow{j} (C, \partial^C) \to 0$$

is a short exact sequence of chain complexes if $0 \to A_n \xrightarrow{i_n} B_n \xrightarrow{j_n} C_n \to 0$ is a short exact seq for each n.

Thm 2.16

A short exact seq of complexes $0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$ induces a long exact seq of homolog groups:

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{j_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(B) \to \cdots$$

where the connecting homomorphism $\partial: H_n(C) \to H_{n-1}(A)$ is defined by diagram chasing: