Ch2 Homology

π_1 is especially useful for studying spaces of low dimension

e.g. we used π_1 to prove $\mathbb{R}^2$ is NOT homeomorphic to $\mathbb{R}^n$, $n \neq 2$

Q: what about high-dimensional spaces? (But π_1 is NOT enough to prove $\mathbb{R}^n \not\cong \mathbb{R}^m$) $n\neq m,\ n,m>2$

Remark

$\exists$ high-dim ver of π_1:

π_1:  / homotopy , higher π_n :  / homotopy

But computations of π_n are very difficult. Even $\pi_n(S^m)$ are highly nontrivial.

"difficult things" $\boxed{\text{cat. of spaces}} \longrightarrow \boxed{\text{cat. of groups}}$ "easier things"

- π_1 — NOT good for high-dim sp
- π_n — too difficult to compute
- (co)homolog — works for high dim sp; more managable than π_n

Remark depends on what you want to study

$\exists$ many types of (co)homology such as

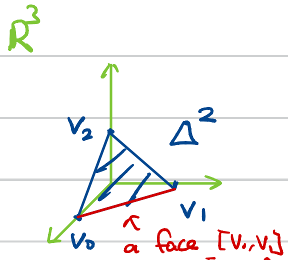
need "triangulation" simplicial, singular (co)homology — for general topo. sp ← will focus on this version

Čech, sheaf cohomology — usual appear in Algebraic Geometry

de Rham cohomology — good for differentiable manifolds (Diff Geo)

Lie algebra cohomology, Hochschild (co)homology, K-theory, etc

§2.1 Singular homology

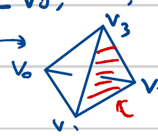


The standard n-simplex Δ^n is

$$\Delta^n = \{ (t_0, t_1, \cdots, t_n) \in \mathbb{R}^{n+1} \mid \Sigma_i\, t_i = 1 ,\ t_i \geq 0\ \forall i \}$$

In textbook, a simplex with vertices $V_0, \cdots, V_n$ is denoted $[V_0, \cdots, V_n]$. So

$$\Delta^n = [V_0, V_1, \cdots, V_n]$$



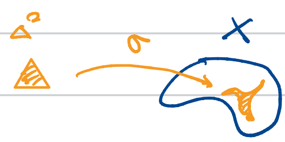
where $V_i = (0, \cdots 0, 1, 0, \cdots, 0)$ ← $(i+1)$-th component

The simplex $[V_0, \cdots, \hat{V_i}, \cdots, V_n]$ ← omit V_i is called a face of $[V_0, \cdots, V_n]$

Let X be a topological space.

A singular n-simplex in X is a continuous map

$$\sigma : \Delta^n \longrightarrow X$$

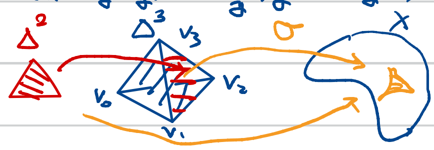


Let $C_n(X)$ be the free abelian group generated by $\{$singular n-simplexes in $X\}$, i.e.

$$C_n(X) := \{ n_1\sigma_1 + \cdots + n_k\sigma_k \mid n_1,\dots,n_k \in \mathbb{Z};\ \sigma_1,\dots,\sigma_k : \Delta^n \to X \}$$

An element $\sum_{i=1}^{k} n_i\sigma_i \in C_n(X)$ is called a (singular) n-chain
$\underbrace{n_i}_{\in\mathbb{Z}}$ ← coeff; can change

The boundary map $\partial_n : C_n(X) \longrightarrow C_{n-1}(X)$ is defined by $\partial_n(\sum_j n_j\sigma_j) = \sum_j n_j \partial_n(\sigma_j)$,



where

$$\partial_n(\sigma) := \sum_i (-1)^i \sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_n]}.$$

Here, $\sigma|_{[v_0,\dots,\hat{v}_i,\dots v_n]} : \Delta^{n-1} \longrightarrow X$ is the $(n-1)$-simplex $\quad \sigma|_{[v_0,v_1,v_2]}$

$$\sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_n]}(t_0,\dots,t_{n-1}) = \sigma(t_0v_0 + \cdots + t_{i-1}v_{i-1} + t_iv_{i+1} + \cdots + t_{n-1}v_n)$$
$$= \sigma(t_0,\dots,t_{i-1},0,t_i,\dots,t_{n-1})$$

Thus, we have

$$\cdots \longrightarrow C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \to 0$$

<u>Lemma</u> (cf. Lemma 2.1) ↙ usually people just write $\partial^2 = 0$

$$\partial_n\partial_{n+1} = 0 \quad \forall n \geq 0$$

pf: computation

Therefore, we have $\operatorname{im}(\partial_{n+1}) \trianglelefteq \ker(\partial_n)$. The quotient group

$$H_n(X) := \ker(\partial_n)/\operatorname{im}(\partial_{n+1})$$

is called the (n-th) singular homology group of X

An element in $\ker(\partial_n)$ is called a (singular) n-cycle

" $\operatorname{im}(\partial_{n+1})$ " (singular) n-boundary

<u>Remark</u>

The data

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots \quad \text{with the property } \partial_n \circ \partial_{n+1} = 0 \ \forall n$$

is called a chain complex

<u>Example (Prop 2.8)</u>

Let X be a point. Then there is only one continuous map $\Delta^n \xrightarrow{e_n} X$ for each n

$\Rightarrow C_n(X) \cong \mathbb{Z}$

<u>Note</u>: $\partial_n(e_n) = \sum_i (-1)^i \underbrace{e_n|_{[v_0,\dots,\hat{v}_i,\dots,v_n]}}_{\text{constant } = e_{n-1}} = \underbrace{e_{n-1} - e_{n-1} + e_{n-1} - \cdots}_{n+1 \text{ terms}} = \begin{cases} 0 & \text{if } n \text{ is odd} \\ e_{n-1} & \text{if } n \text{ is even} \end{cases}$

...

...

So the singular chain complex is isomorphic to

$$\cdots \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \underset{\substack{\ker=0\\ \text{im}=0}}{\overset{n=2}{\mathbb{Z}}} \xrightarrow{\cong} \underset{\substack{\ker=\mathbb{Z}\\ \text{im}=\mathbb{Z}}}{\overset{n=1}{\mathbb{Z}}} \xrightarrow{0} \underset{\substack{\ker=\mathbb{Z}\\ \text{im}=0}}{\overset{n=0}{\mathbb{Z}}} \to 0$$

$$\Rightarrow \quad H_n(*) \cong \begin{cases} \mathbb{Z} & \text{if } n=0 \\ 0 & n>0 \end{cases} \quad \cdots$$

#

Prop 2.6

Let X be a space. If X_α are the path-connected components, then

$$H_n(X) \cong \bigoplus_\alpha H_n(X_\alpha)$$

pf: exercise

Therefore, we can assume X is path-connected when we study general theory of $H_n(X)$

Prop 2.7

If X is nonempty and path-connected, then $H_0(X) \cong \mathbb{Z}$. ($\Rightarrow$ In general, $H_0(\amalg X_\alpha) \cong \bigoplus_\alpha \mathbb{Z}$)

pf

Note that since $\Delta^0 =$ a point, $C_0(X) = \{\sum_{\text{finite}} n_i x_i \mid x_i \in X\}$, and we have the group homo

$$\varepsilon: C_0(X) \to \mathbb{Z}: \varepsilon(\sum n_i x_i) := \sum_i n_i \in \mathbb{Z} \quad \text{(nonempty} \Rightarrow \varepsilon \text{ is onto)}$$

① For $\sigma: \Delta^1 \to X$, $\varepsilon\partial_1(\sigma) = \varepsilon(\sigma|_{[v_1]} - \sigma|_{[v_0]}) = 1-1 = 0$. $\Rightarrow \varepsilon\circ\partial = 0 \Rightarrow \text{im}(\partial) \subseteq \ker\varepsilon$

② $\forall \sum_i n_i x_i \in \ker\varepsilon$, we have $\sum n_i = 0$

$$\Rightarrow \sum_{i=1}^k n_i x_i = n_1(x_1-x_2) + (n_1+n_2)(x_2-x_3) + (n_1+n_2+n_3)(x_3-x_4) + \cdots + \underbrace{(\textstyle\sum_i n_i)}_{=0}\cdot x_k$$

Let $\sigma_i: \Delta^1 \to X$ be a map s.t. $\sigma_i(1,0) = x_{i+1}$, $\sigma_i(0,1) = x_i$ (path-connected $\Rightarrow \exists \sigma_i$)

$$\Rightarrow \partial_1(\sigma_i) = x_i - x_{i+1}$$

$$\Rightarrow \sum_{i=1}^k n_i x_i = \sum_{i=1}^{k-1} (n_1+\cdots+n_i)\partial_1(\sigma_i) \in \text{im}(\partial_1)$$

①+② $\Rightarrow \ker\varepsilon = \text{im}\,\partial_1 \Rightarrow \mathbb{Z} = \text{im}(\varepsilon) \cong C_0(X)/\ker(\varepsilon) = C_0(X)/\text{im}\,\partial_1 = H_0(X)$ #

Def

The **reduced homology group** $\tilde{H}_n(X)$ of a space X (not necessarily path-connected) is

$$\tilde{H}_n(X) = \begin{cases} H_n(X) & \text{if } n \geq 1 \\ \ker(\varepsilon)/\text{im}(\partial_1) & \text{if } n=0 \end{cases}$$

Sometimes convenient

Some homological algebra

A **chain complex** is a sequence of abelian groups (vector spaces, (or R-modules, etc.)) C_n together with a sequence of group homomorphisms (or morphisms) $\partial_n : C_n \to C_{n-1}$ s.t. $\partial_n \circ \partial_{n+1} = 0 \quad \forall n$

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots$$

Or equivalently, $C_\bullet = \bigoplus_n C_n$, $\partial : C_\bullet \to C_{\bullet-1}$ s.t. $\partial^2 = 0$

The (n-th) **homology** of $(C_\bullet, \partial)$ is $H_n(C_\bullet, \partial) = \ker(\partial_n) / \mathrm{im}(\partial_{n+1})$

A **chain map** $\Phi : (C, \partial^C) \to (D, \partial^D)$ is a sequence of gp homos $\Phi_n : C_n \to D_n$ s.t. $\Phi_{n-1} \circ \partial^C_n = \partial^D_n \circ \Phi_n \quad \forall n$. In this case, we say the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial^C_{n+1}"] \arrow[d, "\Phi_{n+1}"'] & C_n \arrow[r, "\partial^C_n"] \arrow[d, "\Psi_n\ \Phi_n"] \arrow[dl, "h_n"'] & C_{n-1} \arrow[r] \arrow[d, "\Phi_{n-1}"] \arrow[dl, "h_{n-1}"'] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r, "\partial^D_{n+1}"'] & D_n \arrow[r, "\partial^D_n"'] & D_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

commutes

Prop

A chain map $\Phi : (C, \partial^C) \to (D, \partial^D)$ induces homomorphisms $\Phi_* : H_n(C, \partial^C) \to H_n(D, \partial^D) \ \forall n$

Def

We say two chain maps $\Phi, \Psi : (C, \partial^C) \to (D, \partial^D)$ are **(chain) homotopic** if $\exists$ seq of homomorphisms $h_n : C_n \to D_{n+1}$ s.t.

$$\Phi_n - \Psi_n = \partial^D_{n+1} \circ h_n + h_{n-1} \circ \partial^C_n$$

Such a seq $h = (h_n) : C \to D$ is called a **(chain) homotopy**

Prop

If the chain maps $\Phi, \Psi : (C, \partial^C) \to (D, \partial^D)$ are homotopic, then the induced maps

$$\Phi_* = \Psi_* : H_n(C) \longrightarrow H_n(D)$$

are equal $\forall n$.

pf

Suppose $h = (h_n)$ is a homotopy operator.

$\forall\ [x] \in H_n(C) = \ker(\partial^C_n) / \mathrm{im}(\partial^C_{n+1})$, $\quad \partial^C_n(x) = 0$

$\Rightarrow \quad \Phi_n(x) - \Psi_n(x) = \partial^D_{n+1}(h_n(x)) + h_{n-1}(\underbrace{\partial^C_n(x)}_{=0}) = \partial^D_{n+1}(-) \in \mathrm{im}(\partial^D_{n+1})$

$\Rightarrow \quad [\Phi_n(x)] = [\Psi_n(x)]$ in $H_n(D) = \ker(\partial^D_n) / \mathrm{im}(\partial^D_{n+1})$ #