$\boxed{\text{Fundamental group of } S^1}$ — use covering spaces to compute $\pi_1(S^1)$

Def

Given a space X, a covering space of X consists of a space $\tilde{X}$ and a map $p: \tilde{X} \to X$ s.t.
$\forall x \in X$, $\exists$ open nbd U of x in X s.t.

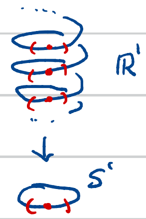
(*) $\quad p^{-1}(U) = \coprod_{\lambda} U_\lambda \subseteq \tilde{X}$

and $\quad p|_{U_\lambda} : U_\lambda \to U$ is a homeomorphism $\forall \lambda$.

Example

Let $p: \mathbb{R}^1 \to S^1 = \{(x,y) \mid x^2+y^2=1\} \subseteq \mathbb{R}^2$ be the map $p(s) := (\cos 2\pi s, \sin 2\pi s)$

Then $(\mathbb{R}^1, p)$ is a covering space of S^1



Two facts about covering spaces (homotopy lifting property, will be proved later)

Let $p: \tilde{X} \to X$ be a covering space.

(a) For each path $f: I \to X$, $f(0) = x_0 \in X$, and each $\tilde{x}_0 \in p^{-1}(x_0)$, $\exists! \, \underset{\text{lifted path}}{\tilde{f}} : I \to \tilde{X}$ s.t.

$\tilde{f}$ is called a lift of f $\quad p \circ \tilde{f} = f \quad$ and $\quad \tilde{f}(0) = \tilde{x}_0$

(b) For each homotopy $F: I \times I \to X$ of paths starting at x_0 and each $\tilde{x}_0 \in p^{-1}(x_0)$,
$\exists!$ lifted homotopy $\tilde{F}: I \times I \to \tilde{X}$ s.t.

$\quad p \circ \tilde{F} = F \quad$ and $\quad \tilde{F}(0, t) \equiv \tilde{x}_0$

$\tilde{F}$ a lift of F

Thm 1.7

$$\pi_1(S^1) \cong \pi_1(S^1, (1,0)) \cong \mathbb{Z} \qquad (S^1 = \{x^2+y^2=1\} \subseteq \mathbb{R}^2)$$

is generated by $[\omega]$, where $\omega(s) := (\cos 2\pi s, \sin 2\pi s)$ is a loop based at $(1,0)$

pf

Let $\omega_n : I \to S^1$ be the loop $\omega_n(s) := (\cos 2\pi n s, \sin 2\pi n s)$.

Step1: A loop based at $(0,1)$ is homotopic to ω_n for some n

Let $f: I \to S^1$ be an arbitrary loop based at $\overset{x_0 = p(0),}{(1,0)} \in S^1$. By (a), $\exists! \, \tilde{f} : I \to \mathbb{R}$ s.t.

$\quad p \circ \tilde{f} = f, \qquad \tilde{f}(0) = 0 \in \mathbb{R}$

Since $p(\tilde{f}(1)) = f(1) = (1,0)$, we have $\tilde{f}(1) = n$ for some $n \in \mathbb{Z} = p^{-1}((1,0)) \subseteq \mathbb{R}$.

Let $\tilde{\omega}_n : I \to \mathbb{R} : \tilde{\omega}_n(s) = n \cdot s \Rightarrow p \circ \tilde{\omega}_n = \omega_n, \quad \tilde{\omega}_n(1) = n = \tilde{f}(1)$

Furthermore, $\tilde{f} \simeq \tilde{\omega}_n$ in $\mathbb{R}$ (eg. $F(s,t) := (1-t)\tilde{f}(s) + t\tilde{\omega}_n(s)$ is a hpt)

Remark
we actually use $\pi_1(\mathbb{R}^1) = 0$ here

$\Rightarrow \quad f = p \circ \tilde{f} \simeq p \circ \tilde{\omega}_n = \omega_n$ in S^1 $\quad$ ($p \circ F : I \times I \xrightarrow{F} \mathbb{R} \xrightarrow{p} S^1$ is a hpt) pf (conti.)

Step 2 $[\omega_n] \neq [\omega_m]$ in $\pi_1(S^1, x_0)$ if $n \neq m$.

Suppose $\omega_n \simeq \omega_m$ and $H: I\times I \to S^1$ is a htp

By (b) $\exists!$ path htp $\tilde{H}: I \times I \to \mathbb{R}$ st. $p\circ\tilde{H} = H$, $\tilde{H}(0,t)\equiv 0$ (Note: $\tilde{H}(1,t) \equiv x_1$, $\exists x_1 \in \mathbb{R}$)

By uniqueness in (a), $\tilde{H}(-,0) = \tilde{\omega}_n$, $\tilde{H}(-,1) = \tilde{\omega}_m \Rightarrow n = \tilde{\omega}_n(1) = x_1 = \tilde{\omega}_m(1) = m$

Step 3 $[\omega_n][\omega_m] = [\omega_{n+m}]$

Let $\tilde{\omega}(s) := \begin{cases} n\cdot 2s, & 0 \le s \le \frac{1}{2} \\ n + m(2s-1), & \frac{1}{2} \le s \le 1 \end{cases}$ ← path in $\mathbb{R}$

$\Rightarrow$ $p\circ\tilde{\omega} = \omega_n \bullet \omega_m$, $\tilde{\omega} \simeq \tilde{\omega}_{n+m}$ in $\mathbb{R}$ $\Rightarrow$ $\omega_n \bullet \omega_m = p\circ\tilde{\omega} \simeq p\circ\tilde{\omega}_{n+m} = \omega_{n+m}$ #

Applications: can prove a few famous thms.

Thm 1.8 (Fundamental Thm of Alg)

Every nonconstant poly with coeff in $\mathbb{C}$ has a root in $\mathbb{C}$

pf

Assume $p(z) = z^n + a_1 z^{n-1} + \cdots + a_n$ has no roots in $\mathbb{C}$.

$\Rightarrow \forall r \ge 0$,

$$f_r(s) := \frac{p(re^{2\pi i s})/p(r)}{|p(re^{2\pi i s})/p(r)|} \quad : I \longrightarrow S^1 = \{|z|=1\} \subseteq \mathbb{C}$$

defines a loop in S^1 based at 1. Note that

① $\forall r$, $\boxed{f_r(s) \simeq f_0(s) \equiv 1 = \omega_0}$ in S^1 (htp: $F(s,t) = f_{tr}(s)$)

② If $r > \max\{1, |a_1| + \cdots + |a_n|\}$ ⊛, then $\boxed{f_r(s) \simeq \omega_n(s)}$ in S^1 because $\left\{ \text{continuity} \atop \forall s, t \in [0,1] \right.$

$H(s,t) := \dfrac{P_t(re^{2\pi i s})/P_t(r)}{|P_t(re^{2\pi i s})/P_t(r)|}$ is a homotopy in S^1, why? Need to check $P_t(re^{2\pi i s}) \neq 0$:

where $P_t(z) = z^n + t(a_1 z^{n-1} + \cdots + a_n)$

If $t \in [0,1]$ $|z| = r$ large ⊛, then $|z^n| > (|a_1| + \cdots + |a_n|)|z^{n-1}| > |a_1 z^{n-1}| + \cdots + |a_n| \ge |a_1 z^{n-1} + \cdots + a_n|$ $\Uparrow$ $P_t(z) \neq 0$

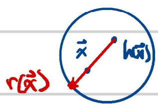
Conclusion: $\omega_0 \simeq f_r \simeq \omega_n$ in S^1 (some large r) $\overset{\text{Thm 1.7}}{\Rightarrow}$ $n = 0$ $\Rightarrow$ $P(z) = $ constant #

Thm 1.9 (A fixed point thm)

Any continuous map $h: D^2 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \le 1\} \to D^2$ has a fixed point

pf

Assume $h(\vec{x}) \neq \vec{x}$ $\forall \vec{x} \in D^2$. Define $r: D^2 \to S^1$ by



exer: r is continuous

$\Rightarrow r(i(\vec{x})) = \vec{x}$, $\forall \vec{x} \in S^1$ Not possible because ⋯ ($i: S^1 \to D^2$ is the inclusion)

$\Rightarrow \pi_1(S^1) \xrightarrow{i_*} \boxed{\pi_1(D^2)}^{=0} \xrightarrow{r_*} \pi_1(S^1)$, $id_* = id$ $\quad$ $id_* = r_* i_* = 0: \mathbb{Z} \to \mathbb{Z}$ ($\to\leftarrow$) #

$f_0(0) \in S^1 \subseteq D^2$ One-line pf

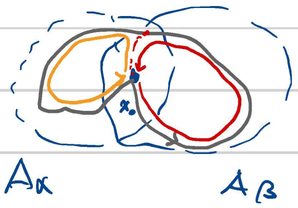
Let f_0 be any loop in S^1 $\Rightarrow$ also a loop in D^2. $\Rightarrow$ $r((1-t)f_0(s) + t\vec{x}_0)$ is a hpt $f_0 \simeq x_0$ is S^1

$\Rightarrow \pi_1(S^1) = 0$ ($\to\leftarrow$) $\Rightarrow \exists$ fixed pt. #

Fundamental group of S^n, $n \ge 2$

Lemma 1.15

Let X be a topo. sp and A_α be open subsets of X. Suppose



(i) $x_0 \in \bigcap_\alpha A_\alpha$

(ii) A_α are path cnt.

(iii) each intersection $A_\alpha \cap A_\beta$ is path cnt.

Then every loop in X based at x_0 is homotopic to a product of loops each of which is contained in a single A_α

pf

Let f be an arbitrary loop in X at x_0.

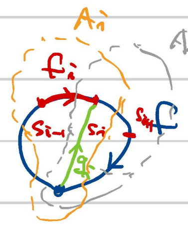
① $\forall s \in I$, $f(s) \in A_\alpha$ for some α. Let V_s be an cnt open nbd of s in I s.t. $\overline{V_s} \subseteq f^{-1}(A_\alpha)$

Note: $\overline{V_s} = [a,b]$, $\exists a,b$

Since I is cpt, $\exists$ finite V_s which cover I.

Let $\{\overset{0}{s_0} < s_1 < \cdots < \overset{=1}{s_m}\}$ be the endpoints of these V_s

$\Rightarrow^{\forall i.}$ $f([s_{i-1}, s_i]) \subseteq A_\alpha$ for some α.



② Let A_i be one of the given A_α s.t. $A_i \supseteq f([s_{i-1}, s_i])$

Let $f_i : I \to A_i$, $f_i(s) = f(s_{i-1} + (s_i - s_{i-1}) \cdot s)$

Since $A_i \cap A_{i+1}$ is p-cnt, $\exists$ path g_i in $A_i \cap A_{i+1}$ s.t. $g_i(0) = x_0$, $g_i(1) = f(s_i)$

$\Rightarrow$ $f \simeq \underbrace{(f_1 \cdot \bar{g}_1)}_{\text{in } A_1} \cdot \underbrace{(g_1 \cdot f_2 \cdot \bar{g}_2)}_{\text{in } A_2} \cdot \cdots \cdot \underbrace{(g_{m-1} \cdot f_m)}_{A_m}$

and the component $g_{i-1} \cdot f_i \cdot \bar{g}_i$ is in A_i #

Prop 1.14

$\pi_1(S^n) = 0$ if $n \ge 2$

pf

Since $S^n = A_1 \cup A_2$, where $S^n = \{x_1^2 + \cdots + x_{n+1}^2 = 1\} \subseteq \mathbb{R}^{n+1}$, $A_2 = S^n - \{(-1,0,\cdots,0)\} \underset{\text{open}}{\subseteq} S^n$, $A_1 = S^n - \{(1,0,\cdots,0)\} \underset{\text{open}}{\subseteq} S^n$

$A_1 \cong A_2 \cong \mathbb{R}^n$ ($\Rightarrow \pi_1(A_1) = 0$, $\pi_1(A_2) = 0$)

$\forall f$ loop in S^n based at $x_0 = (0, \cdots, 0, 1)$, by Lemma 1.15, $\exists$ loops f_1 in A_1, f_2 in A_2 s.t.

$$f \simeq \underset{\substack{\simeq \\ C_{x_0}}}{f_1} \cdot \underset{\substack{\simeq \\ C_{x_0}}}{f_2} \simeq C_{x_0}$$

#

Cor 1.16

$\mathbb{R}^2$ is NOT homeomorphic to $\mathbb{R}^n$ for $n \neq 2$

pf

Suppose $f: \mathbb{R}^2 \to \mathbb{R}^n$ is a homeomorphism. Then $f|_{\mathbb{R}^2 \setminus \{\vec{0}\}} : \mathbb{R}^2 \setminus \{\vec{0}\} \to \mathbb{R}^n \setminus \{f(\vec{0})\}$ is also a homeo

case1 $n=1$

$\mathbb{R}^2 \setminus \{0\}$ is cnt, but $\mathbb{R}^1 \setminus \{f(0)\}$ is NOT $(\to\leftarrow)$

Case 2 $n > 2$

Note that $\mathbb{R}^n \setminus$ a point $\cong S^{n-1} \times \mathbb{R}$ $\quad n-1 \geq 2$ $\quad$ Prop 1.12 $\Rightarrow \pi_1(\mathbb{R}^n \setminus \{f(\vec{0})\}) \cong \pi_1(S^{n-1}) \times \pi_1(\mathbb{R}) = 0$

But $\pi_1(\mathbb{R}^2 \setminus \{\vec{0}\}) \cong \pi_1(S^1 \times \mathbb{R}) \cong \pi_1(S^1) \times \pi_1(\mathbb{R}) \cong \mathbb{Z} \not\cong$ $(\to\leftarrow)$ #

Remark

More generally, one can compute more $\pi_1(X)$ by Van Kampen's thm:

write $\pi_1(X)$ as a free group / some relations

See Thm 1.20 in §1.2

We skip §1.2 here.

§1.3 Covering spaces

Recall

A covering space of X consists of a space $\tilde{X}$ and a map $p: \tilde{X} \to X$ s.t.

$\forall x \in X$, $\exists$ open nbd U of x in X s.t.

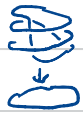
$$(*) \qquad p^{-1}(U) = \coprod_{\lambda} U_\lambda \subseteq \tilde{X}$$

and $p|_{U_\lambda} : U_\lambda \to U$ is a homeomorphism $\forall \lambda$.

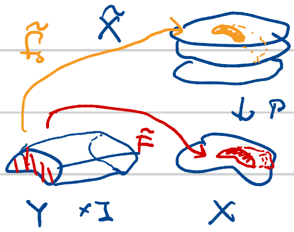
Examples

① $p: \mathbb{R} \to S^1 : s \mapsto (\cos 2\pi s, \sin 2\pi s)$

② $p: S^1 \to S^1 : z \mapsto z^n$ ($S^1 \subseteq \mathbb{C}$)



Lifting properties



Prop 1.30 (homotopy lifting property)

Given a covering sp $p: \tilde{X} \to X$, a homotopy $F: Y \times I \to X$ ($(y,t) \mapsto f_t(y) = F(y,t)$), and a map $\tilde{f}_0 : Y \to \tilde{X}$ lifting f_0, then there exists a unique htp $\tilde{F} : Y \times I \to \tilde{X}$ s.t.

$$p \circ \tilde{F} = F, \quad \tilde{F}(y,0) = \tilde{f}_0(y) \quad \forall y \in Y$$

pf

$\forall y \in Y$, $t \in I$, $\exists U_{y,t} \underset{\text{open}}{\subseteq} X$ ($\ni f_t(y) = F(y,t)$) s.t. $p^{-1}(U_{y,t}) = \coprod_\lambda \tilde{U}_{y,t,\lambda} \subseteq \tilde{X}$ and $p|_{\tilde{U}_{y,t,\lambda}} : \tilde{U}_{y,t,\lambda} \to U_{y,t}$ is a homeo $\forall \lambda$

Let $N_t \times (a_t, b_t)$ be a nbd of (y,t) in $Y \times I$ s.t. $N_t \times (a_t, b_t) \subseteq F^{-1}(U_{y,t})$ ($\ni (y,t)$)

By cptness of I, $\exists\ 0 = t_0 < t_1 < \cdots < t_m = 1$ and $\exists N_y \underset{\text{open}}{\subseteq} Y$ ($\ni y$, $= \bigcap_{\text{finite}} N_t$) s.t. $\forall i$, $N_y \times [t_i, t_{i+1}] \subseteq F^{-1}(U_{y,t})$ for some t

$\Rightarrow$ Inductively, one can construct uniquely

(construct one for each y) $\tilde{F} : N_y \times I \to \tilde{X}$ s.t. $p \circ \tilde{F} = F$, $\tilde{F}(y,0) = \tilde{f}_0(y)$

By uniqueness, $\tilde{F}$ agree on $N_y \cap N_{y'}$, so they glue to one continuous map $\tilde{F} : Y \times I \to \tilde{X}$ s.t. $p \circ \tilde{F} = F$, $\tilde{F}(y,0) = \tilde{f}_0(y)$ ∎

Cor (Statement (a) in §1.1, path lifting property) ← the case $Y = pt$

$\forall$ path $f: I \to X$ starting at $x_0 \in X$, $\forall \tilde{x}_0 \in p^{-1}(x_0)$, $\exists!$ lift $\tilde{f} : I \to \tilde{X}$ starting at $\tilde{x}_0$.

Cor (Statement (b) in §1.1)

For each homotopy $F: I \times I \to X$ of paths starting at x_0 and each $\tilde{x}_0 \in p^{-1}(x_0)$,

$\exists!$ lifted homotopy $\tilde{F} : I \times I \to \tilde{X}$ s.t. $\tilde{F}(0,t) \equiv \tilde{x}_0$ ← case $Y = I$

Prop 1.32

Let $p: \tilde{X} \to X$ be a covering sp., $\tilde{x}_0 \in \tilde{X}$, $x_0 = p(\tilde{x}_0) \in X$. Suppose $\tilde{X}, X$ are p-cnt.

$\Rightarrow \quad \left| \pi_1(X, x_0) / p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \right| = \left| p^{-1}(x_0) \right|$

pf

For $[g] \in \pi_1(X, x_0)$, $\exists!\ \tilde{g}: I \to \tilde{X}$, $\tilde{g}(0) = \tilde{x}_0$, $p \circ \tilde{g} = g \Rightarrow p(\tilde{g}(1)) = g(1) = x_0$

Define a map $\pi_1(X, x_0) / \underbrace{p_*(\pi_1(\tilde{X}, \tilde{x}_0))}_{} =: H \xrightarrow{\Phi} p^{-1}(x_0) : \overset{\text{coset}}{H[g]} \longmapsto \tilde{g}(1)$

well-def: $[h] \in H \Leftrightarrow \tilde{h}$ is a loop $\Leftrightarrow \tilde{h}(1) = \tilde{x}_0$

$\Rightarrow [h][g] = [h \bullet g] \overset{\Phi}{\longmapsto} \widetilde{h \bullet g}(1) \overset{\because \tilde{h}(1) = \tilde{x}_0}{=} (\tilde{h} \bullet \tilde{g})(1) = \tilde{g}(1)$

onto: $\tilde{X}$ is p-cnt $\Rightarrow \forall \tilde{x} \in p^{-1}(x_0)$, $\exists$ path $\tilde{g}: I \to \tilde{X}$, $\tilde{g}(0) = \tilde{x}_0$, $\tilde{g}(1) = \tilde{x} \Rightarrow \Phi(H[p \circ \tilde{g}]) = \tilde{x}$

1-1: Suppose $[g_1], [g_2] \in \pi_1(X, x_0)$, $\tilde{g}_1(1) = \tilde{g}_2(1) \Rightarrow [p \circ (\tilde{g}_1 \bullet \bar{\tilde{g}}_2)][g_2] = [g_1] \Rightarrow H[g_2] = H[g_1]$ \#

Example $\tilde{x}_0 = 0$, $x_0 = (1, 0)$

① $p: \mathbb{R}^1 \to S^1$, $\quad \pi_1(S^1, x_0) / p_*(\underbrace{\pi_1(\mathbb{R}, \tilde{x}_0)}_{= 0}) \cong \mathbb{Z} \overset{\exists \text{ bijection}}{\longleftrightarrow} p^{-1}(x_0) = \{n \mid n \in \mathbb{Z}\} = \mathbb{Z} \subseteq \mathbb{R}$

② $p: S^1 \subseteq \mathbb{C} \longrightarrow S^1 \subseteq \mathbb{C} : z \mapsto z^n$, $\tilde{x}_0 = 1$, $x_0 = 1$

$p^{-1}(1) \overset{n\text{-th roots}}{=} \{ e^{\frac{k}{n} \cdot 2\pi i} \mid k = 0, 1, \cdots, n-1 \}$, $|p^{-1}(1)| = n$

$$\begin{array}{ccc} p_*: \pi_1(S^1, 1) & \longrightarrow & \pi_1(S^1, 1) \\ [e^{2\pi m s \cdot i}] & \longmapsto & [e^{2\pi m n \cdot s \cdot i}] \\ \cong & & \\ \mathbb{Z} & \longrightarrow & \mathbb{Z} : m \longmapsto n \cdot m \end{array}$$

$\Rightarrow \left| \pi_1(S^1, 1) / p_* \pi_1(S^1, 1) \cong \mathbb{Z} / n\mathbb{Z} \right| = n$

Prop 1.33 (A lifting criterion)

Let $p: \tilde{X} \to X$ be a covering sp. and $f: Y \to X$ be a $\overset{\text{cts}}{}$ map. ($\tilde{x}_0 \in p^{-1}(x_0)$, $y_0 \mapsto x_0$)

Suppose Y is p-cnt and locally path-connected (i.e. $\forall y \in Y$, $\exists\ U \underset{\text{open}}{\subseteq} Y$, $y \in U$, U is p-cnt)

Then a lift $\tilde{f}: (Y, y_0) \longrightarrow (\tilde{X}, \tilde{x}_0)$ of f exists iff $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}_0))$

pf: skip

Classification of covering spaces

Def

A space X is semilocally simply-connected if $\forall x \in X$, $\exists$ nbd U of x s.t. the induced map $\pi_1(U, x) \to \pi_1(X, x)$ is trivial.

Remark

Every manifold is locally path-connected and semilocally simply-connected. in differential geometry

Thm 1.38

Let X be path-connected, locally path-connected and semilocally simply-connected.
Then $\exists$ a bijection

$$\left\{\begin{array}{l}\text{basepoint-preserving} \\ \text{iso classes of p-cnt} \\ \text{Covering spaces}\end{array}\right\} \ni \begin{array}{c}(\tilde{X}, \tilde{x}_0) \\ \downarrow p \\ (X, x_0)\end{array} \longmapsto p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \in \left\{\begin{array}{l}\text{Subgroups} \\ \text{of } \pi_1(X, x_0)\end{array}\right\}$$

If basepoints are ignored, this induces a bijection

$$\left\{\begin{array}{l}\text{iso classes of p-cnt} \\ \text{Covering spaces}\end{array} \begin{array}{c}\tilde{X} \\ \downarrow \\ X\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{Conjugacy classes of} \\ \text{subgroups of } \pi_1(X, x_0)\end{array}\right\}$$

In particular, $\exists!$ covering sp $\tilde{X} \to X$ s.t. $\tilde{X}$ is simply-connected. (i.e. $\tilde{X}$: p-cnt, $\pi_1(\tilde{X}) = 0$)
This covering sp is called the universal cover of X.

e.g. $\mathbb{R}^1 \to S^1$ is the universal cover of S^1

Thm

If X is p-cnt, locally p-cnt, semilocally s-cnt, then X has a universal cover.

Summary

1-1 correspondence

$$\begin{array}{lccc}
\text{universal cover of } X & \tilde{X} & \longleftrightarrow & (p_*\pi_1(\tilde{X}) =)\ 0 \\
& \downarrow & & \wedge \\
\text{some covering sp} & X' & \longleftrightarrow & (p_*\pi_1(X') =)\ H \\
& \downarrow & & \wedge \\
& X & \longleftrightarrow & \pi_1(X)
\end{array}$$

$\exists$ such a map by Prop 1.33

Similar to Galois correspondence