Algebraic Topology

Introduction

Main object to study : (topological) space — space equipped with open sets

manifolds, varieties, schemes, ...

Examples of topological spaces : ○, ⊖, ⊝, $\mathbb{R}^n$, (finite-dim) vector spaces, metric spaces,

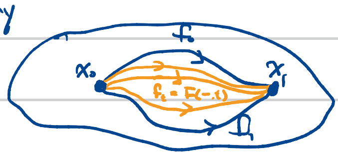
Basic idea of main methods :

main topic , usually lose some info

more easy operations here

difficult things e.g. topological spaces, ...	some construction → e.g. fundamental gp, homology, cohomology, ...	easier things e.g. numbers, groups, vec. spaces, ...
usually, category	functor	category

§1.1 Fundamental group



X: topo. sp. A **path** in X is a continuous map $f: \overset{[0,1]}{\overset{\shortparallel}{I}} \to X$

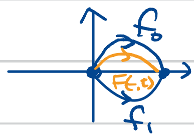
A **homotopy of paths** in X is a continuous map $F: I \times I \to X$ s.t. $F(0,t) = x_0$, $F(1,t) = x_0$ $\forall t \in [0,1]$

If $\exists$ such a F, we say the paths $f_0(s) := F(s,0)$, $f_1(s) := F(s,1)$ are **homotopic**, denoted $f_0 \simeq f_1$ (or $f_0 \sim f_1$)

Example 1.1

$X = \mathbb{R}^2$, $f_0(s) = (2s, 1-(1-s)^2)$, $F(s,t) = \Big(2s, (1-t)\big(1-(1-s)^2\big) + t\big(-1+(1-s)^2\big)\Big)$

$f_1(s) = (2s, -1+(1-s)^2)$



Note: if $\tilde{X} = \mathbb{R}^2 \setminus \{(1,0)\}$, then F does NOT map into $\tilde{X}$. ※ $F(\frac{1}{2}, \frac{1}{2}) = (1,0)$

In fact, f_0 and f_1 are NOT $\boxed{\text{homotopic}}$ in $\tilde{X}$ ← depends on where the paths are

Prop 1.2

The relation of homotopy on paths is an equivalence relation. pf : exercise

Def

The equivalence class of a path f under hpt is called the **homotopy class** of f, denoted [f]

Given two paths $f, g: I \to X$ such that $\boxed{f(1) = g(0)}$, there is a **composition path** (or called **product path**) $f \cdot g : I \to X$, defined by

$$(f \cdot g)(s) := \begin{cases} f(2s), & 0 \le s \le \frac{1}{2} \\ g(2s-1), & \frac{1}{2} \le s \le 1 \end{cases}$$

is continuous

A path f is called a **loop** if $f(0) = f(1)$. Define

$$\pi_1(X, x_0) := \{ [f] \mid f: I \to X,\ f(0) = f(1) = x_0 \}$$

— the **fundamental group** of X at the **basepoint** x_0.

Prop 1.3

$\pi_1(X, x_0)$ with the product $[f][g] := [f \cdot g]$ is a group.

pf

① welldef: If $f_0 \overset{F}{\simeq} f_1$, $g_0 \overset{G}{\simeq} g_1$, then the map $H: I \times I \to X$, $H(s,t) := \begin{cases} F(2s, t), & 0 \le s \le \frac{1}{2} \\ G(2s-1, t), & \frac{1}{2} \le s \le 1 \end{cases}$

is continuous (because $F(2\cdot\frac{1}{2}, t) = F(1,t) = x_0 = G(0,t) = G(2\cdot\frac{1}{2} - 1, t)$ $\forall t \in I$) and a hpt $f_0 \cdot g_0 \simeq f_1 \cdot g_1$

② associativity: Lemma: Let $f, g, h: I \to X$ be paths s.t. $f(1) = g(0)$, $g(1) = h(0)$. Then $(f \cdot g) \cdot h \simeq f \cdot (g \cdot h)$

pf of Lemma: $((f \cdot g) \cdot h)(s) = \begin{cases} f(4s), & 0 \le s \le \frac{1}{4} \\ g(4s-1), & \frac{1}{4} \le s \le \frac{1}{2} \\ h(2s-1), & \frac{1}{2} \le s \le 1 \end{cases}$ $\qquad (f \cdot (g \cdot h))(s) = \begin{cases} f(2s), & 0 \le s \le \frac{1}{2} \\ g(4s-2), & \frac{1}{2} \le s \le \frac{3}{4} \\ h(4s-3), & \frac{3}{4} \le s \le 1 \end{cases}$

Define $H(s,t) := \begin{cases} f((1-t)4s + t\cdot 2s), & 0 \le s \le (1-t)\cdot\frac{1}{4} + t\cdot\frac{1}{2} \\ g((1-t)\cdot(4s-1) + t(4s-2)), & (1-t)\cdot\frac{1}{4} + t\frac{1}{2} \le s \le (1-t)\frac{1}{2} + t\frac{3}{4} \\ h((1-t)\cdot(2s-1) + t(4s-3)), & (1-t)\frac{1}{2} + t\frac{3}{4} \le s \le 1 \end{cases}$ — a homotopy between the 2 path

exer: Check H is continuous

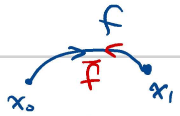
$\Rightarrow ([f][g])[h] = [f]([g][h])$

③ identity: Lemma Let $f: I \to X$ be any path. If $c_{x_1}: I \to X$ be a constant path s.t. $c_{x_1}(s) = f(1)$ $\Rightarrow f \cdot c_{x_1} \simeq f$

If $c_{x_0}(s) \equiv f(0)$, then $c_{x_0} \cdot f \simeq f$

pf: similar as ②

④ inverse: For a path $f: I \to X$, define $\bar{f}(s) := f(1-s)$



$\Rightarrow$ $f \cdot \bar{f} \simeq c_{x_0}$ because the map $\bar{H}(s,t) := \begin{cases} f(2s(1-t)) & 0 \le s \le \frac{1}{2} \\ f((2-2s)(1-t)) & \frac{1}{2} \le s \le 1 \end{cases}$ is a hpt

$\bar{f} \cdot f \simeq c_{x_1}$ because the map $\hat{H}(s,t) := \begin{cases} f(1 - 2s(1-t)) & 0 \le s \le \frac{1}{2} \\ f(1 + (2s-2)(1-t)) & \frac{1}{2} \le s \le 1 \end{cases}$ is a hpt

In particular, $[f]^{-1} = [\bar{f}]$ in $\pi_1(X, x_0)$ $\quad\#$

Remark

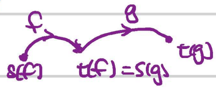
The proof of Prop 1.3 actually implies that the data

$$\begin{array}{c} \Gamma_1 = \{[f] \mid f: I \to X\} \\ s \downarrow \ \downarrow t \quad \uparrow \varepsilon \\ \Gamma_0 = X \end{array}$$

← fundamental groupoid

forms a groupoid, where $s([f]) = f(0)$, $t([f]) = f(1)$, $\varepsilon(x) = c_x$

source map, target map, constant path at x



That is, a product $[f][g] = [f \cdot g]$ is defined if $t([f]) = s([g])$, s.t.

$([f][g])[h] = [f]([g][h])$ if $t([f]) = s([g])$, $t([g]) = s([h])$

$[f]\,\varepsilon(t([f])) = [f] = \varepsilon(s([f]))\,[f]$

$\forall [f]$, $\exists$ $[f]^{-1} (= [\bar{f}])$ s.t. $[f][f]^{-1} = \varepsilon(s([f]))$, $([f]^{-1})[f] = \varepsilon(t([f]))$

$s([f]^{-1}) = t([f])$
$t([f]^{-1}) = s([f])$

Example 1.4 $\quad D^2 = \{x^2+y^2 \le 1\} \subseteq \mathbb{R}^2$

$\forall$ loop $f : I \to \cancel{\mathbb{R}^n}$, the map $F(s,t) := (1-t)f(s) + t\underset{\sim}{x_0}$ ($f(0)=f(1)$) is a hpt between f and C_{x_0} $\Rightarrow$ $[f] = [C_{x_0}]$

$\Rightarrow \pi_1(D^2, x_0) = 0$. Similarly, $\pi_1(\mathbb{R}^n) = 0$.

Prop 1.5

Let $x_0, x_1 \in X$. Suppose $\exists$ path $h : I \to X$ s.t. $h(0) = x_0$, $h(1) = x_1$. Then $\pi_1(X, x_0) \cong \pi_1(X, x_1)$. In particular, $\pi_1(X, x_0)$ is indep of x_0 if X is p-cnt

write $\pi_1(X)$ in this case

pf $\quad$ change-of-basepoint map

Define $\beta_h : \pi_1(X, x_1) \to \pi_1(X, x_0) : \beta_h([f]) := [h \cdot f \cdot \bar{h}]$

$\Rightarrow \beta_h \beta_{\bar{h}} = id_{\pi_1(X,x_0)} \qquad \beta_{\bar{h}} \beta_h = id_{\pi_1(X,x_1)}$ #

Def

A space X is simply connected if X is path-connected and $\pi_1(X) = 0$

Prop 1.12

$\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$ if X, Y are p-cnt.

sketch of pf

Use the property $f : Z \to X \times Y$, $z \mapsto (g(z), h(z))$ is cts $\Leftrightarrow$ g and h are cts, one can show the map

$\pi_1(X \times Y) \to \pi_1(X) \times \pi_1(Y) : [f] \mapsto ([g], [h])$

is an isomorphism #

Induced homomorphisms

Let $\varphi : X \to Y$ be a continuous map s.t. $\varphi(x_0) = y_0$. We will write $\varphi : (X, x_0) \to (Y, y_0)$

Then we have an induced homomorphism

$$\varphi_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0), \quad \varphi_*([f]) := [\varphi \circ f]$$

Easy to check: $\varphi \circ (f \cdot g) = (\varphi \circ f) \cdot (\varphi \circ g)$ $\Rightarrow$ φ_* is a gp homo

Moreover,

- $(\varphi \circ \psi)_* = \varphi_* \circ \psi_*$
- $(id_X)_* = id_{\pi_1(X, x_0)}$

i.e.

Prop

π_1 is a functor from the cat of ptd sp's to the cat of gps

X, Y p-cnt.

$X \cong Y \Rightarrow \pi_1(X) \cong \pi_1(Y)$