Recall

integration of $f dx\wedge dy$ $\overset{-f dy \wedge dx}{=}$ $f dy\wedge dx$ $\leftrightarrow$ orientation

## §3.3 Poincaré duality

### Orientation of manifold

A (topological) manifold of dimension $n$ is a Hausdorff (and 2nd countable) topo. sp which is locally homeomorphic to $\mathbb{R}^n$. A compact mfd (without boundary) is called a closed manifold.

Let $\mathcal{B}(\mathbb{R}^n)$ be the set of ordered bases in $\mathbb{R}^n$

We say $(\vec{v}_1,\cdots,\vec{v}_n)$, $(\vec{w}_1,\cdots,\vec{w}_n)\in\mathcal{B}(\mathbb{R}^n)$ have the same orientation if the matrix of changing coordinates has det $>0$. This is an equivalence relation, temporarily denoted by $\sim$. Note that $\mathcal{B}(\mathbb{R}^n)/\sim$ has 2 elements. An element in $\mathcal{B}(\mathbb{R}^n)/\sim$ is called an orientation of $\mathbb{R}^n$.

Recall that a homeomorphism $\Delta^n\to D^n$ induces a generator of $H_n(D^n,\partial D^n)\xrightarrow{\cong}H_n(\mathbb{R}^n,\mathbb{R}^n\setminus\{0\})$ (induced by inclusion & by 5-lemma)

So for any $\sigma:\Delta^n\hookrightarrow\mathbb{R}^n$, $x\in\mathrm{int}(\mathrm{im}(\sigma))$, $[\sigma]\in H_n(\mathbb{R}^n,\mathbb{R}^n\setminus\{x\})\cong H_n(\Delta^n,\partial\Delta^n)\ni[id]$ is a generator.

A choice of ordered basis $\overset{\beta=}{(\vec{w}_1,\cdots,\vec{w}_n)}$ of $\mathbb{R}^n$ induces $\sigma_\beta:\Delta^n\overset{\subseteq\mathbb{R}^{n+1}}{\hookrightarrow}\mathbb{R}^n:(t_0,\cdots,t_n)\mapsto\sum_{i=1}^n t_i\vec{w}_i+(x-\frac{\sum\vec{w}_i}{n+1})$

By the above remark, $[\sigma_\beta]\in H_n(\mathbb{R}^n,\mathbb{R}^n\setminus\{x\})\cong\mathbb{Z}$ is a generator

**Remark**

Note: $H_n(\mathbb{R}^n,\mathbb{R}^n\setminus\{x\})\cong\tilde{H}_{n-1}(\mathbb{R}^n\setminus\{x\})\cong\tilde{H}_{n-1}(S^{n-1})$. By choosing a suitable iso $H_n(\mathbb{R}^n,\mathbb{R}^n\setminus\{x\})\cong\tilde{H}_{n-1}(S^{n-1})$ a reflection can maps $[\sigma_\beta]$ to $[\sigma_{\beta'}]$ where $\beta'=(w_2,w_1,w_3,\cdots,w_n)$. Since deg(reflection) $=-1$, we have $[\sigma_\beta]=-[\sigma_{\beta'}]$ (idea: $\beta\sim\gamma\leftrightarrow[\sigma_\beta]=c\cdot[\sigma_\gamma]$, $c>0$)

This observation motivates the algebraic topo. version of orientation:

**Def**

A local orientation of $M$ at $x\in M$ is a choice of generator $\mu_x$ of $H_n(M,M\setminus\{x\})\cong\mathbb{Z}$

$$H_n(M,M\setminus\{x\}) \overset{\text{excision}}{\cong} H_n(U,U\setminus\{x\}) \cong H_n(\mathbb{R}^n,\mathbb{R}^n\setminus\{x\})$$

An orientation of an $n$-dimensional mfd $M$ is a function $M\ni x\mapsto\mu_x\in H_n(M,M\setminus\{x\})$ (local orientation)

s.t. $\forall x\in M$, $\exists$ nbd $\mathbb{R}^n\cong U\subseteq M$ containing an open ball $B=B(x;r)\subseteq\mathbb{R}^n$ ($r<\infty$)

s.t. $\exists$ generator $\mu_B\in H_n(M,M\setminus B)\cong H_n(\mathbb{R}^n,\mathbb{R}^n\setminus B)\cong\mathbb{Z}$ with the property

$$\begin{array}{ccc} H_n(M,M\setminus B) & \xrightarrow{\cong} & H_n(M,M\setminus\{y\}) \\ \mu_B & \longmapsto & \mu_y \end{array} \qquad \forall y\in B$$

If an orientation exists for $M$, we say $M$ is orientable

Replacing the coeff gp by $R$ (we will assume $R$ is a commutative ring with $1_R$), one can get definitions of $R$-orientation and $R$-orientable

**Remark** (Prop 3.25)

$\tilde{M}:=\{\mu_x \mid \mu_x \text{ is a local orientation of } M \text{ at } x\}$ can be topologized so that $\tilde{M}\to M$ ($\mu_x\mapsto x$) is a 2-fold covering space. And $M$ is orientable $\Leftrightarrow$ $\tilde{M}$ has 2 connected components.

Thm 3.26

Let $M$ be a closed connected $n$-manifold. Then

(a) If $M$ is $R$-orientable, then the map $H_n(M;R) \to H_n(M, M-\{x\};R) \cong R$ is an iso $\forall x \in M$

(b) If $M$ is NOT $R$-orientable, then the map $H_n(M;R) \to H_n(M, M-\{x\};R) \cong R$ is injective with image $\{r \in R \mid 2r = 0\}$ $\forall x \in M$

(c) $H_i(M;R) = 0$ $\forall i > n$

pf: postponed

Def

An element of $H_n(M;R)$ whose image in $H_n(M, M-\{x\};R)$ is a generator $\forall x \in M$ is called a fundamental class or an orientation class for $M$

Cor (p.236)

A fundamental class exists iff $M$ is closed and $R$-orientable

Construction of fundamental class

Def (p.103)

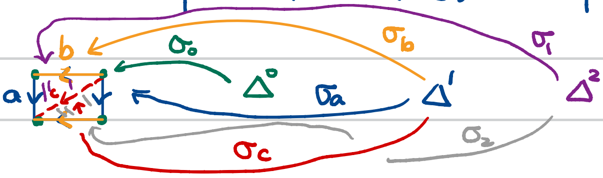
A $\Delta$-complex structure on a space $X$ is a collection of maps $\sigma_\alpha : \Delta^n \to X$, with $n$ depending on $\alpha$, such that:

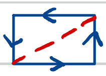
(i) The restriction $\sigma_\alpha|_{\text{int}(\Delta^n)}$ is injective, and each point of $X$ is in the image of exactly one $\sigma_\alpha(\text{int}(\Delta^n))$

(ii) Each restriction of $\sigma_\alpha$ to a face of $\Delta^n$ is one of the maps $\sigma_\beta : \Delta^{n-1} \to X$

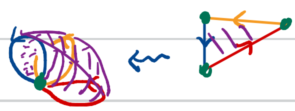
(iii) A set $U \subseteq X$ is open iff $\sigma_\alpha^{-1}(U)$ is open in $\Delta^n$ $\forall \sigma_\alpha$.

Example

① torus $T =$ 


② Klein bottle $K =$ 

Remark

- Any $\Delta$-complex is a CW complex.
- $\sigma_1, \sigma_2$ in Example ① are still NOT honest triangles $\sigma_1(\Delta^2) =$ 

Suppose a closed n-mfd $M$ has a structure of $\Delta$-complex

Let $\sigma_1, \cdots, \sigma_m$ be $n$-simplices of $M$

$\Rightarrow$ In cellular homology, a class $\overset{\mu}{\smile}$ in $H_n(M)$ is represented by $\sum_{i=1}^{m} k_i \sigma_i$

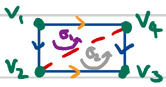
If $\mu \mapsto$ a generator of $H_n(M, M-\{x\})$, $x \in \mathrm{int}(\sigma_i)$, then $k_i = \pm 1$ ($\because$ $[\sigma_i]$ is a generator, $[\sigma_j] \mapsto 0 \in H_n(M, M-x)$ $\forall j \neq i$)

<u>Prop</u> (p.238)

$M$ is orientable iff one can choose $k_i = \pm 1$ so that $d(\sum k_i \sigma_i) = 0$. in cellular homology

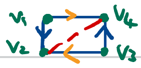
In this case, $\mu = [\sum k_i \sigma_i]$ is a fundamental class of $M$.

<u>Example</u>

① torus $T =$  $\sigma_1 = [v_1\ v_2 v_4]$, $\sigma_2 = [v_2\ v_3\ v_4]$

$\Rightarrow d(\sigma_1 + \sigma_2) = \cancel{[v_2 v_4]} - \cancel{[v_1 v_4]} + \cancel{[v_1 v_2]} + \cancel{[v_3 v_4]} - \cancel{[v_2 v_4]} + \cancel{[v_2 v_3]} = 0$

$\Rightarrow \mu = [\sigma_1 + \sigma_2]$ is a fundamental class of $T$

② Klein bottle $K =$  $\sigma_1 = [v_1\ v_2 v_4]$, $\sigma_2 = [v_2\ v_3\ v_4]$

$\Rightarrow d(\sigma_1 + \sigma_2) = \cancel{[v_2 v_4]} - [v_1\ v_4] + [v_1\ v_2] + \underset{\substack{\parallel \\ [v_1\ v_2]}}{[v_3\ v_4]} - \cancel{[v_2 v_4]} + [v_2\ v_3]$ $\text{''} [v_4\ v_1]$

$= 2[v_1\ v_2] - 2[v_1\ v_4] \neq 0$

Not orientable

<u>Remark</u>

- Any "smooth mfd" is "triangulizable" see e.g. arXiv 1607.08163 see Munkres' book
- NOT every topological mfd is triangulizable
- NOT every CW complex is triangulizable

- $\Delta$-complex and simplicial complex should have same geometric realization by cutting $\Delta$ finer

  $\Delta$ can have same vertices NOT different faces are different simplexes

## <u>Cap product</u>

<u>Def</u>

The cap product is an $R$-bilinear product $\frown : C_k(X;R) \times C^\ell(X;R) \to C_{k-\ell}(X;R)$, $k \geq \ell$

$$\sigma \frown \varphi := \underbrace{\varphi(\sigma|_{[v_0, \cdots, v_\ell]})}_{\in R} \cdot \sigma|_{[v_\ell, \cdots, v_k]}$$

for $\sigma : \Delta^k \to X$, $\varphi \in C^\ell(X;R)$

<u>Lemma</u> (p.240)

$$\partial(\sigma \frown \varphi) = (-1)^\ell (\partial\sigma \frown \varphi - \sigma \frown \delta\varphi)$$

pf: direct computation

As a consequence of the lemma, we have the induced cap product

$$H_k(X;R) \times H^\ell(X;R) \xrightarrow{\frown} H_{k-\ell}(X;R)$$

which is $R$-bilinear

<u>Prop</u> (p.241, "naturality of $\frown$")

For $f : X \to Y$, $\alpha \in H_k(X;R)$, $\varphi \in H^\ell(Y;R)$, we have

$$f_*(\alpha) \frown \varphi = f_*(\alpha \frown f^*(\varphi))$$

$$\begin{tikzcd} H_k(X) \times H^\ell(X) \arrow[r, "\frown"] \arrow[d, "f_*"'] & H_{k-\ell}(X) \arrow[d, "f_*"] \\ H_k(Y) \times H^\ell(Y) \arrow[r, "\frown"] \arrow[u, "f^*"'] & H_{k-\ell}(Y) \end{tikzcd}$$

<u>Prop</u> (p.249, relationship with cup product)

For $\alpha \in C_{k+\ell}(X;R)$, $\varphi \in C^k(X;R)$, $\psi \in C^\ell(X;R)$,

$$\psi(\alpha \frown \varphi) = (\varphi \smile \psi)(\alpha)$$

<u>pf</u>

For $\sigma : \Delta^{k+\ell} \to X$,

$$\begin{aligned} \psi(\sigma \frown \varphi) &= \psi(\underbrace{\varphi(\sigma|_{[v_0,\dots,v_k]})}_{\in R} \cdot \sigma|_{[v_k,\dots,v_{k+\ell}]}) \quad \leftarrow \psi \text{ is } R\text{-linear} \\ &= \varphi(\sigma|_{[v_0,\dots,v_k]}) \cdot \psi(\sigma|_{[v_k,\dots,v_{k+\ell}]}) \\ &= (\varphi \smile \psi)(\sigma) \end{aligned}$$

//

<u>Poincaré duality</u>

<u>Thm 3.30</u> (Poincaré duality)

If $M$ is a closed $R$-orientable $n$-mfd with fundamental class $[M] \in H_n(M;R)$, then the map

$$D : H^k(M;R) \longrightarrow H_{n-k}(M;R), \quad D(\varphi) := [M] \frown \varphi$$

is an isomorphism $\forall k$

<u>Example</u>

| | | $H_0$ | $H_1$ | $H_2$ |
|---|---|---|---|---|
| | torus $= T$ | $\mathbb{Z}$ | $\mathbb{Z}^2$ | $\mathbb{Z}$ |
| NOT orientable | $\mathbb{RP}^2$ | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $0$ |
| NOT orientable | Klein bottle | $\mathbb{Z}$ | $\mathbb{Z} \oplus \mathbb{Z}_2$ | $0$ |

$\cong$ different different

$\leftarrow$ closed $\rightarrow$ orientable

$$\begin{array}{cccccc} & H_0 & & & & H_n \\ S^n : & \mathbb{Z}, & 0, & \cdots & 0, & \mathbb{Z} \end{array} \qquad \cong$$

$$\begin{array}{cccccccccc} & H_0 & H_1 & H_2 & & & H_{2n-2} & H_{2n-1} & H_{2n} \\ \mathbb{CP}^n : & \mathbb{Z}, & 0, & \mathbb{Z}, & 0, & \cdots & \mathbb{Z}, & 0, & \mathbb{Z} \end{array} \qquad \cong$$