Example 3.7

$$\Rightarrow h(\alpha\cup\beta)(Z) = [h(\alpha\cup\beta)((BAD)) - h(\alpha\cup\beta)((BCD))]$$

$$= [h(\alpha)(\underset{a}{(BA)})\cdot h(\beta)(\underset{b}{(AD)}) - h(\alpha)(\underset{b}{(BC)})\cdot h(\beta)(\underset{a}{(CD)})]$$

$$= 1\cdot 1 - 0\cdot 0 = 1$$

So $\alpha\cup\beta = \zeta \in H^2(T;\mathbb{Z}) \cong \mathrm{Hom}(H_2(T),\mathbb{Z})$ is a generator of $H^2(T;\mathbb{Z})$ ($\zeta(Z)=1$)

Conclusion

$$H^\bullet(T;\mathbb{Z}) \cong \mathbb{Z}\oplus\mathbb{Z}^2\oplus\mathbb{Z}$$

deg: 0, 1, 2

generator: 1, α,β, ζ

with the relations

$\alpha\cup\beta = \zeta$, $\alpha\cup\alpha = 0$, $\beta\cup\beta = 0$

$\alpha\cup\alpha = (-1)^{1\cdot 1}\alpha\cup\alpha \Rightarrow 2(\alpha\cup\alpha) = 0$ in $H^2(T;\mathbb{Z})\cong\mathbb{Z}$ $\Rightarrow \alpha\cup\alpha = 0$

1 is the identity element.

$= \zeta\cup\alpha = \zeta\cup\beta$

$\alpha\cup\zeta = \beta\cup\zeta = 0$, $\zeta\cup\zeta = 0$ ← degree reason #

exer

Compute the cohomology ring of closed surfaces (Example 3.7, Example 3.8)

Algebraic structures behind homology / cohomology

Def

Let R be a ring. A (left) R-module M consists of an abelian gp $(M,+)$ and an operation (scalar product) $\cdot : R\times M \to M$ s.t. $\forall r,s\in R$, $x,y\in M$, we have

(i) $r\cdot(x+y) = r\cdot x + r\cdot y$

(ii) $(r+s)\cdot x = r\cdot x + s\cdot x$

(iii) $(rs)\cdot x = r\cdot(s\cdot x)$

(iv) $1_R\cdot x = x$ (if R has an identity element 1_R)

Homomorphisms, submodules, generators, finitely generated, kernel, quotient are defined as usual

(Homomorphisms: also say "R-linear map")

Example

① If $R=\mathbb{k}$ is a field, then $\mathbb{k}$-module = $\mathbb{k}$-vector space

② $\mathbb{Z}$-module = abelian group: $n\cdot x = \underbrace{x+x+\cdots+x}_{n\text{ times}}$, $(-n)\cdot x = \underbrace{(-x)+\cdots+(-x)}_{n\text{ time}}$, $n\in\mathbb{N}$

③ $C_n(X)$ is an abelian gp $\Rightarrow$ a $\mathbb{Z}$-module

④ $C_n(X;R)$ is an R-module: For $s, r_i\in R$, $\sum r_i\sigma_i \in C_n(X;R)$, we have $s\cdot(\sum r_i\sigma_i) = \sum(sr_i)\sigma_i$

∂ is a homo of R-mod

⑤ $H_n(X;R)$ is an R-module: $s\cdot[\sum r_i\sigma_i] = [\sum(sr_i)\sigma_i]$ (← quotient module)

⑥ $C^n(X;R)$ and $H^n(X;R)$ are R-module: For $\varphi\in C^n(X;R)$, $r\in R$, $r\cdot\varphi\in C^n(X;R)$, $(r\cdot\varphi)(\sigma) = r\varphi(\sigma)$

d is a homo of R-mod; $r\cdot[\varphi] = [r\cdot\varphi]$

Def

An (associative) algebra A over R (or R-algebra) is an R-module together with a (associative) binary operation $A \times A \to A$ which is bilinear:

$$(rx+sy)\cdot z = r(x\cdot z) + s(y\cdot z), \quad x\cdot(ry+sz) = r(x\cdot y) + s(x\cdot z)$$

and

$$r(x\cdot y) = (rx)\cdot y$$

Very often, an algebra is assumed to have identity 1_A and an alg mor keeps the id elem

Example

① R is an algebra over R ② The poly ring $R[x]$ is an R-alg.
③ $(C^{\bullet}(X;R), \smile)$ is an R-algebra

Def

A graded ring ($\mathbb{Z}_+$-graded) (resp. graded algebra) is a ring (resp. alg) A with a decomposition $A = \bigoplus_{k=0}^{\infty} A_k$ of additive subgps (resp. submods) A_k s.t. $\forall a \in A_k, b \in A_l$, $a\cdot b \in A_{k+l}$

To indicate that an element $a \in A$ lies in A_k, we write $|a| = k$ and say the degree of a is k. We say a is homogeneous if $a \in A_k$ for some k.

A graded ring is said to be commutative (or graded commutative ...) if $ab = (-1)^{|a|\cdot|b|} ba$

A differential graded algebra (= dga) (over R) is a graded R-alg A together with an R-linear operator $d: A \to A$ of degree one (i.e. $|d(a)| = 1 + |a|$ $\forall$ homogeneous $a \in A$) s.t.

$$d \circ d = 0 \quad \text{and} \quad d(ab) = (da)b + (-1)^{|a|} a(db)$$

($|a| = |d| \cdot |a|$) ← "Koszul sign convention"

Prop

If (A, d) is a dga, then $\ker d / \operatorname{im} d$ is a graded ring/alg

pf: See the remark after Lemma 3.6

Example

① The singular cochain complex $(C^{\bullet}(X;R), \smile, \delta)$ is a dga
② The singular cohomology $(H^{\bullet}(X;R), \smile)$ is a commutative graded alg if R is commutative
③ The polynomial ring $R[x_1, \cdots, x_n]$ is a graded alg. ($(R[x_1,\cdots,x_n])_k = \{$homogeneous poly of deg $k\}$)
④ The exterior algebra $\Lambda^{\bullet} V = TV / \langle x\otimes y + y \otimes x \mid x, y \in V \rangle$ is a graded commutative alg over k (V = vec. sp)
⑤ The differential forms $(\Omega^{\bullet}(M), \wedge, d_{dR})$ form a cdga i.e. commutative dga (over $\mathbb{R}$)

Recall

integration of $fdx \wedge dy$ $= -fdy\wedge dx$ ⟷ orientation

$fdy \wedge dx$

§3.3 Poincaré duality

Orientation of manifold

A (topological) **manifold** of dimension n is a Hausdorff (and 2nd countable) topo. sp which is locally homeomorphic to $\mathbb{R}^n$. A compact mfd (without boundary) is called a **closed manifold**.

Let $\mathcal{B}(\mathbb{R}^n)$ be the set of ordered bases in $\mathbb{R}^n$.

We say $(\vec{v}_1, \cdots, \vec{v}_n), (\vec{w}_1, \cdots, \vec{w}_n) \in \mathcal{B}(\mathbb{R}^n)$ have the same orientation if the matrix of changing coordinates has det >0. This is an equivalence relation, temporarily denoted by $\sim$. Note that $\mathcal{B}(\mathbb{R}^n)/_\sim$ has 2 elements. An element in $\mathcal{B}(\mathbb{R}^n)/_\sim$ is called an **orientation** of $\mathbb{R}^n$.

Recall that a homeomorphism $\Delta^n \to D^n$ induces a generator of $H_n(D^n, \partial D^n) \xrightarrow{\cong} H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{\vec{0}\})$ (induced by inclusion, ↓ by 5-lemma)

So for any $\sigma: \Delta^n \hookrightarrow \mathbb{R}^n$, $x \in \mathrm{im}(\sigma)$, $[\sigma] \in H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) \cong H_n(\Delta^n, \partial\Delta^n) \ni [id]$ is a generator.

A choice of ordered basis $\beta = (\vec{w}_1, \cdots, \vec{w}_n)$ of $\mathbb{R}^n$ induces $\sigma_\beta: \Delta^n \hookrightarrow \mathbb{R}^n$ ($\subseteq \mathbb{R}^{n+1}$): $(t_0, \cdots, t_n) \mapsto \sum_{i=0}^{n} t_i \vec{w}_i + (x - \frac{\sum \vec{w}_i}{n+1})$

By the above remark, $[\sigma_\beta] \in H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) \cong \mathbb{Z}$ is a generator

Remark

Note: $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) \cong \tilde{H}_{n-1}(\mathbb{R}^n \setminus \{x\}) \cong \tilde{H}_{n-1}(S^{n-1})$. By choosing a suitable iso $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) \cong \tilde{H}_{n-1}(S^{n-1})$ a reflection can maps $[\sigma_\beta]$ to $[\sigma_{\beta'}]$ where $\beta' = (w_2, w_1, w_3, \cdots, w_n)$. Since deg(reflection) $= 1$, we have $[\sigma_\beta] = -[\sigma_{\beta'}]$ (idea: $\beta \sim \gamma \iff [\sigma_\beta] = c\cdot[\sigma_\gamma]$, $c>0$)

This observation motivates the algebraic topo. version of orientation:

Def

A **local orientation** of M at $x \in M$ is a choice of generator μ_x of $H_n(M, M \setminus \{x\}) \cong \mathbb{Z}$ (excision: $\cong H_n(U, U \setminus \{x\}) \cong \mathbb{R}^n$)

An **orientation** of an n-dimensional mfd M is a function $M \ni x \mapsto \mu_x \in H_n(M, M \setminus \{x\})$ (local orientation)

s.t. $\forall x \in M$, $\exists$ nbd $\mathbb{R}^n \cong U \subseteq M$ containing an open ball $B = B(x; r) \subseteq \mathbb{R}^n$ ($r \neq \infty$)

s.t. $\exists$ generator $\mu_B \in H_n(M, M \setminus B) \cong H_n(\mathbb{R}^n, \mathbb{R}^n \setminus B) \cong \mathbb{Z}$ with the property

$$\begin{array}{ccc} H_n(M, M\setminus B) & \xrightarrow{\cong} & H_n(M, M \setminus \{y\}) \\ \mu_B & \longmapsto & \mu_y \end{array} \quad \forall y \in B$$

If an orientation exists for M, we say M is **orientable**

Replacing the coeff gp by R (we will assume R is a commutative ring with 1_R), one can get definitions of **R-orientation** and **R-orientable**