Special topic : de Rham thm

§1. Manifold

Geometry concerns properties of spaces.

e.g. earth → 

- think earth as the solution set of $x^2+y^2+z^2=1$ ⇝ Algebraic Geometry
- study earth by maps, longitude 經度 and latitude 緯度 ⇝ Differential Geometry (today's topic)

In Differential Geometry, people mainly study (differentiable) manifolds which is a type of spaces one can use techniques similar to longitude and latitude (coordinates)

Def

A differentiable manifold of dimension n is a set M and a family of injective maps $\phi_\alpha : U_\alpha \to M$ ($U_\alpha \subseteq \mathbb{R}^n$ open) such that

(i) $\bigcup_\alpha \phi_\alpha(U_\alpha) = M$

(ii) $\forall \alpha, \beta$ with $\phi_\alpha(U_\alpha) \cap \phi_\beta(U_\beta) = W \neq \phi$, the sets $\phi_\alpha^{-1}(W)$ and $\phi_\beta^{-1}(W)$ are open sets in $\mathbb{R}^n$ and the transition maps $\phi_\beta^{-1} \circ \phi_\alpha$ are differentiable

Meaning

- a pair (U_α, ϕ_α) is a (local) coordinate chart 地圖 — see google map for a chart of earth
- a family $\{(U_\alpha, \phi_\alpha)\}$ with $\bigcup_\alpha \phi_\alpha(U_\alpha) = M$ is called an atlas 地圖集
- transition maps $\phi_\beta^{-1} \circ \phi_\alpha$ describe how charts are connected in an atlas

Example

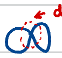
① a point • is a manifold of dim 0 { ② The graph of a function $y=f(x)$ is a mfd of dim 1

③ $S^1 = \{x^2+y^2=1\} =$  $\subseteq \mathbb{R}^2$ is a manifold of dim 1

$(0, 2\pi) \to (\cos\theta, \sin\theta)$ $\quad$ $(\cos\theta, \sin\theta) \leftarrow (-\pi, \pi)$

Note $[0, 2\pi] \xrightarrow[\phi]{(\cos\theta, \sin\theta)}$ ○ is NOT good because $\phi(0) = \phi(2\pi)$ (2 pts on a map represent 1 pt in real world!!)

← different from open subset in $\mathbb{R}^1$ ⟷

④  is NOT a mfd. (self-intersections cause problems when you differentiate functions on a mfd)

⑤ $S^2 = \{x^2+y^2+z^2=1\} =$  $\subseteq \mathbb{R}^3$ is a mfd of dim 2

⑥ Any open subset in $\mathbb{R}^n$ is a mfd of dim n

⑦ $S^n = \{x_0^2 + \cdots + x_n^2 = 1\} \subseteq \mathbb{R}^{n+1}$ is a mfd of dim n

Remark

One can assume M is a subset (submanifold) of $\mathbb{R}^m$

§2. De Rham thm

Smooth singular homology:

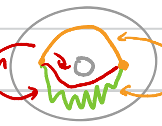
Let M be a smooth mfd. $C_p^\infty(M) :=$ free abelian gp generated by $\{$ smooth maps $\Delta^p \to M \}$

$\Rightarrow$ we have the inclusion $\iota: C_p^\infty(M) \hookrightarrow C_p(M)$ which is a chain map

Thm (Thm 18.7 in Lee's book " Introduction to Smooth Manifolds")

For any smooth mfd M, the map $\iota_*: H_p^\infty(M) \longrightarrow H_p(M)$ induced by inclusion is an isomorphism

idea of pf

close enough homotopic  far, not homotopic

Use Whitney Approximation Thm (Thm 6.26 in [Lee])

Suppose M and N are smooth mfds, and $F: N \longrightarrow M$ is a continuous map. Then F is homotopic to a smooth map $N \to M$. Furthermore, if F is smooth in $A \subseteq_{\text{closed}} N$, then the homotopy can be taken to be relative to A

to construct a "smoothing operator" $s: C_p(M) \longrightarrow C_p^\infty(M)$ s.t.

- s is a chain map
- $s \circ \iota = \text{id}$ $\leftarrow$ complicated
- $\iota \circ s$ is chain homotopic to id. #

Define a map $I: H_{dR}^p(M) \longrightarrow H^p(M;\mathbb{R}) \underset{\text{universal coeff thm}}{\cong} \operatorname{Hom}_{\mathbb{R}}(H_p(M;\mathbb{R}), \mathbb{R}) \underset{\text{previous thm}}{\cong} \operatorname{Hom}_{\mathbb{R}}(H_p^\infty(M;\mathbb{R}), \mathbb{R})$

$$I([\omega])([c]) = \sum c_i \int_{\sigma_i} \omega = \sum c_i \int_{\Delta^p} \sigma_i^* \omega \qquad \text{for } c = \sum c_i \sigma_i$$

Stoke's Thm [Thm 18.12, Lee] : $\int_{\partial c} \omega = \int_c d\omega$ $\Rightarrow$ I is well-defined

De Rham Thm [Thm 18.14, Lee]

$\forall$ smooth mfd M, $\forall p$, the map $I: H_{dR}^p(M) \longrightarrow H^p(M;\mathbb{R})$ is an isomorphism.

sketch of pf

Let's say a mfd M is "de Rham" if the homo I is an iso

Step1: Prove every convex open subset of $\mathbb{R}^n$ is de Rham

Step2: Use Mayer-Vietoris seq to show "if M has a finite de Rham cover, then M is de Rham" (open cover s.t. all U_i, $U_{i_1} \cap \cdots \cap U_{i_k}$ are de Rham)

Step3: Show "if M has a de Rham basis, then M is de Rham" (remove finite assumption) (a de Rham cover which is also a basis)

Step4: Since every M is locally diffeomorphic to $\mathbb{R}^n$, one can show M is de Rham #