§3.2 Cup product

Let R be a ring. For cochains $\varphi \in C^k(X;R)$ and $\psi \in C^\ell(X;R)$, the cup product $\varphi \smile \psi \in C^{k+\ell}(X;R)$ is the cochain whose value on $\sigma: \Delta^{k+\ell} \to X$ is given by

$$(\varphi \smile \psi)(\sigma) := \varphi(\sigma|_{[v_0,\cdots,v_k]}) \cdot \psi(\sigma|_{[v_k,\cdots,v_{k+\ell}]})$$

product in R

where $\sigma|_{[v_0,\cdots,v_k]}: \Delta^k \subset \mathbb{R}^{k+1} \to X: (t_0,\cdots,t_k) \mapsto \sigma(t_0,\cdots,t_k,0,\cdots,0)$, $\sigma|_{[v_k,\cdots,v_{k+\ell}]}: \Delta^\ell \to X: (t_0,\cdots,t_\ell) \mapsto \sigma(0,\cdots,0,t_0,\cdots,t_\ell)$

Lemma 3.6

$$\delta(\varphi \smile \psi) = (\delta\varphi) \smile \psi + (-1)^k \varphi \smile (\delta\psi)$$

for $\varphi \in C^k(X;R)$, $\psi \in C^\ell(X;R)$

pf

For $\sigma: \Delta^{k+\ell+1} \to X$, we have

Recall: $(\delta\varphi)(\sigma) = \sum_{i=0}^{k+1} (-1)^i \varphi(\sigma|_{[v_0,\cdots,\hat{v}_i,\cdots,v_{k+1}]})$

$$(\delta\varphi \smile \psi)(\sigma) = \sum_{i=0}^{k+1} (-1)^i \varphi(\sigma|_{[v_0,\cdots,\hat{v}_i,\cdots,v_{k+1}]})\, \psi(\sigma|_{[v_{k+1},\cdots,v_{k+\ell+1}]})$$

$$(-1)^k(\varphi \smile \delta\psi)(\sigma) = \sum_{i=k}^{k+\ell+1} (-1)^i \varphi(\sigma|_{[v_0,\cdots,v_k]})\, \psi(\sigma|_{[v_k,\cdots,\hat{v}_i,\cdots,v_{k+\ell+1}]})$$

$$\Rightarrow (\delta\varphi \smile \psi)(\sigma) + (-1)^k(\varphi \smile \delta\psi)(\sigma) = (\varphi \smile \psi)(\partial\sigma) = \delta(\varphi \smile \psi)(\sigma) \qquad \#$$

Remark

$\smile$ is associative and distributive on $C^\bullet(X;R)$

Lemma 3.6 $\Rightarrow$ $(C^\bullet(X;R), \delta, \smile)$ is a "different graded algebra"

Thus, one has a well-defined product

- if $\delta\varphi = 0$, $\delta\psi = 0$, then $\delta(\varphi \smile \psi) = \delta\varphi \smile \psi \pm \varphi \smile \delta\psi = 0$
- if $\delta\psi = 0$, then $[\delta\varphi \smile \psi] = [\delta(\varphi\smile\psi) - \pm(\varphi \smile \delta\psi)] = [\delta(\varphi\smile\psi)] = 0$

$$H^k(X;R) \times H^\ell(X;R) \xrightarrow{\smile} H^{k+\ell}(X;R): [\varphi] \smile [\psi] := [\varphi \smile \psi]$$

If R has an identity 1_R, then the class $1 \in H^0(X;R)$ defined by $X \ni x \mapsto 1_R$ is an identity for $\smile$

Prop 3.10

For a map $f: X \to Y$, the induced maps $f^*: H^n(Y;R) \to H^n(X;R)$ satisfy

$$f^*(\alpha \smile \beta) = f^*(\alpha) \smile f^*(\beta)$$

pf

This comes from the cochain formula $f^\#(\varphi) \smile f^\#(\psi) = f^\#(\varphi \smile \psi)$:

$$\begin{aligned}(f^\#\varphi \smile f^\#\psi)(\sigma) &= f^\#\varphi(\sigma|_{[v_0,\cdots,v_k]})\, f^\#\psi(\sigma|_{[v_k,\cdots,v_{k+\ell}]}) \\ &= \varphi(f\sigma|_{[v_0,\cdots,v_k]})\, \psi(f\sigma|_{[v_k,\cdots,v_{k+\ell}]}) \\ &= (\varphi\smile\psi)(f\sigma) \\ &= f^\#(\varphi\smile\psi)(\sigma) \qquad \#\end{aligned}$$

Thm 3.11

Suppose R is commutative. Then $\forall\, \alpha \in H^k(X;R)$, $\beta \in H^l(X;R)$,

$$\alpha \cup \beta = (-1)^{kl}\, \beta \cup \alpha$$

pf

Suppose $\varphi \in C^k(X;R)$, $\psi \in C^l(X;R)$ s.t. $\alpha = [\varphi]$, $\beta = [\psi]$. Recall that for $\sigma : \Delta^{k+l} \to X$,

$$(\varphi \cup \psi)(\sigma) = \varphi(\sigma|_{[v_0,\cdots,v_k]}) \cdot \psi(\sigma|_{[v_k,\cdots,v_{k+l}]}) \qquad R \text{ is commutative}$$

$$(\psi \cup \varphi)(\sigma) = \psi(\sigma|_{[v_0,\cdots,v_l]}) \cdot \varphi(\sigma|_{[v_l,\cdots,v_{k+l}]}) = \varphi(\sigma|_{[v_l,\cdots,v_{k+l}]}) \cdot \psi(\sigma|_{[v_0,\cdots,v_l]})$$

Note: need to shuffle v_i

Let $P : C_n(X) \to C_n(X)$ be the operator defined by

$$P(\sigma) := \varepsilon_n \cdot \bar{\sigma}$$

where $\sigma : \Delta^n \to X$,

$$\bar{\sigma} : \Delta^n \xrightarrow{\subset \mathbb{R}^{n+1}} X, \qquad \bar{\sigma}(t_0 v_0 + \cdots + t_n v_n) := \sigma(t_0 v_n + \cdots + t_n v_0) = \sigma|_{[v_n,\cdots,v_0]} \qquad \Sigma t_i = 1,\ t_i \geq 0$$

$$\varepsilon_n := (-1)^{1+2+\cdots+n} = (-1)^{\frac{n(n+1)}{2}} \quad \leftarrow \text{the sign of the permutation } (n\ n{-}1 \cdots 2\ 1\ 0)$$

Claim:

$P : C_n(X) \to C_n(X)$ is a ① chain map, ② chain homotopic to id

If the claim is true, then

- $P^* = \mathrm{id} : H^n(X;R) \to H^n(X;R)$
- Since

$$\varepsilon_{k+l} = (-1)^{1+\cdots+(k+l)} = (-1)^{1+\cdots+k}\,(-1)^{kl+(1+\cdots+l)} = (-1)^{kl}\,\varepsilon_k \varepsilon_l$$

we have

$$(P^*\varphi \cup P^*\psi)(\sigma) = \varphi(P(\sigma|_{[v_0,\cdots,v_k]})) \cdot \psi(P(\sigma|_{[v_k,\cdots,v_{k+l}]}))$$

$$= \varepsilon_k \cdot \varepsilon_l\, \varphi(\sigma|_{[v_k,\cdots,v_0]}) \cdot \psi(\sigma|_{[v_{k+l},\cdots,v_k]})$$

$$P^*(\psi \cup \varphi)(\sigma) = (\psi \cup \varphi)(P(\sigma)) = \varepsilon_{k+l}\, (\psi \cup \varphi)(\sigma|_{[v_{k+l},\cdots,v_0]})$$

$$= \varepsilon_{k+l}\, \psi(\sigma|_{[v_{k+l},\cdots,v_k]}) \cdot \varphi(\sigma|_{[v_k,\cdots,v_0]})$$

$\because$ R is commutative

$$= (-1)^{kl}\, \varepsilon_k \varepsilon_l\, \varphi(\sigma|_{[v_k,\cdots,v_0]}) \cdot \psi(\sigma|_{[v_{k+l},\cdots,v_k]}) = (-1)^{kl}\,(P^*\varphi \cup P^*\psi)(\sigma)$$

$$\Rightarrow [P^*\psi \cup \varphi] = P^*(\alpha \cup \beta) = \alpha \cup \beta$$

$$= (-1)^{kl}\,[P^*\psi] \cup [P^*\varphi] = (-1)^{kl}\, P^*\beta \cup P^*\alpha = (-1)^{kl}\, \beta \cup \alpha$$

pf of Claim:

① $\partial P = P\partial$: for $\sigma : \Delta^n \to X$,

$$\partial P(\sigma) = \varepsilon_n\, \partial(\sigma|_{[v_n,\dots,v_0]}) = \varepsilon_n \sum_{i=0}^{n} (-1)^i \sigma|_{[v_n,\dots \widehat{v_{n-i}} \dots v_0]}$$

$$P\partial(\sigma) = P\Big(\sum_{i=0}^{n} (-1)^i \sigma|_{[v_0,\dots \hat{v}_i \dots v_n]}\Big)$$

$$= \varepsilon_{n-1} \sum_{i=0}^{n} (-1)^i \sigma|_{[v_n \dots \hat{v}_i \dots v_0]} \overset{i \to n-i}{=} \varepsilon_{n-1} \cdot \sum_{i=0}^{n} (-1)^{n-i} \sigma|_{[v_n \dots \widehat{v_{n-i}} \dots v_0]}$$

$$= \partial P(\sigma) \qquad \varepsilon_{n-1} \cdot (-1)^{n-i} = (-1)^n \cdot \varepsilon_{n-1} \cdot (-1)^{-i} = \varepsilon_n \cdot (-1)^{-i} = \varepsilon_n \cdot (-1)^i$$

② chain homotopy: Write v_i for vertices in $\Delta^n \times \{0\}$, w_i for corresponding vertices in $\Delta^n \times \{1\}$, $\pi = pr_1 : \Delta^n \times I \to \Delta^n$

Define $P : C_n(X) \longrightarrow C_{n+1}(X)$ by

$$P(\sigma) := \sum_{i=0}^{n} (-1)^i \varepsilon_{n-i} (\sigma\pi)|_{[v_0,\dots,v_i, w_n,\dots,w_i]}$$

Here,

$$(\sigma\pi)|_{[v_0,\dots,v_i,w_n,\dots,w_i]}(t_0,\dots,t_{n+1}) = \sigma\pi\Big(\sum_{j=0}^{i} t_j v_j + \sum_{k=i+1}^{n+1} t_k w_{n+i+1-k}\Big)$$

$$= \sigma\big(t_0 v_0 + \dots + (t_i + t_{n+1}) v_i + t_n v_{i+1} + \dots + t_{i+1} v_n\big)$$

Check $\partial P + P\partial = P - id$:

(i) $$(\partial P)(\sigma) = \sum_{j \le i} \boxed{(-1)^i (-1)^j \varepsilon_{n-i}} (\sigma\pi)|_{[v_0,\dots,\hat{v}_j,\dots v_i, w_n,\dots w_i]}$$

$$+ \sum_{j \ge i} (-1)^i (-1)^{n+1+i-j} \varepsilon_{n-i} (\sigma\pi)|_{[v_0 \dots v_i, w_n, \dots \hat{w}_j, \dots w_i]},$$

where the $0 \le i = j \le n$ terms give

$$\varepsilon_n \cdot (\sigma\pi)|_{[w_n,\dots,w_0]} + \sum_{i=1}^{n} \varepsilon_{n-i} \cdot (\sigma\pi)|_{[v_0,\dots v_{i-1}, w_n,\dots,w_i]}$$

let $k = i+1$ $\quad = (-1)^{n+k} \cdot \varepsilon_{n-k+1} = (-1)^{n+k} \cdot (-1)^{n-k+1} \varepsilon_{n-k} = -\varepsilon_{n-k}$

$$+ \sum_{i=0}^{n-1} (-1)^{n+1+i} \varepsilon_{n-i} (\sigma\pi)|_{[v_0,\dots,v_i,w_n,\dots,w_{i+1}]} - (\sigma\pi)|_{[v_0,\dots,v_n]}$$

$$= \sum_{k=1}^{n} -\varepsilon_{n-k} (\sigma\pi)|_{[v_0,\dots v_{k-1}, w_n,\dots, w_k]}$$

$$= \varepsilon_n\, \sigma|_{[v_n,\dots,v_0]} - \sigma|_{[v_0,\dots,v_n]} = P(\sigma) - id(\sigma)$$

(ii) $$(P\partial)(\sigma) = P\Big(\sum_{i=0}^{n} (-1)^i \sigma|_{[v_0,\dots \hat{v}_i \dots v_n]}\Big)$$

$$= \sum_{j<i} (-1)^i (-1)^j \underbrace{\boxed{\varepsilon_{n-1-j}}}_{= \varepsilon_{n-j} \cdot (-1)^{n-j}} (\sigma\pi)|_{[v_0 \dots v_j, w_n, \dots \hat{w}_i \dots w_j]}$$

$$+ \sum_{j>i} (-1)^i (-1)^{j-1} \varepsilon_{n-j} (\sigma\pi)|_{[v_0,\dots \hat{v}_i \dots v_j, w_n \dots w_j]}$$

$$\overset{i \leftrightarrow j}{=} \sum_{j<i} \boxed{(-1)^{i-1} (-1)^j \varepsilon_{n-i}} (\sigma\pi)|_{[v_0,\dots,\hat{v}_j,\dots v_i, w_n,\dots,w_i]}$$

$$+ \sum_{j>i} (-1)^j (-1)^i (-1)^{n-i} \varepsilon_{n-i} (\sigma\pi)|_{[v_0,\dots,v_i,w_n,\dots,\hat{w}_j,\dots,w_i]}$$

So $\partial P + P\partial = P - id$ #

Conclusion

Let R be a ring. The coh $(H^\bullet(X;R), \cup)$ is a ring with the property
$\cup : H^i(X;R) \times H^j(X;R) \longrightarrow H^{i+j}(X;R)$ — a graded ring

Furthermore, if R is commutative, then
$\alpha \cup \beta = (-1)^{ij}\, \beta \cup \alpha \qquad \forall \alpha \in H^i(X;R),\ \beta \in H^j(X;R)$

Such a ring is called a commutative graded ring. We will explore these structures later.

Example $(R=\mathbb{Z},\ X=S^n)$

$\overset{n\geq 1}{X} = S^n$. Recall $H_k(S^n) = \begin{cases} \mathbb{Z}, & k=0,n \\ 0, & \text{otherwise} \end{cases}$ is free $\forall k \Rightarrow \mathrm{Ext}(H_{k-1}(S^n),\mathbb{Z}) = 0$

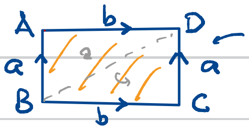
universal coeff thm $\Rightarrow$ $H^k(S^n;\mathbb{Z}) \cong \mathrm{Hom}(H_k(S^n),\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & k=0,n \\ 0, & \text{otherwise} \end{cases}$

$H^\bullet(S^n;\mathbb{Z}) \cong \overset{0}{\mathbb{Z}} \oplus 0 \oplus \cdots \oplus \overset{n}{\mathbb{Z}} \oplus 0 \oplus \cdots$ distribution law

$\underset{a\cdot 1}{} \quad \cup \quad b\cdot[\sigma] \overset{}{=} (ab)(1\cup[\sigma]) = (ab)[\sigma]$ $\qquad [\sigma]\cup[\sigma] \in H^{2n}(S^n) = 0 \Rightarrow [\sigma]\cup[\sigma] = 0$

So, as a ring, $H^\bullet(S^n;\mathbb{Z}) \cong (\overset{ac}{\mathbb{Z}} \oplus \overset{bd}{\underset{n}{\mathbb{Z}}}, \cup)$, where
$(a,b) \cup (c,d) = (ac, ad+bc)$

Example 3.7 $(R=\mathbb{Z},\ X=T$. See Example 3.7 for orientable close surfaces$)$

$X = T =$  $\leftarrow \square = S \cong S^1 \vee S^1 \subseteq T$

Recall that $\cong \tilde{H}_2(T/S) \cong H_2(D^2/S^1) \cong H_2(D^2, S^1)$

$\cdots \to 0 = H_2(S) \to H_2(T) \overset{\cong}{\to} \boxed{H_2(T,S)} \overset{0}{\to} H_1(S) \to \cdots$

$\Rightarrow H_2(T)$ is generated by the class induced by quotient map. $\quad$ exer (cf. computation of generator of $H_n(S^n)$)

need this order for later computation $\quad (BAD) - (BCD) \quad \sim$ a homeomorphism $\Delta^2 \to$ [hatched square]

where $(BAD) : \Delta^2 \to T$ is induced by the affine map $\begin{matrix} v_0 \mapsto B \\ v_1 \mapsto A \\ v_2 \mapsto D \end{matrix}$, and (BCD) is defined similarly

Let $z \in H_2(T)$ be the class induced by $(BAD) - (BCD)$

Since generators: $H_0(T) \cong \mathbb{Z}$, $H_1(T) \cong \overset{a}{\mathbb{Z}} \oplus \overset{b}{\mathbb{Z}}$, $H_2(T) \cong \overset{z}{\mathbb{Z}}$ are free, it follows from universal coeff thm that
$H^0(T;\mathbb{Z}) \cong \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \cong \mathbb{Z}$, $\quad H^1(T;\mathbb{Z}) \cong \mathrm{Hom}(\mathbb{Z}^2,\mathbb{Z}) \cong \mathbb{Z}^2$, $\quad H^2(T;\mathbb{Z}) \cong \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \cong \mathbb{Z}$

Let $\{\alpha,\beta\} \in H^1(T;\mathbb{Z}) \underset{h}{\overset{\cong}{\to}} \mathrm{Hom}(\mathbb{Z}^2,\mathbb{Z})$ be the dual basis of $\{a,b\}$, i.e.
$h(\alpha)(a) = 1,\ h(\alpha)(b) = 0,\ h(\beta)(a) = 0,\ h(\beta)(b) = 1$

Example 3.7

$$\Rightarrow h(\alpha\cup\beta)(Z) = \left[h(\alpha\cup\beta)((BAD)) - h(\alpha\cup\beta)((BCD))\right]$$

$$= \left[h(\alpha)(\underbrace{(BA)}^{a}) \cdot h(\beta)(\underbrace{(AD)}^{b}) - h(\alpha)(\underbrace{(BC)}^{b}) \cdot h(\beta)(\underbrace{(CD)}^{a})\right]$$

$$= 1\cdot 1 - 0\cdot 0 = 1$$

So $\alpha\cup\beta = \overset{\zeta(Z)=1}{\zeta} \in H^2(T;\mathbb{Z}) \cong \mathrm{Hom}(H_2(T),\mathbb{Z})$ is a generator of $H^2(T;\mathbb{Z})$

Conclusion

$$H^\bullet(T;\mathbb{Z}) \cong \underset{\text{deg}: 0}{\mathbb{Z}} \oplus \underset{1}{\mathbb{Z}^2} \oplus \underset{2}{\mathbb{Z}}$$

generator: 1, α,β, ζ

$\alpha\cup\alpha = (-1)^{1\cdot 1}\alpha\cup\alpha \Rightarrow 2(\alpha\cup\alpha) = 0$ in $H^2(T;\mathbb{Z}) \cong \mathbb{Z}$ $\Rightarrow \alpha\cup\alpha = 0$

with the relations

$\alpha\cup\beta = \zeta$, $\alpha\cup\alpha = 0$, $\beta\cup\beta = 0$

$= \zeta\cup\alpha = \zeta\cup\beta$

1 is the identity element. $\alpha\cup\zeta = \beta\cup\zeta = 0$, $\zeta\cup\zeta = 0$ ← degree reason #

exer

Compute the cohomology ring of closed surfaces (Example 3.7, Example 3.8)

Algebraic structures behind homology / cohomology

Def

Let R be a ring. A (left) R-module M consists of an abelian gp $(M,+)$ and an operation (scalar product) $\cdot : R\times M \to M$ s.t. $\forall\, r,s\in R$, $x,y\in M$, we have

(i) $r\cdot(x+y) = r\cdot x + r\cdot y$

(ii) $(r+s)\cdot x = r\cdot x + s\cdot x$

(iii) $(rs)\cdot x = r\cdot(s\cdot x)$

(iv) $1_R\cdot x = x$ (if R has an identity element 1_R)

Homomorphisms, submodules, generators, finitely generated, kernel, quotient are defined as usual

← also say "R-linear map"

Example

① If $R=\mathbb{k}$ is a field, then $\mathbb{k}$-module $=$ $\mathbb{k}$-vector space

② $\mathbb{Z}$-module $=$ abelian group: $n\cdot x = \overbrace{x+x+\cdots+x}^{n\text{ times}}$, $(-n)\cdot x = \overbrace{(-x)+\cdots+(-x)}^{n\text{ time}}$, $n\in\mathbb{N}$

③ $C_n(X)$ is an abelian gp $\Rightarrow$ a $\mathbb{Z}$-module

④ $C_n(X;R)$ is an R-module: For $s, r_i\in R$, $\sum r_i\sigma_i \in C_n(X;R)$, we have $s\cdot(\sum r_i\sigma_i) = \sum (sr_i)\sigma_i$

← quotient module, ∂ is a homo of R-mod

⑤ $H_n(X;R)$ is an R-module: $s\cdot[\sum r_i\sigma_i] = [\sum (sr_i)\sigma_i]$

⑥ $C^n(X;R)$ and $H^n(X;R)$ are R-module: For $\varphi\in C^n(X;R)$, $r\in R$, $r\cdot\varphi\in C^n(X;R)$, $(r\cdot\varphi)(\sigma) = r\varphi(\sigma)$

d is a homo of R-mod $\qquad r\cdot[\varphi] = [r\cdot\varphi]$