② Relative coh groups and long exact seq of a pair (X,A): We first dualize the short exact seq

$$0 \to C_n(A) \xrightarrow{i} C_n(X) \xrightarrow{j} C_n(X,A) \to 0$$

by applying $\mathrm{Hom}(-,G)$ to get

$$0 \leftarrow C^n(A;G) \xleftarrow{i^*} C^n(X;G) \xleftarrow{j^*} C^n(X,A;G) \leftarrow 0 \qquad \circledast$$

where $C^n(X,A;G) := \mathrm{Hom}(C_n(X,A),G)$.

Lemma $\circledast$ is a short exact seq of cochain complexes

pf

i^* is onto: Since $C_n(A)$, $C_n(X)$ are free, we can extend $\alpha \in \mathrm{Hom}(C_n(A),G)$ to $\hat{\alpha}: C_n(X) \to G$ by $\hat{\alpha}(\sigma)=0$ $\forall \sigma \notin C_n(A)$, $\sigma: \Delta^n \to X$, $i^*(\hat{\alpha}) = \alpha$

$\ker i^* = \mathrm{im}\, j^*$: $\beta \in \ker i^* \Leftrightarrow \beta(\sigma)=0 \ \forall \sigma \in C_n(A) \Leftrightarrow \exists \gamma: C_n(X)/C_n(A) \to G$ s.t. $\beta = \gamma \circ j$

j^* is 1-1: true because j is onto

Compatibility between i^*, j^* and $\delta = \partial^*$: direct checks #

Therefore, we have the long exact seq $\swarrow$ $H^n(X,A;G) = \ker/\mathrm{im}$ at $C^n(X,A;G)$

$$\cdots \to H^n(X,A;G) \xrightarrow{j^*} H^n(X;G) \xrightarrow{i^*} H^n(A;G) \xrightarrow{\delta} H^{n+1}(X,A;G) \to \cdots$$

Similarly, one also has $(\tilde{H}^n(X,A;G) := H^n(X,A;G)\ \forall n,\ A \neq \emptyset)$

$$\cdots \to \tilde{H}^n(X,A;G) \xrightarrow{j^*} \tilde{H}^n(X;G) \xrightarrow{i^*} \tilde{H}^n(A;G) \xrightarrow{\delta} \tilde{H}^{n+1}(X,A;G) \to \cdots \qquad \circledast\circledast$$

and

$$\cdots \to \tilde{H}^n(X,A;G) \to \tilde{H}^n(X,B;G) \to \tilde{H}^n(A,B;G) \to \tilde{H}^{n+1}(X,A;G) \to \cdots$$

Note: as before, the seq $\circledast\circledast$ gives an iso $\tilde{H}^n(X;G) \cong H^n(X,\{x_0\};G)$, $x_0 \in X$.

Prop (p. 200)

The connecting homomorphisms $\delta: H^n(A;G) \to H^{n+1}(X,A;G)$ and $\partial: H_{n+1}(X,A) \to H_n(A)$ are dual to each other in the sense that the diagram

$$\begin{array}{ccc} H^n(A;G) & \xrightarrow{\delta} & H^{n+1}(X,A;G) \\ \downarrow h & & \downarrow h \\ \mathrm{Hom}(H_n(A),G) & \xrightarrow{\partial^*} & \mathrm{Hom}(H_{n+1}(X,A),G) \end{array}$$

Commutes

pf: exer

Prop

Since $C_n(X,A)$ are free, it follows from Thm 3.2 that the seq's

$$0 \to \mathrm{Ext}(H_{n-1}(X,A),G) \to H^n(X,A;G) \to \mathrm{Hom}(H_n(X,A),G) \to 0$$

are split and exact

③ Induced homomorphisms For $f: X\to Y$, we have the chain map $f_\#: C_n(X) \to C_n(Y)$.

Its dual $f^\#: C^n(Y;G) \to C^n(X;G)$ is a cochain map

$\Rightarrow$ we have $f^*: H^n(Y;G) \to H^n(X;G)$

Similarly, for $f: (X,A) \to (Y,B)$, we have $f^*: H^n(Y,B;G) \to H^n(X,A;G)$

Prop: $(fg)^* = g^* f^*$, $id^* = id$

Remark

A map $f: (X,A) \to (Y,B)$ induces a commutative diagram

$$\begin{array}{ccccccccc} 0 \to & \mathrm{Ext}(H_{n-1}(X,A), G) & \to & H^n(X,A;G) & \xrightarrow{h} & \mathrm{Hom}(H_n(X,A), G) & \to 0 \\ & \uparrow (f_*)^* & \circlearrowleft & \uparrow f^* & \circlearrowleft & \uparrow (f_*)^* & \\ 0 \to & \mathrm{Ext}(H_{n-1}(Y,B), G) & \to & H^n(Y,B;G) & \xrightarrow{h} & \mathrm{Hom}(H_n(Y,B), G) & \to 0 \end{array}$$

④ Homotopy invariance Recall that if $f\simeq g: (X,A) \to (Y,B)$, then (Thm 2.10) $\exists$ chain homotopy

$P: C_n(X,A) \to C_{n+1}(Y,B)$ s.t. $g_\# - f_\# = \partial P + P\partial$

$\xRightarrow{\text{take dual}}$ $g^\# - f^\# = P^*\delta + \delta P^*$: $C_n(Y,B;G) \to C_n(X,A;G)$

$\Rightarrow$ Prop

If $f\simeq g: (X,A) \to (Y,B)$, then $f^* = g^*: H_n(Y,B;G) \to H_n(X,A;G)$

⑤ Excision

Thm

Suppose $Z \subseteq A \subseteq X$, $\mathrm{cl}(Z) \subset A$. Then the inclusion $i: (X-Z, A-Z) \hookrightarrow (X,A)$ induces isomorphisms

$$i^*: H^n(X,A;G) \xrightarrow{\cong} H^n(X-Z, A-Z;G)$$

pf

By excision of homology, $i_*: H_n(X-Z, A-Z) \to H_n(X,A)$ are iso's.

$$\begin{array}{ccccccc} 0 \to & \mathrm{Ext}(H_{n-1}(X-Z, A-Z), G) & \to & H^n(X-Z, A-Z;G) & \to & \mathrm{Hom}(H_n(X-Z,A-Z), G) & \to 0 \\ \wr\wr & \wr\wr \uparrow (i_*)^* & & \uparrow i^* & & \wr\wr \uparrow (i_*)^* & \wr\wr \\ 0 \to & \mathrm{Ext}(H_{n-1}(X,A), G) & \to & H^n(X,A;G) & \to & \mathrm{Hom}(H_n(X,A), G) & \to 0 \end{array}$$

$\Rightarrow$ by five-lemma, i^* are iso's $\#$

⑥ **Cellular cohomology** For a CW cx X, one has the cellular cochain complex $(H^n(X^n,X^{n-1};G), d_n)$ defined as in the diagram

⊛

$$\begin{tikzcd}
& & H^{n-1}(X^{n-1};G) \arrow[r] \arrow[dr, "\delta_{n-1}"] & H^{n-1}(X^{n-2};G) \overset{\circledast}{=} 0 & & & \\
\cdots \arrow[r] & H^{n-1}(X^{n-1},X^{n-2};G) \arrow[ur, "j_{n-1}"] \arrow[rr, "d_{n-1}"] & & H^n(X^n,X^{n-1};G) \arrow[rr, "d_n"] \arrow[dr, "j_n"] & & H^{n+1}(X^{n+1},X^n;G) \arrow[r] & \cdots \\
& & 0 = H^n(X^{n+1},X^n;G) \arrow[r] & H^n(X^{n+1};G) \arrow[r] & H^n(X^n;G) \arrow[ur, "\delta_n"] \arrow[r] & 0 &
\end{tikzcd}$$

Thm 3.5

(i) $H^n(X;G) \cong \ker d_n / \operatorname{im} d_{n-1}$. (ii) Furthermore, the cellular cochain cx $(H^n(X^n,X^{n-1};G), d_n)$ is iso to the dual of the cellular chain cx $\Rightarrow$ one can apply universal coeff thm to ⤴ for computing cellular coh

pf

(i) By universal coeff thm, free $\Rightarrow$ Ext $=0$

$$0 \to \operatorname{Ext}(\underbrace{H_{k-1}(X^n,X^{n-1})}_{\text{free}}, G) \to H^k(X^n,X^{n-1};G) \to \operatorname{Hom}(\underbrace{H_k(X^n,X^{n-1})}_{=0 \text{ if } k\neq n}, G) \to 0$$

$\Rightarrow H^k(X^n,X^{n-1};G) = 0$ for $k \neq n$ $\Rightarrow H^k(X^n;G) \cong H^k(X^{n-1};G)$ $\forall k \neq n, n-1$

$\Rightarrow H^k(X^n;G) = 0$ if $k > n$ ⊛

By universal coeff thm, (assume $k \leq n+1$) 0 by pf of Lemma 2.34 (c)

$$0 \to \operatorname{Ext}(\underbrace{H_{k-1}(X,X^{n+1})}_{=0}, G) \to H^k(X,X^{n+1};G) \to \operatorname{Hom}(\underbrace{H_k(X,X^{n+1})}_{=0}, G) \to 0$$

$\Rightarrow H^k(X,X^{n+1};G) = 0$ if $k \leq n+1$ $\Rightarrow$ $\boxed{H^n(X;G) \cong H^n(X^{n+1};G)}$

So the diagram ⊛ implies

$$H^n(X;G) \cong H^n(X^{n+1};G) \cong \ker \delta_n \cong j_n^{-1}(\ker \delta_n)/\ker j_n = \ker d_n / \operatorname{im} \delta_{n-1} = \ker d_n / \operatorname{im} d_{n-1}$$

(ii) We have the diagram

cellular coboundary map

$$\begin{tikzcd}
H^k(X^k,X^{k-1};G) \arrow[r] \arrow[d, "h"', "\text{is}"] & H^k(X^k;G) \arrow[r, "\delta"] \arrow[d, "h"] & H^{k+1}(X^{k+1},X^k;G) \arrow[d, "h", "\text{is}"'] \\
\operatorname{Hom}(H_k(X^k,X^{k-1}),G) \arrow[r] & \operatorname{Hom}(H_k(X^k),G) \arrow[r, "\partial^*"] & \operatorname{Hom}(H_{k+1}(X^{k+1},X^k),G)
\end{tikzcd}$$

iso by universal coeff thm

dual of cellular boundary map

This diagram commutes because of the naturality of h and by a discussion of the long exact seq. of (X,A) $\Rightarrow$ (ii) is ok #

⑦ **Mayer-Vietoris sequences** If $X = \operatorname{int}(A) \cup \operatorname{int}(B)$, then we have the long exact seq.

$$\cdots \to H^n(X;G) \to H^n(A;G) \oplus H^n(B;G) \to H^n(A\cap B;G) \to H^{n+1}(X;G) \to \cdots$$

exer: Read p. 203, 204 for the details and other versions of M-V seq

§3.2 Cup product

Let R be a ring. For cochains $\varphi \in C^k(X;R)$ and $\psi \in C^l(X;R)$, the cup product $\varphi \cup \psi \in C^{k+l}(X;R)$ is the cochain whose value on $\sigma : \Delta^{k+l} \to X$ is given by

$$(\varphi \cup \psi)(\sigma) := \varphi(\sigma|_{[v_0,\cdots,v_k]}) \cdot \psi(\sigma|_{[v_k,\cdots,v_{k+l}]})$$

product in R

where $\sigma|_{[v_0,\cdots,v_k]} : \Delta^k \subset \mathbb{R}^{k+1} \to X : (t_0,\cdots,t_k) \mapsto \sigma(t_0,\cdots,t_k,0,\cdots,0)$, $\sigma|_{[v_k,\cdots,v_{k+l}]} : \Delta^l \to X : (t_0,\cdots,t_l) \mapsto \sigma(0,\cdots,0,t_0,\cdots,t_l)$

☆ **Lemma 3.6**

$$\delta(\varphi \cup \psi) = (\delta\varphi) \cup \psi + (-1)^k \varphi \cup (\delta\psi)$$

for $\varphi \in C^k(X;R)$, $\psi \in C^l(X;R)$

pf

Recall: $(\delta\varphi)(\sigma) = \sum_{i=0}^{k+1} (-1)^i \varphi(\sigma|_{[v_0,\cdots,\hat{v}_i,\cdots,v_{k+1}]})$

For $\sigma : \Delta^{k+l+1} \to X$, we have

$$(\delta\varphi \cup \psi)(\sigma) = \sum_{i=0}^{k+1} (-1)^i \varphi(\sigma|_{[v_0,\cdots,\hat{v}_i,\cdots,v_{k+1}]})\, \psi(\sigma|_{[v_{k+1},\cdots,v_{k+l+1}]})$$

$$(-1)^k(\varphi \cup \delta\psi)(\sigma) = \sum_{i=k}^{k+l+1} (-1)^i \varphi(\sigma|_{[v_0,\cdots,v_k]})\, \psi(\sigma|_{[v_k,\cdots,\hat{v}_i,\cdots,v_{k+l+1}]})$$

$$\Rightarrow (\delta\varphi \cup \psi)(\sigma) + (-1)^k(\varphi \cup \delta\psi)(\sigma) = (\varphi \cup \psi)(\partial\sigma) = \delta(\varphi \cup \psi)(\sigma) \qquad \#$$

Remark

$\cup$ is associative and distributive on $C^*(X;R)$

Lemma 3.6 $\Rightarrow$ $(C^*(X;R), \delta, \cup)$ is a "different graded algebra"

Thus, one has a well-defined product

$$H^k(X;R) \times H^l(X;R) \xrightarrow{\cup} H^{k+l}(X;R) : [\varphi] \cup [\psi] \mapsto [\varphi \cup \psi]$$

- if $\delta\varphi = 0$, $\delta\psi = 0$, then $\delta(\varphi \cup \psi) = \delta\varphi \cup \psi \pm \varphi \cup \delta\psi = 0$
- if $\delta\psi = 0$, then $[\delta\varphi \cup \psi] = [\delta(\varphi \cup \psi) - \pm(\varphi \cup \delta\psi)] = [\delta(\varphi \cup \psi)] = 0$

If R has an identity 1_R, then the class $1 \in H^0(X;R)$ defined by $X \ni x \mapsto 1_R$ is an identity for $\cup$

Prop 3.10

For a map $f : X \to Y$, the induced maps $f^* : H^n(Y;R) \to H^n(X;R)$ satisfy

$$f^*(\alpha \cup \beta) = f^*(\alpha) \cup f^*(\beta)$$

pf

This comes from the cochain formula $f^\#(\varphi) \cup f^\#(\psi) = f^\#(\varphi \cup \psi)$:

$$\begin{aligned}(f^\#\varphi \cup f^\#\psi)(\sigma) &= f^\#\varphi(\sigma|_{[v_0,\cdots,v_k]})\, f^\#\psi(\sigma|_{[v_k,\cdots,v_{k+l}]}) \\ &= \varphi(f\sigma|_{[v_0,\cdots,v_k]})\, \psi(f\sigma|_{[v_k,\cdots,v_{k+l}]}) \\ &= (\varphi \cup \psi)(f\sigma) \\ &= f^\#(\varphi \cup \psi)(\sigma) \qquad \#\end{aligned}$$