§3.1 Cohomology

Remark
There is also a product in homology called cross product, but it's not as useful as cup product

Cohomology is a kind of dual notion of homology.
A big difference between homology and coh:
Cohomology is equipped a natural product, called cup product, which is useful in many topics, such as characteristic classes.

Dual complex:

Let $\cdots \to C_n \xrightarrow{\partial} C_{n-1} \to \cdots \to C_1 \xrightarrow{\partial} C_0 \to 0$ be a chain complex. G = ab gp

Denote $C_n^* = \mathrm{Hom}(C_n, G)$, $\delta = \delta_n = \partial^* : C_n^* \to C_{n+1}^* : f \mapsto f \circ \partial$ ← called coboundary map

$C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{f} G$

Note that $(\alpha\beta)^* = \beta^*\alpha^*$, $id^* = id$, $0^* = 0$

Lemma $\delta \circ \delta = 0$

pf $\delta^2(f) = f \circ \partial \circ \partial = 0$

We have the cochain complex (i.e. $\delta : D_n \to D_{n+1}$, $\delta^2 = 0$)

$$\cdots \leftarrow C_{n+1}^* \xleftarrow{\delta_n} C_n^* \xleftarrow{\delta_{n-1}} C_{n-1}^* \leftarrow \cdots \leftarrow C_1^* \leftarrow C_0^* \leftarrow 0$$

The gp $H^n(C; G) := \ker(\delta_n)/\mathrm{im}(\delta_{n-1})$ is called the cohomology group of (C_n^*, δ).

Remark
One can ① take homology then take dual or ② take dual then take cohomology

① $\mathrm{Hom}(H_n(C, \partial); G)$

② $\cdots \leftarrow C_{n+1}^* \xleftarrow{\delta} C_n^* \xleftarrow{\delta} C_{n-1}^* \leftarrow \cdots$, $H^n(C; G) = \ker(\delta)/\mathrm{im}(\delta)$

These two are different in general. The relationship between them is the content of "universal coeff thm"

Example①

Consider $0 \to \overset{1}{\mathbb{Z}} \xrightarrow{\times m} \overset{0}{\mathbb{Z}} \to 0$

⇒ homology: $H_0 = \mathbb{Z}_m$, $H_1 = 0$

dual of homology: $\mathrm{Hom}(H_0, \mathbb{Z}) = \mathrm{Hom}(\mathbb{Z}_m, \mathbb{Z}) = 0$, $\mathrm{Hom}(H_1, \mathbb{Z}) \cong 0$

$f(mx) \leftrightarrow f(m) = mf(1)$ ⟵ $f(x) \leftrightarrow f(1)$

dual cx: $0 \leftarrow \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z} \xleftarrow{\times m} \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z} \leftarrow 0$

coh of dual cx: $H^0 \cong 0$, $H^1 \cong \mathbb{Z}_m$

NOT isomorphic

Universal coefficient theorem

Let H, G be arbitrary ab gps. Suppose $\{x_\alpha\}_{\alpha\in\Lambda}$ generates H. Let

F_0 = free ab gp gen. by $\{x_\alpha\}_{\alpha\in\Lambda}$

$F_1 = \ker(F_0 \twoheadrightarrow H)$ ← also a free ab gp

this part is called a free resolution of H

exer Read p. 193 for the definition of free resolutions

Then the seq

$$\boxed{0 \to F_1 \xhookrightarrow{f_1} F_0} \xrightarrow{f_0} H \to 0$$

is an exact seq. Furthermore, we have

exact here

$$0 \leftarrow \mathrm{Hom}(F_1, G) \xleftarrow{f_1^*} \mathrm{Hom}(F_0, G) \xleftarrow{f_0^*} \mathrm{Hom}(H, G) \leftarrow 0$$

exer

$\ker(f_1^*) = \mathrm{im}(f_0^*)$

which is a cochain cx, but NOT exact at $\mathrm{Hom}(F_1, G)$ in general. Define

$$\mathrm{Ext}(H,G)\ (= \mathrm{Ext}^1(H,G)) := \mathrm{Hom}(F_1,G)/\mathrm{im}(f_1^*)$$

Lemma (Lemma 3.1, p. 194)

$\mathrm{Ext}(H,G)$ is indep of the choices of F_1, F_0

pf: skip. exer: Read Lemma 3.1 and its proof.

Properties of $\mathrm{Ext}(H,G)$ (p. 195)

exer

Prove these properties

- $\mathrm{Ext}(H\oplus H', G) \cong \mathrm{Ext}(H,G) \oplus \mathrm{Ext}(H',G)$
- $\mathrm{Ext}(H,G) = 0$ if H is free
- $\mathrm{Ext}(\mathbb{Z}_n, G) \cong G/nG$

(p. 191)

$h: H^n(C;G) \to \mathrm{Hom}(H_n(C), G)$,

$h([\varphi])([c]) = \varphi(c)$

Lemma h is well-def

pf. $\cdot\ \delta\varphi = 0 = \varphi\partial \Rightarrow \varphi|_{\mathrm{im}\,\partial} = 0 \Rightarrow \varphi: H_n(C) \to G$

$\cdot\ (\delta\psi)(c) = \psi\cdot\partial(c) = 0$ #

Thm 3.2 (Universal coefficient theorem for cohomology)

Let C be a chain cx of free ab gps. Then the seq.

$$0 \to \mathrm{Ext}(H_{n-1}(C), G) \to H^n(C;G) \xrightarrow{h} \mathrm{Hom}(H_n(C), G) \to 0$$

$\exists F$, $F \circ (\to) = \mathrm{id}$

is a split exact seq.

i.e.

$H^n(C;G) \cong \mathrm{Ext}(H_{n-1}, G) \oplus \mathrm{Hom}(H_n, G)$

pf: p. 191 ~ 195, skip here.

Example

$$C: \quad 0 \to \underset{1}{\mathbb{Z}} \xrightarrow{\times m} \underset{0}{\mathbb{Z}} \to 0, \quad \begin{array}{l} H_0 = \mathbb{Z}_m \\ H_1 = 0 \end{array} \Rightarrow \mathrm{Ext}(H_0, \mathbb{Z}) \cong \mathbb{Z}_m$$

$$0 \to \mathrm{Ext}(H_0(C), \mathbb{Z}) \cong \mathbb{Z}_m \to H^1(C;\mathbb{Z}) \cong \mathbb{Z}_m \to \mathrm{Hom}(H_1, \mathbb{Z}) = 0 \to 0$$

$$0 \to \mathrm{Ext}(H_{-1}, \mathbb{Z}) = 0 \to H^0(C;\mathbb{Z}) = 0 \to \mathrm{Hom}(H_0, \mathbb{Z}) = 0 \to 0$$

Cohomology of spaces

Let X be a space, G be an ab gp, $C_n(X)$ = singular chain

We define the group $C^n(X;G)$ of singular n-cochains with coefficients in G to be the dual gp $\mathrm{Hom}(C_n(X), G)$

The coboundary map $\delta : C^n(X;G) \longrightarrow C^{n+1}(X;G)$ is the dual ∂^* of the boundary map, i.e., $\forall\ \varphi \in C^n(X;G)$, $\sigma : \Delta^{n+1} \longrightarrow X$,

$$(\delta\varphi)(\sigma) = \sum_i (-1)^i \varphi(\sigma|_{[v_0, \cdots, \hat{v_i}, \cdots v_{n+1}]})$$

Since $\partial^2 = 0 \Rightarrow \delta^2 = 0$, we can define the cohomology group $H^n(X;G)$ with coefficients in G to be the quotient $\ker\delta / \mathrm{im}\,\delta$ at $C^n(X;G)$ in the cochain complex

$$\cdots \leftarrow C^{n+1}(X;G) \xleftarrow{\delta} C^n(X;G) \xleftarrow{\delta} C^{n-1}(X;G) \leftarrow \cdots \leftarrow C^0(X;G) \leftarrow 0$$

Elements of $\ker\delta$ are cocycles, and elements of $\mathrm{im}\,\delta$ are coboundaries.

By Thm 3.2, we have the following

Thm

The seq

$$0 \longrightarrow \mathrm{Ext}(H_{n-1}(X), G) \longrightarrow H^n(X;G) \longrightarrow \mathrm{Hom}(H_n(X), G) \longrightarrow 0$$

is split exact.

Remark

- Since $\mathrm{Ext}(H_{-1}(X), G) = 0$, we have $H^0(X;G) \cong \mathrm{Hom}(H_0(X); G)$
- Since $H_0(X) = \bigoplus_{\text{path components}} \mathbb{Z}$ is free, we have $\mathrm{Ext}(H_0(X); G) = 0$.

  $\Rightarrow H^1(X;G) \cong \mathrm{Hom}(H_1(X), G)$
- If F is a field, then by a generalized form of universal coeff thm

  $$H^n(X;F) \cong \mathrm{Hom}_F(H_n(X;F), F)$$

Most thms for singular homology have cohomological versions:

① Reduced cohomology group $\tilde{H}^n(X;G)$ can be defined by dualizing the augmented chain complex $\cdots \to C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$, then taking $\ker/\mathrm{im}$. So we have

$$\tilde{H}^n(X;G) = H^n(X;G) \quad \forall n > 0, \qquad \tilde{H}^0(X;G) \cong \mathrm{Hom}(\tilde{H}_0(X), G) \text{ by universal coeff thm}$$

By the construction, one can see that $\tilde{H}^0(X;G) = \{\text{functions } X \to G \text{ that are constant on path-components}\} \big/ \{X \to G \text{ constant on all } X\}$

② Relative coh groups and long exact seq of a pair (X,A): We first dualize the short exact seq

$$0 \to C_n(A) \xrightarrow{i} C_n(X) \xrightarrow{j} C_n(X,A) \to 0$$

by applying $\mathrm{Hom}(-,G)$ to get

$$0 \leftarrow C^n(A;G) \xleftarrow{i^*} C^n(X;G) \xleftarrow{j^*} C^n(X,A;G) \leftarrow 0 \qquad \circledast$$

where $C^n(X,A;G) := \mathrm{Hom}(C_n(X,A),G)$.

<u>Lemma</u> $\circledast$ is a short exact seq of cochain complexes

pf

$i^*(\hat{\alpha}) = \alpha$ $\Uparrow$ $\quad \sigma: \Delta^n \to X,\ \forall \sigma \notin C_n(A)$

<u>i^* is onto</u>: Since $C_n(A)$, $C_n(X)$ are free, we can extend $\alpha \in \mathrm{Hom}(C_n(A),G)$ to $\hat{\alpha}: C_n(X) \to G$ by $\hat{\alpha}(\sigma)=0$

<u>$\ker i^* = \mathrm{im}\, j^*$</u>: $\beta \in \ker i^* \Leftrightarrow \beta(\sigma)=0\ \forall \sigma \in C_n(A) \Leftrightarrow \exists \gamma: C_n(X)/C_n(A) \to G$ s.t. $\beta = \gamma \circ j$

<u>j^* is 1-1</u>: true because j is onto

<u>Compatibility between i^*, j^* and $\delta = \partial^*$</u>: direct checks #

Therefore, we have the long exact seq $\swarrow$ $H^n(X,A;G) = \ker/\mathrm{im}$ at $C^n(X,A;G)$

$$\cdots \to H^n(X,A;G) \xrightarrow{j^*} H^n(X;G) \xrightarrow{i^*} H^n(A;G) \xrightarrow{\delta} H^{n+1}(X,A;G) \to \cdots$$

Similarly, one also has $(\tilde{H}^n(X,A;G) := H^n(X,A;G)\ \forall n,\ A \neq \emptyset)$

$$\cdots \to \tilde{H}^n(X,A;G) \xrightarrow{j^*} \tilde{H}^n(X;G) \xrightarrow{i^*} \tilde{H}^n(A;G) \xrightarrow{\delta} \tilde{H}^{n+1}(X,A;G) \to \cdots \qquad \circledast\circledast$$

and

$$\cdots \to \tilde{H}^n(X,A;G) \to \tilde{H}^n(X,B;G) \to \tilde{H}^n(A,B;G) \to \tilde{H}^{n+1}(X,A;G) \to \cdots$$

Note: as before, the seq $\circledast\circledast$ gives an iso $\tilde{H}^n(X;G) \cong H^n(X,\{x_0\};G)$, $x_0 \in X$.

<u>Prop</u> (p. 200)

The connecting homomorphisms $\delta: H^n(A;G) \to H^{n+1}(X,A;G)$ and $\partial: H_{n+1}(X,A) \to H_n(A)$ are dual to each other in the sense that the diagram

$$\begin{array}{ccc} H^n(A;G) & \xrightarrow{\quad\delta\quad} & H^{n+1}(X,A;G) \\ \downarrow h & & \downarrow h \\ \mathrm{Hom}(H_n(A),G) & \xrightarrow{\partial^*} & \mathrm{Hom}(H_{n+1}(X,A),G) \end{array}$$

commutes

<u>pf</u>: exer

<u>Prop</u>

Since $C_n(X,A)$ are free, it follows from <u>Thm 3.2</u> that the seq's

$$0 \to \mathrm{Ext}(H_{n-1}(X,A),G) \to H^n(X,A;G) \to \mathrm{Hom}(H_n(X,A),G) \to 0$$

are split and exact