Example 2.31

Lemma

(i) (Property (g), p134) A reflection of S^n (wrt a n-dim vec. subsp in $\mathbb{R}^{n+1}$) has degree -1

(ii) (Property (f), p134) $\deg(a) = (-1)^{n+1}$

Generator of $H_n(S^n)$ (Example 2.23, p125):

Recall from the good pair $(D^n, S^{n-1} = \partial D^n)$, we have

$$\cdots \longrightarrow \tilde{H}_n(D^n) = 0 \longrightarrow \underline{H_n(D^n, S^{n-1})} \xrightarrow{\cong} \tilde{H}_{n-1}(S^{n-1}) \longrightarrow \tilde{H}_{n-1}(D^n) = 0 \to \cdots$$

$$\downarrow \text{IIS} \qquad H_n(D^n/S^{n-1}, S^{n-1}/S^{n-1}) \cong \tilde{H}_n(S^n) \qquad n \geq 1$$

two points $x_1, x_{-1} = \pm 1$ ↙ in $\ker(\varepsilon)$

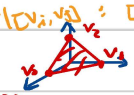
$n=0$: $\tilde{H}_0(S^0) = \langle [x_1 - x_{-1}] \rangle \cong \mathbb{Z}$, where x_i is the constant map $\Delta^0 \longrightarrow \{x_i\} \subseteq S^0$

$n=1$: Let $\sigma : \Delta^1 \longrightarrow D^1$ be a homeomorphism — generator — See Example 2.23, p.125

$$H_1(D^1, S^0) \ni \underset{\text{generator}}{[\sigma]} \xmapsto[\cong]{\partial} [\sigma(1) - \sigma(0)] = [x_1 - x_{-1}] \in \tilde{H}_0(S^0)$$
or $[x_{-1} - x_1]$

$$\downarrow \text{IIS}$$
$$H_1(D^1/S^0, S^0/S^0) \cong H_1(S^1) \ni [\omega], \quad \omega(x) = e^{\pi i \sigma(x)}$$

Note: $\sigma \in \ker(\partial : C_2(D^2, S^1) \to C_1(D^2, S^1))$
because $\sigma|_{[v_i, v_j]} : [v_i, v_j] \to S^1$



$n=2$: Let $\sigma : \Delta^2 \longrightarrow D^2$ be a homeomorphism

from $\sigma(v_1)$ to $\sigma(v_2)$ ↙ ↙ from $\sigma(v_0)$ to $\sigma(v_2)$

$$H_2(D^2, S^1) \ni [\sigma] \xmapsto[\cong]{\partial} \left[\sum (-1)^i \sigma|_{[\ldots \hat{v}_i \ldots]}\right] = \left[\sigma|_{[v_1, v_2]} - \sigma|_{[v_0, v_2]} + \sigma|_{[v_0, v_1]}\right] \in H_1(S^1)$$

Note: If we choose $\tilde{\tau} : \Delta^1 \longrightarrow S^1$: $\tilde{\tau}(t, 1-t) = \begin{cases} \sigma|_{[v_1, v_2]}(2t), & 0 \leq t \leq \frac{1}{2} \\ \sigma|_{[v_0, v_2]}(2-2t), & \frac{1}{2} \leq t \leq 1 \end{cases}$, then

Not in $\ker(\partial)$!: $\tilde{\tau}^{-1} + (\sigma|_{[v_1, v_2]} - \sigma|_{[v_0, v_2]}) \in \operatorname{im}(\partial : C_2(S^1) \to C_1(S^1))$

because $\boxed{\text{exer}}$ $\exists\ \omega : \Delta^2 \longrightarrow S^1$ s.t. $\omega|_{[v_1, v_2]} = \sigma|_{[v_1, v_2]}$, $\omega|_{[v_0, v_2]} = \sigma|_{[v_0, v_2]}$,
$\omega|_{[v_0, v_1]} = \tilde{\tau}^{-1}$ $\Rightarrow \sigma|_{[v_1, v_2]} - \sigma|_{[v_0, v_2]} + \tilde{\tau}^{-1} = \partial\omega \Rightarrow$ equation holds.

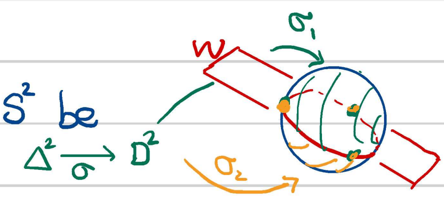
Similarly, if we choose $\tau : \Delta^1 \to v_0 \triangle v_1 \xrightarrow{\sigma} S^1$, then we have $\partial\tau = 0$ and
$$[\tau] = [\sigma|_{[v_1, v_2]} - \sigma|_{[v_0, v_2]} + \sigma|_{[v_0, v_1]}] \quad \text{in } H_1(S^1)$$
which is a generator by the case $n=1$.

Since $\partial : H_2(D^2, S^1) \longrightarrow H_1(S^1) : [\sigma] \mapsto [\tau]$ is an iso, we conclude that $[\sigma]$ is a generator of $H_2(D^2, S^1)$ and $\Delta^2 \xrightarrow[\cong]{\sigma} D^2 \longrightarrow D^2/\partial D^2 \cong S^2$ induces a homology class which generates $H_2(S^2)$

Another generator for $H_2(S^2)$:

Let W be a 2-dim vec. subsp in $\mathbb{R}^3$, and $\sigma_1, \sigma_2 : \Delta^2 \longrightarrow S^2$ be maps as in picture. Denote

$$\Delta_1^2 = \sigma_1(\Delta^2), \quad \Delta_2^2 = \sigma_2(\Delta^2)$$



($\sigma_1 = r_W \circ \sigma_2$, r_W = reflection along W)

Example 2.31

Note that we have

by long exact seq of (S^2, Δ_2^2)

$(\Delta_1^2, \partial\Delta_1^2) \cong (D^2, S^1)$

$[\sigma_1]$ is a generator here by a previous discussion

$$\begin{array}{ccccc} \tilde H_2(S^2) & \xrightarrow{\cong} & H_2(S^2, \Delta_2^2) & \xleftarrow{\cong} & H_2(\Delta_1^2, \partial\Delta_1^2) \\ \Psi & & \downarrow\wr\!\text{S} & & \downarrow\wr\!\text{S} \\ [\sigma_1 - \sigma_2] & & H_2(S^2/\Delta_2^2, \Delta^2/\Delta_2^2) & \xleftarrow{\cong} & H_2(\Delta_1^2/\partial\Delta_1^2, \partial\Delta_1^2/\partial\Delta_1^2) \end{array}$$

Prop 2.22: for good (X,A), $H(X,A) \cong H(X/A, A/A)$

Note: $\partial(\sigma_1 - \sigma_2) = 0$ because $\sigma_1|_{[v_i, v_j]} = \sigma_2|_{[v_i, v_j]}$

$[\;] = [\;]$

So $[\sigma_1 - \sigma_2]$ is a generator of $\tilde H_2(S^2)$

general n:

Inductively, one can show that

① a homeomorphism $\sigma : \Delta^n \longrightarrow D^n$ induces a generator $[\sigma]$ of $H_n(D^n, S^{n-1})$

② Cut S^n by a n-dim vec. subsp W in $\mathbb{R}^{n+1}$. Each piece $\cong D^n$.

We have

$\Delta^n \xrightarrow{\sigma} D^n$, σ_1, σ_2

$\sigma_1 = r_W \circ \sigma_2$

$\sigma_1|_{\partial\Delta} = \sigma_2|_{\partial\Delta}$ ($\Rightarrow \partial(\sigma_1 - \sigma_2) = 0$)

$[\sigma_1 - \sigma_2]$ is a generator of $H_n(S^n)$

Proof of Lemma

(i) A reflection $r_W : S^n \to S^n$ gives us a generator $[\sigma_1 - \sigma_2]$ of $H_n(S^n)$, and

$(r_W)_* : H_n(S^n) \longrightarrow H_n(S^n) : [\sigma_1 - \sigma_2] \mapsto [\sigma_2 - \sigma_1] = -[\sigma_1 - \sigma_2]$

So $\deg(r_W) = -1$

(ii) $a : S^n \to S^n : \vec v \mapsto -\vec v$ can be written as $a = r_1 \circ \cdots \circ r_{n+1}$, where

r_i = reflection along $\{(x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_i = 0\}$

So $\deg(a) = \deg(r_1) \cdot \deg(r_2) \cdots \deg(r_{n+1}) = (-1)^{n+1}$ #

Conclusion of Example 2.31:

The map

$$S^n \to \mathbb{R}P^n \to \mathbb{R}P^n/\mathbb{R}P^{n-1} \cong S^n$$

has degree $1 + (-1)^{n+1}$

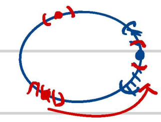
Example 2.32

Let $f : S^1 \overset{\subseteq \mathbb{C}}{\longrightarrow} S^1$, $f(z) = z^k$.

① $k=0$: $f = \text{constant} \Rightarrow \deg(f) = 0$ (Property (b), p134)

② $k>0$: for $y \in S^1$, $f^{-1}(y)$ has k points $x_1, \cdots, x_k$

e.g.



$k=3$

$y=1$

a map from ((•)) to ((•)) is homotopic to rotation which has degree $+1$ (exer)

$$\Rightarrow \deg(f) = \sum_{i=1}^{k} \deg f|_{x_i} = \sum_{i=1}^{k} +1 = k.$$

③ $k<0$: $f(z) = (z^{-1})^{(-k)>0}$

The map $z \mapsto z^{-1}$ is the reflection along $\operatorname{Im}(z)=0 \Rightarrow \deg = -1$

$$\Rightarrow \deg(f) = (-1)\cdot \underbrace{(-k)}_{\leftarrow \text{ by ②}} = k$$

Conclusion:

$$\deg(S^1 \to S^1 : z \mapsto z^k) = k.$$

Remark

If $f : S^n \to S^n$ is differentiable, then $\deg f$ is closely related to its differentials.

See Milnor's book "Topology from the differentiable viewpoint"

Homology of $\mathbb{RP}^n$ (Example 2.42):

Recall from Example 0.4 that

$$\mathbb{RP}^n = e^0 \cup_{\varphi_1} e^1 \cup_{\varphi_2} \cdots \cup_{\varphi_n} e^n,$$

$\varphi_k : \partial D^k = S^{k-1} \longrightarrow \mathbb{RP}^{k-1}$ "$= e^0 \cup \cdots \cup e^{k-1}$" is the quotient map

By Example 2.31, the degree of

$$\cong S^{k-1} \xrightarrow{\varphi_k} \mathbb{RP}^{k-1} \xrightarrow{\delta} \mathbb{RP}^{k-1}/\mathbb{RP}^{k-2} \cong S^{k-1}$$

is $1 + (-1)^k = \begin{cases} 2 & k \text{ even} \\ 0 & k \text{ odd} \end{cases}$

Recall:

$$\tilde{H}_{k-1}(S^{k-1}) \cong H_{k-1}(D^{k-1}, \partial D^{k-1})$$

$H_{k-2}(S^{k-2})$ ‖S

So the cellular complex of $\mathbb{RP}^n$ is

$$\begin{cases} 0 \to \overset{e^n}{\mathbb{Z}} \xrightarrow{2} \overset{e^{n-1}}{\mathbb{Z}} \xrightarrow{0} \cdots \xrightarrow{2} \overset{e^1}{\mathbb{Z}} \xrightarrow{0} \overset{e^0}{\mathbb{Z}} & \text{if } n \text{ is even} \\ 0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \cdots \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} & \text{if } n \text{ is odd} \end{cases}$$

$$\Rightarrow H_k(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & \text{for } k=0,\ k=n \text{ odd} \\ \mathbb{Z}_2 & \text{for } k \text{ odd},\ 0<k<n \\ 0 & \text{otherwise} \end{cases}$$

#

exer: Compute other examples in p.141 ~ 146

Euler characteristic

Recall that a finitely generated abelian group is isomorphic to

$\mathbb{Z}^r \oplus G$

for some number r, some finite abelian group G.

The number r is called the rank of this abelian group.

Def

Suppose a space X has finitely generated homology groups. Then the number

$$\chi(X) := \sum_n (-1)^n \operatorname{rank} H_n(X)$$

is called the Euler characteristic of X

Thm 2.44

Let X be a finite CW complex, and $c_n = \#$ of n-cells in a fixed CW str of X.

$\Rightarrow$

$$\chi(X) = \sum_n (-1)^n c_n \leftarrow \text{independent of the choice of CW str.}$$

pf

Let

$$0 \to C_k \xrightarrow{d_k} C_{k-1} \to \cdots \to C_1 \xrightarrow{d_1} C_0 \xrightarrow{\text{“}d_0\text{”}} 0$$

$\leftarrow$ Note: All the C_i's are finitely generated by assumption ($\mathbb{Z}^{c_i}$)

be the corresponding cellular complex

Since $H_n(X) = \ker(d_n)/\operatorname{im}(d_{n+1})$, we have

$$\operatorname{rank} H_n(X) = \operatorname{rank}\ker(d_n) - \operatorname{rank}\operatorname{im}(d_{n+1})$$

By isomorphism thm of groups, $C_n/\ker(d_n) = \operatorname{im}(d_n)$

$\Rightarrow$ $\operatorname{rank}\operatorname{im}(d_n) = \operatorname{rank} C_n - \operatorname{rank}\ker(d_n) = c_n - \operatorname{rank}\ker(d_n)$

So $\operatorname{rank} H_n(X) = \operatorname{rank}\ker(d_n) + \operatorname{rank}\ker(d_{n+1}) - c_{n+1}$

$$\Rightarrow \sum_n (-1)^n \operatorname{rank} H_n(X) = (\underbrace{\operatorname{rank}\ker(d_0)}_{=c_0} + \operatorname{rank}\cancel{\ker(d_1)} - c_1) - (\operatorname{rank}\cancel{\ker(d_1)} + \cancel{\operatorname{rank}}(d_2) - c_2) + \cdots$$

$$= \sum_n (-1)^n c_n \qquad \#$$

Example

“ $v-e+f$ ”

①  $c_0 = 4 =$ vertice, $c_1 = 5 =$ edges, $c_2 = 2 =$ faces, $\chi = 4-5+2 = \underline{1}$ $\leftarrow$ same $\rightarrow$  $c_0=4$, $c_1=4$, $c_2=1$, $\chi = 4-4+1 = \underline{1}$

② (See Example 2.36 & 2.37)

$M_g =$ orientable closed surface of genus g

$N_g =$ nonorientable " "

by Example 2.36 & 2.37 $\Rightarrow$ $\chi(M_g) = 2-2g$

$\chi(N_g) = 2-g$

Mayer-Vietoris seq: another seq useful in computation

Version I (p.149)

exer
Prove the analogous seq is exact for reduced homologies

Suppose $A, B \subseteq X$ are subspaces s.t. $X = \mathrm{int}(A) \cup \mathrm{int}(B)$. Then

$$\cdots \to H_n(A\cap B) \xrightarrow{\Phi} H_n(A) \oplus H_n(B) \xrightarrow{\Psi} H_n(X) \xrightarrow{\partial} H_{n-1}(A\cap B) \to \cdots \qquad \circledast$$

is exact, where $\Phi([x]) = ([x], -[x])$, $\Psi([x],[y]) = [x+y]$

pf

Let $C_n(A+B)$ be the subgp of $C_n(X)$ generated by $C_n(A) \cup C_n(B)$.

$C_n(A+B)$ is a subcomplex of $(C_n(X), \partial)$. By Prop 2.21 ($C^{\mathcal{U}}(X) \overset{\sim}{\hookrightarrow} C(X)$), the inclusion $(C_n(A+B), \partial) \hookrightarrow (C_n(X), \partial)$ induces iso of homologies.

The short exact seq

$$x \mapsto (x,-x) \qquad (x,y) \mapsto x+y$$

$$0 \longrightarrow C_n(A\cap B) \xrightarrow[\varphi]{} C_n(A) \oplus C_n(B) \xrightarrow[\psi]{} C_n(A+B) \longrightarrow 0$$

induces the Mayer-Vietoris seq. #

Version II (p149)

Suppose $X = A \cup B$ s.t. A and B are deformation retracts of nbds U and V with $U \cap V$ deformation retracting onto $A \cap B$. Then the seq $\circledast$ is exact.

Lemma (Five lemma, p.129)

In a commutative diagram of abelian gps as at the right, if the 2 rows are exact and $\alpha, \beta, \delta, \varepsilon$ are isos, then γ is also an iso

exact

$$\begin{array}{ccccccccc} A & \to & B & \to & C & \to & D & \to & E \\ {\scriptstyle\alpha}\downarrow\wr & & {\scriptstyle\beta}\downarrow\wr & & {\scriptstyle\gamma}\downarrow & & {\scriptstyle\delta}\downarrow\wr & & {\scriptstyle\varepsilon}\downarrow\wr \\ A' & \to & B' & \to & C' & \to & D' & \to & E' \end{array}$$

pf: diagram chasing. exer.

pf of Ver II:

Under the assumptions, the proof of Ver I implies

$$\begin{array}{ccccccccccc} \cdots \to & H_n(A\cap B) & \to & H_n(A)\oplus H_n(B) & \to & H_n(A+B) & \to & H_{n-1}(A\cap B) & \to & H_{n-1}(A)\oplus H_{n-1}(B) & \to \cdots \\ & \wr\downarrow & & \wr\downarrow & & \downarrow\wr \text{ (five-lemma)} & & \wr\downarrow & & \wr\downarrow & \\ \cdots \to & H_n(U\cap V) & \to & H_n(U)\oplus H_n(V) & \to & H_n(U+V) & \to & H_{n-1}(U\cap V) & \to & H_{n-1}(U)\oplus H_{n-1}(V) & \to \cdots \end{array}$$

$$\underbrace{H_n(U+V)}_{\cong\, H_n(X) \text{ by Prop 2.21}}$$

#

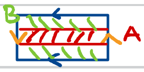
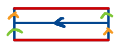
Example 2.46

Take $X = S^m \subseteq \mathbb{R}^{m+1}$, $A = \{x_{m+1} \geq 0\}$, $B = \{x_{m+1} \leq 0\}$ $\Rightarrow X = A \cup B$.

$\Rightarrow \cdots \to \underset{0}{\underset{\shortparallel}{\tilde{H}_n(A) \oplus \tilde{H}_n(B)}} \to \underset{\tilde{H}_n(S^m)}{\underset{\shortparallel}{\tilde{H}_n(X)}} \xrightarrow{\cong} \underset{\tilde{H}_{m-1}(S^{m-1})}{\underset{\shortparallel}{\tilde{H}_{n-1}(A \cap B)}} \to \underset{0}{\underset{\shortparallel}{\tilde{H}_{n-1}(A) \oplus \tilde{H}_{n-1}(B)}} \to \cdots$ is exact

$\Rightarrow \tilde{H}_n(S^m) \cong \tilde{H}_{n-1}(S^{m-1})$ #

Example 2.47

Let $K =$ Klein bottle $=$  $A =$  $\cong$ Möbius band $\cong$  $= B$

$A \cap B = S^1$

$\cdots \to \overset{0}{\overset{\shortparallel}{\tilde{H}_2(A) \oplus \tilde{H}_2(B)}} \to \tilde{H}_2(K) \to \overset{\mathbb{Z}}{\overset{\text{IIS}}{\tilde{H}_1(A \cap B)}} \xrightarrow{\Phi} \overset{\mathbb{Z} \oplus \mathbb{Z}}{\overset{\shortparallel}{\tilde{H}_1(A) \oplus \tilde{H}_1(B)}} \to \tilde{H}_1(K) \to \tilde{H}_0(A \cap B) = 0$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad 1 \longmapsto (2, -2)$

$\Rightarrow H_2(K) = \ker(\Phi) = 0$, $H_1(K) \cong \mathbb{Z} \oplus \mathbb{Z} / \operatorname{im}(\Phi) \cong \mathbb{Z} \oplus \mathbb{Z}_2$ #

Homology with coefficients

Let G be an abelian gp (or a ring).

$$C_n(X;G) := \{ \textstyle\sum n_i \sigma_i \mid n_i \in G,\ \sigma_i : \Delta^n \to X \},\quad \partial(\textstyle\sum n_i \sigma_i) = \sum_{i,j} (-1)^j n_i\, \sigma_i|_{[v_0, \cdots, \hat{v}_j, \cdots, v_n]}$$

The resulting homology groups $H_n(X;G)$ are called homology groups with coefficients in G

By the same constructions and arguments, we have

- relative homology with coeff. G: $H_n(X, A; G)$
- reduced " " : $\tilde{H}_n(X;G)$
- long exact seq. of relative homologies
- excision thm
- Mayer-Vietoris seq.
- cellular homology: $H_n^{CW}(X;G) =$ homology of $\cdots \to H_n(X^n, X^{n-1}; G) \xrightarrow{d_n} H_{n-1}(X^{n-1}, X^{n-2}; G) \to \cdots$

exer (Lemma 2.49)

Prove that

① $H_n(X;G) \cong H_n^{CW}(X;G)$

② $d_n(\sum_\alpha n_\alpha e_\alpha^n) = \sum_{\alpha,\beta} d_{\alpha\beta}\, n_\alpha\, e_\beta^{n-1}$