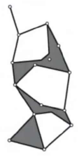
. . .

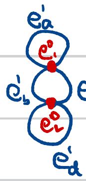
- $\Delta_{\alpha\beta} : \partial D^n_\alpha \longrightarrow S^{n-1}_\beta = \partial D^n_\alpha \xrightarrow{\varphi_\alpha} X^{n-1} \xrightarrow{q} X^{n-1}/X^{n-2} \xrightarrow{q_\beta} S^{n-1}_\beta$

By the commutativity of the diagram, we have
$$d_n(e^n_\alpha) = \Sigma_\beta d_{\alpha\beta} e^{n-1}_\beta \qquad \#$$

Example (See the board of week 5)



homotopic to



$X_0 = \,:$

$X_1 = X$

cellular cx:

$e^1_a \longmapsto e^0_1 - e^0_1 = 0$ ← endpoint - startpoint

$e^1_d \longmapsto e^0_2 - e^0_2 = 0$

$0 \to \mathbb{Z}^4 \xrightarrow{d_1} \mathbb{Z}^2 \to 0$

$e^1_b \longmapsto e^0_1 - e^0_2$

$e^1_c \longmapsto e^0_1 - e^0_2$ ← orientation NOT important here because will generate a subgp

$H_0(X) = \mathbb{Z}^2/\langle(1,-1)\rangle \cong \mathbb{Z}$

$\Rightarrow H_1(X) = \ker(d_1)$

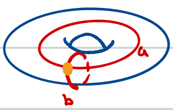
$= \langle (1,0,0,0), (0,0,0,1), (0,1,-1,0) \rangle$

$\cong \mathbb{Z}^3$

other $H_k = 0$ #

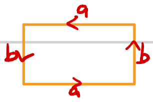
Example 2.36 (orientable closed surface)

$\boxed{\text{Remark: will explain later: } \deg(S^1 \to S^1 : z \mapsto z^n) = n}$



$T = M_1$
$= e^0 \cup e^1_a \cup e^1_b \cup e^2$

$0 \to \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^2 \xrightarrow{d_1} \mathbb{Z} \to 0$, $d_1 = 0$



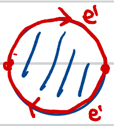
$\partial D^2 = S^1 \to X^1 =$  $\longrightarrow X^1/X^1 \setminus e_a \cong$  has deg $1-1=0$

Similarly, $\deg(S^1 \to X^1/X^1 \setminus e_b \cong S^1_b) = 0 \quad \Rightarrow d_2 = 0$

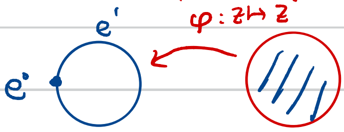
$$\Rightarrow H_k(T) = \begin{cases} \mathbb{Z}, & k=0,2 \\ \mathbb{Z}^2, & k=1 \\ 0, & \text{otherwise} \end{cases}$$

exer: Compute homology of orientable surface of genus g (see picture on p.5)

Example 2.37 (nonorientable closed surface)

① $N_1 \cong \mathbb{R}P^2 \cong$  (Also see wiki: Boy's surface) $\cong e^0 \cup e^1 \cup_\varphi e^2$

See Example 0.4



" $\varphi : z \mapsto z^2$ "

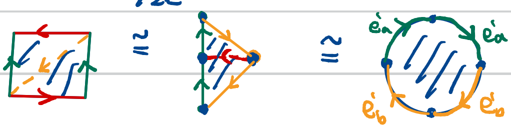
$\deg(\varphi) = 2$

Cellular complex of $\mathbb{R}P^2$:

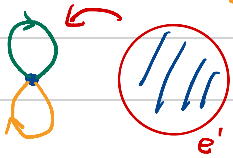
$$0 \to \underset{e^2}{\mathbb{Z}} \xrightarrow{\times 2} \underset{e^1}{\mathbb{Z}} \xrightarrow{0} \underset{e^0}{\mathbb{Z}} \to 0$$

$\Rightarrow H_0(\mathbb{R}P^2) \cong \mathbb{Z}$, $H_1(\mathbb{R}P^2) \cong \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2$, $H_2(\mathbb{R}P^2) = 0$

" $\varphi : e^1 \mapsto e^1_a e^1_a e^1_b e^1_b$ "

② $N_2 \cong$ Klein bottle $K \cong$  $\cong e^0 \cup e^1_a \cup e^1_b \cup_\varphi e^2$

See wiki: Klein bottle



Cellular complex of K:

$$0 \to \underset{e^2}{\mathbb{Z}} \xrightarrow{1 \mapsto (2,2)} \underset{e^1_a \quad e^1_b}{\mathbb{Z} \oplus \mathbb{Z}} \xrightarrow{0} \underset{e^0}{\mathbb{Z}}$$

$\Rightarrow H_0(K) \cong \mathbb{Z}$, $H_1(K) \cong \mathbb{Z}^2/\langle(2,2)\rangle \cong \mathbb{Z} \oplus \mathbb{Z}_2$, $H_2(K) = 0$

$(a, b) \longmapsto (a-b, b)$

$(x+y, y) \longleftrightarrow (x, y)$

exer:

$H_k(N_g) = ?$

N_g = closed nonorientable surface of genus g

See Example 2.37

Further reading:

Classification of closed surfaces < orientable, genus g, $g \geq 0$; nonorientable, genus g, $g \geq 1$

Google: classification of surfaces

Summary of computation method:

Step1 Find a CW structure (key1) of the target space X.

Step2 Compute the boundary map of cellular complex by the degrees of attaching maps (key2)

Step3 Compute the cellular homology groups

Next goal: Compute $H_k(\mathbb{R}P^n)$

Computation of degree

Let $f: S^n \to S^n$, $n>0$. Suppose $\exists y \in S^n$ s.t. $f^{-1}(y) = \{x_1, \cdots, x_m\}$ is a finite set. ($\leftarrow$ no other point maps to y via f)

Let $U_1, \cdots, U_m$ be disjoint nbds of $x_1, \cdots, x_m$, and V be a nbd of y s.t. $f(U_i) \subseteq V$ $\forall i = 1, \dots, m$.

$\Rightarrow$ $f(U_i - \{x_i\}) \subseteq V - \{y\}$ and

$$\begin{tikzcd}
 & H_n(U_i, U_i - \{x_i\}) \arrow[r, "f_*"] \arrow[dl, "\cong"'] \arrow[d, "k_i"] & H_n(V, V-\{y\}) \arrow[d, "\cong"] \\
H_n(S^n, S^n - \{x_i\}) & H_n(S^n, S^n - f^{-1}(y)) \arrow[l, "P_i"'] \arrow[r, "f_*"] & H_n(S^n, S^n - \{y\}) \\
 & H_n(S^n) \arrow[ul, "\cong"] \arrow[u, "j"] \arrow[r, "f_*"] & H_n(S^n) \arrow[u, "\cong"']
\end{tikzcd}$$

(*)

$\tilde{H}_n(S^n - f(x_i))$ by excision $\to \cong$; by excision ($H_n(V, V-\{y\}) \cong H_n(S^n, S^n-\{y\})$)

by long exact seq of $(S^n, S^n - x_i)$; $\tilde{H}_n(S^n - \{x_i\}) = 0$

by long exact seq of $(S^n, S^n - y)$

Remark

Commutativity of (*)

$$\begin{tikzcd}
H_n(U_i, U_i - x_i) \arrow[r, "f_*"] \arrow[d, "\cong"] & H_n(V, V-y) \arrow[d, "\cong"] \\
H_n(S^n, S^n - x_i) \arrow[d, "\cong"] & \\
H_n(S^n) \arrow[r, "f_*"] & H_n(S^n) \arrow[u, "\cong"']
\end{tikzcd}$$

commutativity of f_*

Not commute in force

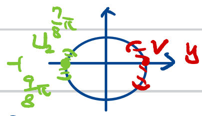
Thus, $H_n(U_i, U_i - x_i) \cong H_n(S^n) \cong \mathbb{Z} \cong H_n(V, V-y)$ and $\exists!$ $d \in \mathbb{Z}$ s.t.

$H_n(U_i, U_i - x_i) \xrightarrow{f_*} H_n(V, V-y)$: $f_*(x) = dx$ under these iso

This number d is called the **local degree** of f at x_i, written $\deg f|_{x_i}$.

Example

$f: S^1 \to S^1 : z \mapsto z^2$ ($S^1 \subseteq \mathbb{C}$), $y = 1$ $\Rightarrow$ $f^{-1}(y) = \{\pm 1\}$ $x_1 = 1, x_2 = -1$



Let $V = \{e^{2\pi i\theta} \mid \frac{-\pi}{4} < \theta < \frac{\pi}{4}\}$, $U_1 = \{e^{2\pi i\theta} \mid \frac{-\pi}{8} < \theta < \frac{\pi}{8}\}$ ($+1$), $U_2 = \{\frac{7}{8}\pi < \theta < \frac{9}{8}\pi\}$ (-1)

Since $f|_{U_i}: U_i \to V$ is a homeo, the mor. $f_*: H_1(U_i, U_i - x_i) \to H_1(V, V-y)$ is iso $\forall i$

$\Rightarrow$ $\deg f|_{x_i} = \pm 1$ $\forall i$,   Q: $\deg f|_{x_i} = +1$? -1 ?

$\tilde{\omega}: \Delta^1 \to U_2$

$\tilde{\omega}(t, 1-t) = e^{2\pi i(\frac{1}{2} + \frac{1}{8}(t - \frac{1}{2}))}$

$$\begin{tikzcd}
{[\tilde{\omega}]}\ H_1(U_2, U_2 - \{-1\}) \arrow[r, "f_*"] \arrow[d, "\cong"] & H_1(V, V - \{1\}) \arrow[d, "\cong"] \\
{[\omega]} \in H_1(S^1, S^1 - \{-1\}) & H_1(S^1, S^1 - \{1\}) \\
\langle \omega \rangle = H_1(S^1) \arrow[u, "\cong"] & H_1(S^1) \arrow[u, "\cong"] \\
\mathbb{Z} \arrow[u, phantom, "\cong"] & \mathbb{Z} \arrow[u, phantom, "\cong"]
\end{tikzcd}$$

$(f_*\tilde{\omega})(t, 1-t) = e^{2\pi i(\frac{1}{4}(t - \frac{1}{2}))}$ $\mapsto$ $[\omega]$ $\leftarrow$ $[\omega]$

$\Rightarrow$ $\deg f|_{x_2} = +1$

Similarly $\deg f|_{x_1} = +1$.

#

$\{(t, 1-t) \in \mathbb{R}^2\}$ $=$ $\omega: \Delta^1 \to S^1$: $\omega(t, 1-t) = e^{2\pi i t}$

$\omega = \phi \circ \tilde{\omega}$

$\phi: e^{2\pi i\theta} \mapsto e^{2\pi i(8(\theta - \frac{1}{2}) + \frac{1}{2})}$

id on homology

$\phi \simeq$ shift relative $\{-1\}$

Prop 2.30

Let $f: S^n \to S^n$, $n>0$. Suppose $\exists y \in S^n$ s.t. $f^{-1}(y) = \{x_1, \cdots, x_m\}$ is a finite set.

Then $\deg f = \sum_{i=1}^{m} \deg f|_{x_i}$

pf.

① By excision,

$$H_n(S^n, S^n - f^{-1}(y)) \cong H_n\left(\coprod_{i=1}^{m} U_i, \coprod_{i=1}^{m} (U_i - \{x_i\})\right) \cong \bigoplus_{i=1}^{m} \underbrace{H_n(U_i, U_i - \{x_i\})}_{\cong \mathbb{Z}} \cong \mathbb{Z}^m$$

② Recall from diagram ⊛ :

$$\begin{tikzcd} & H_n(U_i, U_i - x_i) \arrow[dl, "\cong"'] \arrow[d, "k_i"] \\ \mathbb{Z} \cong H_n(S^n) \cong H_n(S^n, S^n - x_i) & H_n(S^n, S^n - f^{-1}(y)) \arrow[l, "p_i"'] \end{tikzcd}$$

$H_n(U_i, U_i - x_i) \cong H_n(S^n)$

$i \neq j$ and

$$\begin{tikzcd} \underbrace{H_n(U_i, U_i)}_{0} \arrow[d] \arrow[r, phantom, ""] & H_n(U_i, U_i - x_i) \arrow[l] \arrow[d, "k_i"] \\ H_n(S^n, S^n - x_j) & H_n(S^n, S^n - f^{-1}(y)) \arrow[l, "p_j"'] \end{tikzcd}$$

$p_j k_i = 0$

$x_j \notin U_i$, $U_i - x_j \subset S^n - x_j$

So k_i and p_j can be considered as the inclusions and projections of direct sum respectively.

③ Since

$$\begin{tikzcd} \underset{\mathbb{Z}}{\overset{\cong}{H_n(S^n, S^n - x_i)}} & H_n(S^n, S^n - f^{-1}(y)) \arrow[l, "p_i"'] \\ & H_n(S^n) \cong \mathbb{Z} \ni 1 \arrow[ul, "\cong"] \arrow[u, "j"] \end{tikzcd}$$

commutes, we have $p_i j(1) = 1 \quad \forall i$

$\Rightarrow j(1) = (1, 1, \cdots, 1) = \sum_i k_i(1) \in H_n(S^n, S^n - f^{-1}(y))$

④ Therefore,

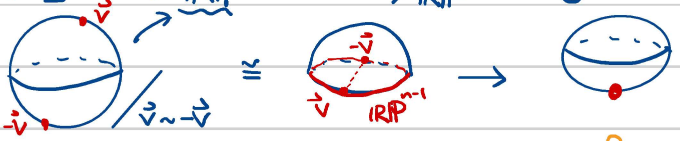
$$f_*(\underbrace{j(1)}_{\uparrow j}) = \sum_i f_*(k_i(1)) = \sum_i f_*(1) = \sum_i (\deg f|_{x_i}) \cdot 1 \in H_n(S^n, S^n - y)$$

$H_n(V, V-y) \cong H_n(S^n, S^n - y)$

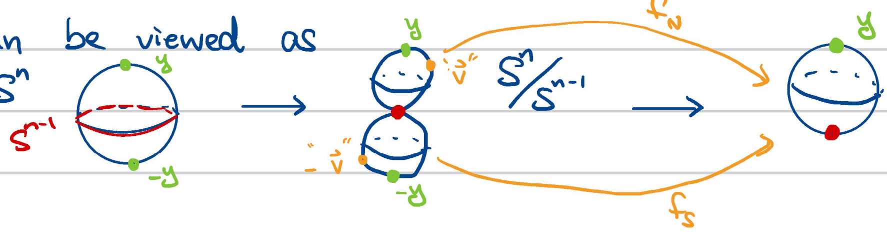
$\Rightarrow \quad H_n(S^n) \ni 1 \xrightarrow{f_*} f_*(1) = \sum_i (\deg f|_{x_i}) \cdot 1 \in \longrightarrow H_n(S^n) \cong H_n(S^n, S^n - y)$ #

Example 2.31

Let $f: S^n \to \underset{\sim}{\mathbb{R}P^n} \to \mathbb{R}P^n / \mathbb{R}P^{n-1} \cong S^n$



f can be viewed as



$\deg f = \deg f|_y + \deg f|_{-y} = \deg f_N + \deg f_S$

$f_N \simeq id_{S^n}$, $\quad f_N \simeq a$, $\quad a: S^n \to S^n : \vec{v} \to -\vec{v}$

Q: $\deg(a) = ?$

Example 2.31

Lemma

(i) (Property (e), p134) A reflection of S^n (wrt a n-dim $\overset{\text{vec.}}{\text{subsp}}$ in $\mathbb{R}^{n+1}$) has degree -1

(ii) (Property (f), p134) $\deg(a) = (-1)^{n+1}$

Generator of $H_n(S^n)$ (Example 2.23, p125):

Recall from the good pair $(D^n, S^{n-1} = \partial D^n)$, we have

$$\cdots \longrightarrow \tilde{H}_n(D^n) = 0 \longrightarrow \underline{H_n(D^n, S^{n-1})} \xrightarrow{\cong} \tilde{H}_{n-1}(S^{n-1}) \longrightarrow \tilde{H}_{n-1}(D^n) = 0 \to \cdots$$

$$H_n(D^n/S^{n-1}, S^{n-1}/S^{n-1}) \cong \tilde{H}_n(S^n) \qquad n \geq 1$$

two points x_1, x_{-1} $= \pm 1$, in $\ker(\varepsilon)$

$n=0$: $\tilde{H}_0(S^0) = \langle [x_1 - x_{-1}] \rangle \cong \mathbb{Z}$, where x_i is the constant map $\Delta^0 \longrightarrow \{x_i\} \subseteq S^0$

$n=1$: Let $\sigma: \Delta^1 \longrightarrow D^1$ be a homeomorphism   See Example 2.23, p.125

generator   generator

$$H_1(D^1, S^0) \ni [\sigma] \underset{\cong}{\overset{\partial}{\longmapsto}} [\sigma(1) - \sigma(0)] = [x_1 - x_{-1}] \in \tilde{H}_0(S^0)$$

or $[x_{-1} - x_1]$

$$H_1(D^1/S^0, S^0/S^0) \cong H_1(S^1) \ni [\omega], \quad \omega(x) = e^{\pi i \sigma(x)}$$

Note: $\sigma \in \ker(\partial: C_2(D^2, S^1) \to C_1(D^2, S^1))$ because $\sigma|_{[v_i, v_j]}: [v_i, v_j] \to S^1$

$n=2$: Let $\sigma: \Delta^2 \longrightarrow D^2$ be a homeomorphism

from $\sigma(v_1)$ to $\sigma(v_2)$ from $\sigma(v_0)$ to $\sigma(v_2)$

$$H_2(D^2, S^1) \ni [\sigma] \underset{\cong}{\overset{\partial}{\longmapsto}} [\Sigma (-1)^i \sigma|_{[\dots \hat{v}_i \dots]}] = [\sigma|_{[v_1, v_2]} - \sigma|_{[v_0, v_2]} + \sigma|_{[v_0, v_1]}] \in H_1(S^1)$$

Note: If we choose $\tilde{\tau}: \Delta^1 \longrightarrow S^1$: $\tilde{\tau}(t, 1-t) = \begin{cases} \sigma|_{[v_1, v_2]}(2t), & 0 \leq t \leq \frac{1}{2} \\ \sigma|_{[v_0, v_2]}(2-2t), & \frac{1}{2} \leq t \leq 1 \end{cases}$, then

Not in $\ker(\partial)$!: $\tilde{\tau}^{-1} + (\sigma|_{[v_1, v_2]} - \sigma|_{[v_0, v_2]}) \in \operatorname{im}(\partial: C_2(S^1) \to C_1(S^1))$

because $\boxed{\text{exer}}$ $\exists\ \omega: \Delta^2 \longrightarrow S^1$ s.t. $\omega|_{[v_1, v_2]} = \sigma|_{[v_1, v_2]}$, $\omega|_{[v_0, v_2]} = \sigma|_{[v_0, v_2]}$,

$\omega|_{[v_0, v_1]} = \tilde{\tau}^{-1}$ $\Rightarrow$ $\sigma|_{[v_1, v_2]} - \sigma|_{[v_0, v_2]} + \tilde{\tau}^{-1} = \partial\omega \Rightarrow$ equation holds.

Similarly, if we choose $\tau: \Delta^1 \to v_0 \to v_1 \xrightarrow{\sigma} S^1$, then we have $\partial\tau = 0$ and

$$[\tau] = [\sigma|_{[v_1, v_2]} - \sigma|_{[v_0, v_2]} + \sigma|_{[v_0, v_1]}] \quad \text{in } H_1(S^1)$$

which is a generator by the case $\underline{n=1}$.

Since $\partial: H_2(D^2, S^1) \longrightarrow H_1(S^1): [\sigma] \mapsto [\tau]$ is an iso, we conclude that $[\sigma]$ is a generator of $H_2(D^2, S^1)$ and $\Delta^2 \underset{\cong}{\overset{\sigma}{\longrightarrow}} D^2 \longrightarrow D^2/\partial D^2 \cong S^2$ induces a homology class which generates $H_2(S^2)$

Another generator for $H_2(S^2)$:

Let W be a 2-dim vec. subsp in $\mathbb{R}^3$, and $\sigma_1, \sigma_2: \Delta^2 \longrightarrow S^2$ be maps as in picture. Denote

$$\Delta_1^2 = \sigma_1(\Delta^2), \quad \Delta_2^2 = \sigma_2(\Delta^2)$$

($\sigma_1 = r_W \circ \sigma_2$, r_W = reflection along W)