11/11

Def

Let X be a CW complex

By Lemma 2.34, $H_n(X^n, X^{n-1}) \cong \mathbb{Z}^{r_n}$, $r_n = \#$ of n-cells

Define

$$d_n := j_{n-1}\circ\partial : \underset{\substack{\| \\ \mathbb{Z}^{r_n}}}{H_n(X^n,X^{n-1})} \xrightarrow{\partial} H_{n-1}(X^{n-1}) \xrightarrow{j_{n-1}} \underset{\substack{\| \\ \mathbb{Z}^{r_{n-1}}}}{H_{n-1}(X^{n-1},X^{n-2})}$$

$$\cdots \to H_n(X^{n-1}) \to H_n(X^n) \to H_n(X^n,X^{n-1}) \to H_{n-1}(X^{n-1}) \to H_{n-1}(X^n) \to \cdots$$

$$\cdots \to H_n(X^{n-1},X^{n-2}) \to H_{n-1}(X^{n-2}) \to H_{n-1}(X^{n-1}) \to H_{n-1}(X^{n-1},X^{n-2}) \to H_{n-2}(X^{n-2}) \to \cdots$$

Lemma

$$d_n\circ d_{n+1} = 0$$

pf

$$\begin{tikzcd}
H_{n+1}(X^{n+1},X^n) \arrow[r, "\partial"] \arrow[rr, bend left, "d_{n+1}"] & H_n(X^n) \arrow[r, "j_n"] & H_n(X^n,X^{n-1}) \arrow[d, "\partial"] \arrow[dr, bend left, "d_n"] & \\
& & H_{n-1}(X^{n-1}) \arrow[r, "j_{n-1}"] & H_{n-1}(X^{n-1},X^{n-2})
\end{tikzcd} = 0$$

$$\cdots \to H_n(X^{n-1}) \to H_n(X^n) \to \underset{\cong \mathbb{Z}^{r_n}}{H_n(X^n,X^{n-1})} \to H_{n-1}(X^{n-1}) \to \cdots \qquad r_n = \#\text{ of } n\text{-cells}$$

$\#$

The complex $(H_n(X^n,X^{n-1}), d_n)$ is called the cellular chain complex, and the the corresponding homology

$$H_n^{CW}(X)\ \big(= H_n(X)\big) = \ker(d_n) / \mathrm{im}(d_{n+1})$$

is called the cellular homology of X

Thm 2.35

$$H_n^{CW}(X) \cong H_n(X)$$

pf

$$\begin{tikzcd}
 & 0 = H_n(X^{n-1}) \arrow[r] & H_n(X^n) \arrow[r, "\text{onto}", "i"'] \arrow[dr, "j", "1\text{-}1"'] & \overset{H_n(X)}{\overset{\shortparallel}{\boxed{H_n(X^{n+1})}}} \arrow[r] & H_n(X^{n+1}, X^n) = 0 & & \\
\cdots \arrow[r] & H_{n+1}(X^{n+1}, X^n) \arrow[ur, "\partial"] \arrow[rr, "d_{n+1}"] & & H_n(X^n, X^{n-1}) \arrow[rr, "d_n"] \arrow[dr, "\partial"] & & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r] & \cdots \\
0 = H_{n+1}(X^n) \arrow[r] & H_{n+1}(X^{n+1}) \arrow[u, "1\text{-}1"] & & & H_{n-1}(X^{n-1}) \arrow[ur, "j_{n-1}", "1\text{-}1"'] \arrow[r, "\text{onto}"] & H_{n-1}(X^n) \arrow[d] & \\
 & & & & & H_{n-1}(X^n, X^{n-1}) = 0 &
\end{tikzcd}$$

Since $H_n(X^n) \xrightarrow{i} H_n(X^{n+1})$ is onto, we have

$$H_n(X) \cong H_n(X^n) / \ker i$$

$$= H_n(X^n) / \partial\big(H_{n+1}(X^{n+1}, X^n)\big)$$

$\because j$ 1-1 $\cong j\big(H_n(X^n)\big) / \overset{d_{n+1}}{\overset{\shortparallel}{\boxed{j\partial}}}\big(H_{n+1}(X^{n+1}, X^n)\big)$

$$= \ker \partial / \mathrm{im}(d_{n+1})$$

Δ 1-1

$$\overset{\Delta \text{ 1-1}}{\cong} \ker(j_{n-1} \circ \partial) \big/ \operatorname{im} d_{n+1}$$

$$= \ker d_n \big/ \operatorname{im} d_{n+1} = H_n^{CW}(X) \quad ✱$$

<u>Example</u>

Recall from <u>Example 0.6</u>:

$$\mathbb{CP}^n \cong e^0 \cup e^2 \cup \cdots \cup e^{2n}$$

$\Rightarrow$ cellular complex:

$$0 \to \underset{\substack{\cong \\ H_{2n}(X^{2n}, X^{2n-1})}}{\overset{2n}{\mathbb{Z}}} \to 0 \to \mathbb{Z} \to 0 \to \cdots \to \overset{2}{\mathbb{Z}} \to \overset{1}{0} \to \overset{0}{\mathbb{Z}} \qquad , X = \mathbb{CP}^n$$

$$\Rightarrow H_k(\mathbb{CP}^n) \cong \begin{cases} \mathbb{Z} & , k = 0, 2, 4, \cdots, 2n \\ 0 & , \text{otherwise.} \end{cases} \qquad \#$$

<u>Remark</u>

Let X be a CW cx. Then

① $H_n(X) = 0$ if X has NO n-cells.

② $H_n(X)$ can be generated by k elements if X has k n-cells

③ $H_n(X) \cong \mathbb{Z}^k$ if X has k n-cells, NO $(n-1)$-cells, NO $(n+1)$-cells.

<u>Formula of d_n: $H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-1}, X^{n-2})$</u>

Def (degree)

Let $f: S^n \to S^n$, $n>0$.

Since $H_n(S^n) \cong \mathbb{Z}$, the induced map f_* is of the for

$$f_*: \underset{\displaystyle \mathbb{Z}}{\underset{\cong}{H_n(S^n)}} \longrightarrow \underset{\displaystyle \mathbb{Z}}{\underset{\cong}{H_n(S^n)}} : f_*(\alpha) = d\,\alpha, \quad \underset{f_*(1)}{\underset{\shortparallel}{d}} \in \mathbb{Z}$$

The number d is called the degree of f

Properties of degree (p. 134):

Let $f, g: S^n \to S^n$. Then

(a) $\deg(\mathrm{id}_{S^n}) = 1$

(b) If $f: S^n \to S^n$ is NOT onto, then

$\exists\, x_0 \in S^n - f(S^n)$

$\Rightarrow \quad f = \quad S^n \xrightarrow{f} S^n - \{x_0\} \hookrightarrow S^n$

$\Rightarrow \quad f_* = \quad H_n(S^n) \longrightarrow \underset{\displaystyle 0}{\underset{\shortparallel}{H_n(S^n - \{x_0\})}} \longrightarrow H_n(S^n)$

$\Rightarrow \quad f_* = 0 \quad \Rightarrow \quad \boxed{\deg(f) = 0}$

(c) $f \simeq g \iff \deg(f) = \deg(g)$

"$\Rightarrow$" is from "homotopy invariance"

"$\Leftarrow$" : Cor 4.25

(d) $\deg(f \circ g) = \deg(f)\deg(g)$

exer: derive other properties on p. 134

Cellular Boundary Formula

The boundary map $d_n : \underbrace{H_n(X^n, X^{n-1})}_{\substack{\cong \\ \text{free abelian gp} \\ \text{generated by } n\text{-cells} \\ e^n_\alpha \ (\cong \mathbb{Z}^{r_n})}} \to \underbrace{H_{n-1}(X^{n-1}, X^{n-2})}_{\substack{\cong \\ \text{free ab gp} \\ \text{gen. by } (n-1)\text{-cells} \\ e^{n-1}_\beta \ (\cong \mathbb{Z}^{r_{n-1}})}}$

satisfies

$$d_n(e^n_\alpha) = \sum_\beta d_{\alpha\beta}\, e^{n-1}_\beta \qquad n > 1$$

where $d_{\alpha\beta}$ is the degree of the map

$$\partial D^n_\alpha \cong S^{n-1}_\alpha \xrightarrow[\text{map}]{\text{attaching}} X^{n-1} \longrightarrow X^{n-1} / X^{n-1} \setminus e^{n-1}_\beta \cong S^{n-1}_\beta$$

$\underline{n=1}$:

$$d_1 : H_1(X^1, X^0) \longrightarrow H_0(X^0)$$
$$[\gamma] \longmapsto [\gamma(1) - \gamma(0)]$$

$\gamma : [0,1] \to X^1$

$\gamma(0), \gamma(1) \in X^0$

If X_0 = a point, then $d_1 = 0$

$d_{\alpha\beta} \cdot e^{n-1}_\beta$

" e^{n-1}_β "

pf " e^n_α "

$$\begin{tikzcd}
H_n(D^n_\alpha, \partial D^n_\alpha) \arrow[r, "\partial", "\cong"'] \arrow[d, "\Phi_{\alpha *}"] & \tilde H_{n-1}(\partial D^n_\alpha) \arrow[r, "\Delta_{\alpha\beta *}"] \arrow[d, "\varphi_{\alpha *}"] & \tilde H_{n-1}(S^{n-1}_\beta) \\
H_n(X^n, X^{n-1}) \arrow[r, "\partial"] \arrow[dr, "d_n"'] & \tilde H_{n-1}(X^{n-1}) \arrow[r, "q_*"] \arrow[d, "j"] & \tilde H_{n-1}(X^{n-1}/X^{n-2}) \arrow[u, "q_{\beta *}"'] \arrow[d, "\cong"] \\
 & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r, "\cong", "\text{Prop 2.22}"'] & H_{n-1}(X^{n-1}/X^{n-2}, X^{n-2}/X^{n-2})
\end{tikzcd}$$

$\vee_\alpha S^{n-1}_\beta$

" e^{n-1}_β " $\oplus$

e^n_α

e^{n-1}_β

$d_n(e^n_\alpha)$

where:

- Φ_α is the characteristic map of e^n_α
- φ_α " attaching " e^n_α

φ_α " attaching " e_α

- $q : X^{n-1} \longrightarrow X^{n-1}/X^{n-2}$ is the quotient map
- $q_\beta :$

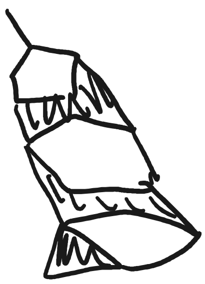
$$\begin{tikzcd} X^{n-1}/X^{n-2} \arrow[r] \arrow[rr, bend left] & X^{n-1}/X^{n-1}-e_\beta^{n-1} & S_\beta^{n-1} = D_\beta^{n-1}/\partial D_\beta^{n-1} \arrow[l, "\cong"'] \\ & X^{n-1} \arrow[u] & D_\beta^{n-1} \arrow[l, "\Phi"] \arrow[u] \end{tikzcd}$$

- $\Delta_{\alpha\beta} : \partial D_\alpha^n \xrightarrow{\varphi_\alpha} X^{n-1} \xrightarrow{q} X^{n-1}/X^{n-2} \xrightarrow{q_\beta} S_\beta^{n-1}$

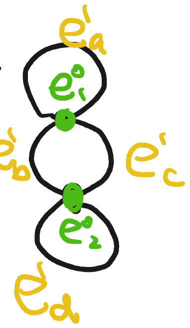
By the commutivity of the diagram, we have

$$d_n(e_\alpha^n) = \sum_\beta d_{\alpha\beta} e_\beta^{n-1} \qquad \#$$

<u>Example</u> (See the board of week 5)



$X =$



$\simeq X_1$ $\qquad X_0 = :$

cellular cx :

$e_a^1 \longmapsto e_1^0 - e_1^0 = 0$

$e_d^1 \longmapsto e_2^0 - e_2^0 = 0$

$$0 \longrightarrow \mathbb{Z}^4 \xrightarrow{d_1} \mathbb{Z}^2 \longrightarrow 0$$

$e_b^1 \longmapsto \pm(e_1^0 - e_2^0)$

$e_c^1 \longmapsto \pm(e_1^0 - e_2^0)$

$$\Rightarrow H_0(X) \cong \mathbb{Z}^2/\langle(1,-1)\rangle \cong \mathbb{Z}$$

$$H_1(X) \cong \ker(d_1) = \left\langle \begin{matrix} (1,0,0,0),\ (0,0,0,1), \\ (0,1,\pm1,0) \end{matrix} \right\rangle \cong \mathbb{Z}^3$$

$H_k(X) = 0$ for other k $\qquad \#$

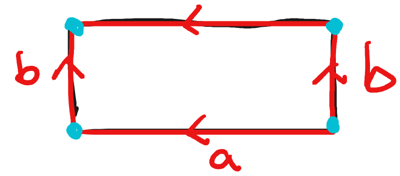
<u>Example 2.36</u> (orientable closed $\overset{M_g}{\text{surface}}$)

$T = M_1$
$= e^0 \cup e_a^1 \cup e_b^1$
$\cup\ e^2$

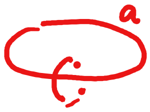
a

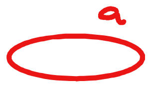
$\boxed{\begin{array}{l}\text{Remark (will explain later)} \\ \deg(S^1 \to S^1 : z \mapsto z^m) = m\end{array}}$

cellular complex :

$=$ 

$$0 \to \overset{e}{\mathbb{Z}} \xrightarrow[=0]{d_2} \mathbb{Z}^2 \xrightarrow{d_1 = 0} \mathbb{Z} \to 0$$

$\partial D^2 = S^1 \longrightarrow X^1 =$ 

$\longrightarrow X^1/X^1 - e_a^1 =$ 

has degree $1 - 1 = 0$

Similarly

$\partial D^2 \longrightarrow X^1/X^1 - e_b^1$ has degree 0

So

$$d_2(e^2) = \underbrace{\boxed{d_a}}_{=0} \cdot e_a^1 + \underbrace{\boxed{d_b}}_{=0} \cdot e_b^1 = 0$$

$$\Rightarrow \quad H_k(\mathbb{T}) = \begin{cases} \mathbb{Z}, & k = 0, 2 \\ \mathbb{Z}^2, & k = 1 \\ 0, & k > 2 \end{cases}$$

$\cancel{A}$

exer: Compute homology of other orientable closed surface of genus g (see picture on p5)