Algebraic Topology 11/4

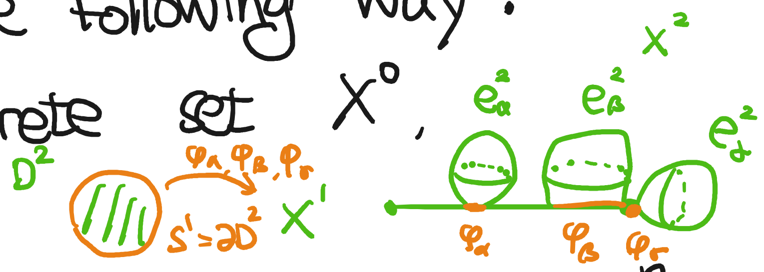
§2.2 Computation and applications

CW complex (p.5, p.519)

Def

A CW complex (or cell complex) is a space X constructed in the following way:

① Start with a discrete set X^0, the 0-cell of X

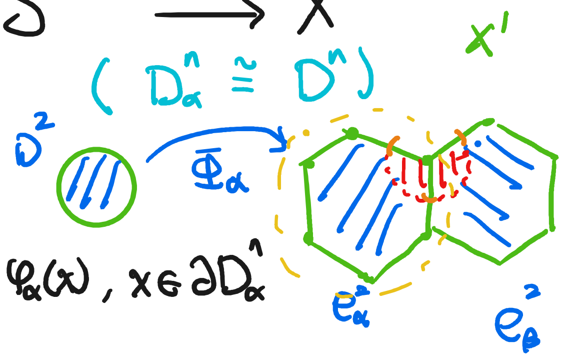


② Inductively, form the n-skeleton X^n from X^{n-1} by attaching n-cells e^n_α via maps $\varphi_\alpha : S^{n-1} \longrightarrow X^{n-1}$

$(D^n_\alpha \cong D^n)$

That is,

$$X^n = X^{n-1} \sqcup_\alpha D^n_\alpha \Big/ x \sim \varphi_\alpha(x),\ x \in \partial D^n_\alpha$$



Remark

the cell e^n_α is homeomorphic to $D^n_\alpha \setminus \partial D^n_\alpha$ under the quotient map

③ $X = \cup_n X^n$ with the weak topology:

$A \subseteq X$ is open (or $\overset{\text{resp.}}{\text{closed}}$) iff $A \cap X^n$ is open (or closed) in X^n for each

For each cell e^n_α, the map

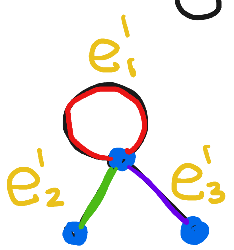
$$\Phi_\alpha : D^n = D^n_\alpha \hookrightarrow X^{n-1} \amalg_\alpha D^n_\alpha \twoheadrightarrow X^n \hookrightarrow X$$

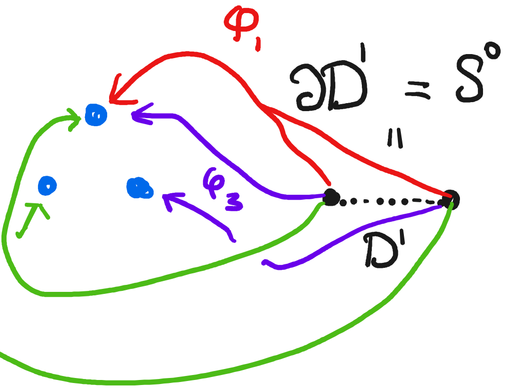
is called the characteristic map of e^n_α.

If $X = X^n$ for some n, then X is said to be finite-dimensional, and the smallest such n is called the dimension of X.

Example 0.1

A 1-dim CW cx $X = X^1$ is sometimes called a "graph" in algebraic topology

e.g. $X =$ 

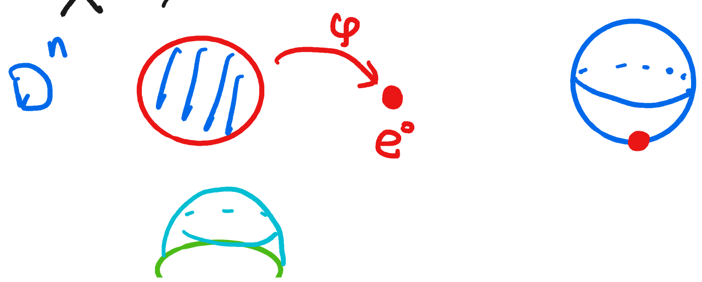
$\;\; = X^1$, $X^0 =$ 

Example 0.3

The sphere S^n has the structure of CW cxes with 2 cell, e^0 and e^n

$X^0 = \bullet \quad X^1 = \cdots = X^{n-1}$, $\quad \varphi : \partial D^n = S^{n-1} \longrightarrow \bullet = X^{n-1} = X^0$

$X^n = S^n$



"$S^n = e^0 \cup e^n$"

Example 0.4

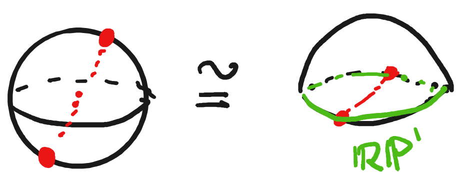
The real projective n-space $\mathbb{R}P^n$ is

$$\mathbb{R}P^n = \{\text{1-dim real vector subsp in } \mathbb{R}^{n+1}\}$$

$$= \mathbb{R}^{n+1} \setminus \{0\} \Big/ \vec{v} \sim \lambda \vec{v} \text{ for some } \lambda \neq 0$$

$$= S^n \Big/ \vec{v} \sim -\vec{v}$$

$\mathbb{R}P^2$



$$= D^n \Big/ \vec{v} \sim -\vec{v},\ \vec{v} \in \partial D^n \cong S^{n-1}$$

$\Rightarrow$ $\mathbb{R}P^n = \mathbb{R}P^{n-1} \cup e^n$, $\varphi_n : \partial D^n = S^{n-1} \longrightarrow \mathbb{R}P^{n-1}$ is the quotient map

$\Rightarrow$ inductively,

$$\mathbb{R}P^n = e^0 \cup_{\varphi_1} e^1 \cup_{\varphi_2} e^2 \cup \cdots \cup_{\varphi_n} e^n$$

CW complex structure

Example 0.5

$$\mathbb{R}P^\infty = \bigcup_n \mathbb{R}P^n = e^0 \cup_{\varphi_1} e^1 \cup_{\varphi_2} e^2 \cup_{\varphi_3} \cdots$$

$\mathbb{R}P^n \hookrightarrow \mathbb{R}P^{n+1}$

Example 0.5

The complex projective n-space $\mathbb{C}P^n$ is

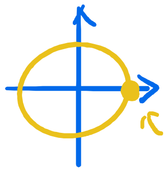
$$\mathbb{C}P^n = \{\text{1-dim complex vec. subsp in } \mathbb{C}^{n+1}\}$$

$$= \left. \mathbb{C}^{n+1} \setminus \{0\} \middle/ \vec{v} \sim \lambda \vec{v},\ \lambda \overset{\neq 0}{\in} \mathbb{C} \right.$$

$\mathbb{C}^{n+1} \supseteq$

$$= \left. S^{2n+1} \middle/ \vec{v} \sim \lambda \vec{v},\ |\lambda| = 1,\ \vec{v} \in S^{2n+1} \right.$$

For $(z_1, \cdots, z_{n+1}) \in S^{2n+1} \subseteq \mathbb{C}^{n+1}$,



$\mathbb{C} \ni \lambda,\ |\lambda| = 1$

$\leftarrow$ only positive number > 0

① if $z_{n+1} \neq 0$, then

$$(z_1, \cdots, z_{n+1}) \sim \underline{(\lambda z_1, \cdots, \lambda z_n, |z_{n+1}|)}$$

where $\lambda = \dfrac{|z_{n+1}|}{z_{n+1}}$

the unique vector in $[z_1 : \cdots : z_{n+1}]$ s.t. $z_{n+1} > 0$

And

$S^{2n+1}/\sim_{|\lambda|=1}$

$$[\vec{w} : \sqrt{1 - |\vec{w}|^2}] \in \{ [z_1 : \cdots : z_{n+1}] \in \mathbb{CP}^n \mid z_{n+1} > 0 \} \ni [z_1 : \cdots : z_{n+1}]$$

$$\vec{w} \in \quad \cong \quad \underbrace{D^{2n} \setminus \partial D^{2n}}_{\cong \mathbb{R}^{2n} \cong \mathbb{C}^n} \qquad (z_1, \cdots, z_n)$$

$\vec{w} = (z_1, \cdots, z_n)$

② if $z_{n+1} = 0$, then

$$\{ [z_1 : \cdots : z_n : 0] \in \mathbb{CP}^n \} \cong \mathbb{CP}^{n-1}$$

So $\mathbb{CP}^n = \underbrace{\mathbb{CP}^{n-1}}_{\text{② } z_{n+1} = 0} \cup \underbrace{e^{2n}}_{\text{① } z_{n+1} \neq 0}$, where

$\varphi : \partial D^{2n} \cong S^{2n-1} \longrightarrow \mathbb{CP}^n$ is the quotient map

$\Rightarrow \mathbb{CP}^n = e^0 \cup e^2 \cup e^4 \cup \cdots \cup e^{2n}$

Similarly, one also has

$$\mathbb{CP}^\infty = e^0 \cup e^2 \cup \cdots$$

Def

A <u>subcomplex</u> of a CW complex X is a closed subspace $A \subset X$ which is a union of cells of X.

A pair (X, A) of a CW cx X and a subcx A is called a <u>CW pair</u>

exer

A CW pair is a good pair (the assumption of <u>Thm</u> 2.13)

Remark (p519 ~)

"CW" refers to the following 2 properties of CW cxes:

① closure-finiteness:

The closure of each cell meets only finite cells

② weak topology

Cor 2.24

If a CW complex X is the union of subcxes A, B, then $(B, A\cap B) \hookrightarrow (X, A)$ induces iso

$$H_n(B, A\cap B) \longrightarrow H_n(X, A) \qquad \forall n$$

pf

Since CW pairs are good pairs, <u>Prop 2.22</u> implies

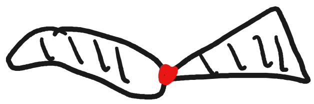
$$H_n(B, A\cap B) \xrightarrow{\cong} H_n(B/A\cap B,\ A\cap B/A\cap B) \xleftarrow{\cong} \tilde{H}_n(B/A\cap B)$$

$\because\ B/A\cap B \xrightarrow{\cong} X/A$ $\rightarrow$ $\|$?S is a homeo

$$H_n(X,A) \xrightarrow{\cong} H_n(X/A,\ A/A) \xleftarrow{\cong} \tilde{H}_n(X/A)$$

Wedge sum

$X \vee Y$:



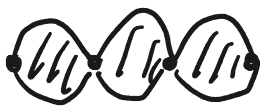
Given spaces X, Y with $x_0 \in X$, $y_0 \in Y$, the **wedge sum** $X \vee Y$ is

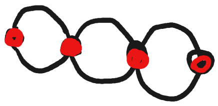
$$X \vee Y = X \sqcup Y / x_0 \sim y_0$$

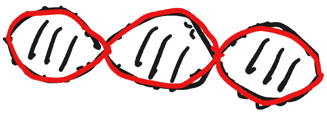
If we are given X_α, $x_\alpha \in X_\alpha$ $\alpha \in \Lambda$, then

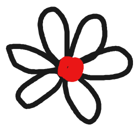
$$\bigvee_\alpha X_\alpha := \coprod_{\alpha \in \Lambda} X_\alpha / x_\alpha \sim x_\beta,\ \alpha, \beta \in \Lambda$$

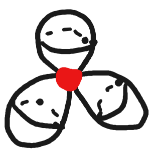
Example

Suppose X is the CW cx 

$X^0 = \cdot\ \cdot\ \cdot\ \cdot$ $\qquad X^1 =$  $\leftarrow$ 6 1-cells

$X^2 =$ 

$\Rightarrow$ $X^1/X^0 =$  $= \overbrace{S^1 \vee \cdots \vee S^1}^{6}$

$X^2/X^1 =$  $= S^2 \vee S^2 \vee S^2$

In general,

$$X^n / X^{n-1} = \bigvee_\alpha S^n_\alpha, \quad \alpha: \text{indexed by } n\text{-cells}$$

Cor 2.25

For a wedge sum $\bigvee_\alpha X_\alpha$, the inclusions

For a wedge sum $\bigvee_\alpha X_\alpha$, the

$$i_\alpha : X_\alpha \hookrightarrow \bigvee_\alpha X_\alpha$$

induce an iso

$$\bigoplus_\alpha i_\alpha : \bigoplus_\alpha \tilde{H}_n(X_\alpha) \xrightarrow{\cong} \tilde{H}_n(\bigvee_\alpha X_\alpha)$$

provided the pairs $(X_\alpha, \{x_\alpha\})$ are good.

pf:
Apply Prop 2.22 to $(X, A) = (\coprod_\alpha X_\alpha, \coprod_\alpha \{x_\alpha\})$ #

Cellular homology

Lemma 2.34

If X is a CW complex, then

(a) $H_k(X^n, X^{n-1}) \cong \begin{cases} 0 & \text{if } k \neq n \\ \mathbb{Z}^{r_n} & \text{if } k = n \end{cases}$

where $r_n = \#$ of n-cells in X

(b) $H_k(X^n) = 0$ for $k > n$.

In particular, if X is finite-dim, then
$H_k(X) = 0$ for $k > \dim X$

(c) The map $H_k(X^n) \to H_k(X)$ induce by $X^n \hookrightarrow X$
is $\begin{cases} \text{an iso for } k < n \\ \text{surjective for } k = n. \end{cases}$

pf
(a) is true because (X^n, X^{n-1}) is good and

$$H_k(X^n, X^{n-1}) \cong \tilde{H}_k(X^n / X^{n-1}) \cong \tilde{H}_k(\bigvee S^n_\alpha)$$

$$= \bigoplus \tilde{H}_k(S^n) \cong \bigoplus \mathbb{Z} = \mathbb{Z}^{r_n}$$

$$\cong \bigoplus_\alpha H_k(S_\alpha) = \bigoplus_\alpha \mathbb{Z} = \mathbb{Z} \qquad \#$$

(b) We have the long exact seq for (X^n, X^{n-1}):

$$\cdots \to \underbrace{H_{k+1}(X^n, X^{n-1})}_{=0 \text{ if } k\neq n-1} \to H_k(X^{n-1}) \to H_k(X^n) \to \underbrace{H_k(X^n, X^{n-1})}_{\overset{0 \text{ if}}{\| \; k\neq n}} \to \cdots$$

If $k > n \geq 0$,

$$H_k(X^n) \cong H_k(X^{n-1}) \cong H_k(X^{n-2}) \cong \cdots$$

$$\cong H_k(X^0) = 0 \qquad \#$$

(c) skip here (p. 138-139)

Let X be a CW complex.

By Lemma 2.34, $H_n(X^n, X^{n-1}) \cong \mathbb{Z}^{r_n}$, $r_n = \#$ of n-cells ;

Define

$$d_n := j \circ \partial : \underbrace{H_n(X^n, X^{n-1})}_{\overset{\mathbb{Z}^{r_n}}{\cong}} \longrightarrow H_{n-1}(X^{n-1}) \longrightarrow \underbrace{H_{n-1}(X^{n-1}, X^{n-2})}_{\overset{\cong}{\mathbb{Z}^{r_{n-1}}}}$$

$$\cdots \to H_n(X^n) \to H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-1}) \to H_{n-1}(X^n) \to \cdots$$

$$H_{n-1}(X^{n-2}) \to H_{n-1}(X^{n-1}) \to H_{n-1}(X^{n-1}, X^{n-2}) \to H_{n-2}(X^{n-2})$$

next ...

Lemma

$$d_n \circ d_{n+1} = 0$$

and $H_n^{CW}(X) := \ker(d_n) / \operatorname{im}(d_{n+1}) \overset{\text{Thm}}{\cong} H_n(X)$