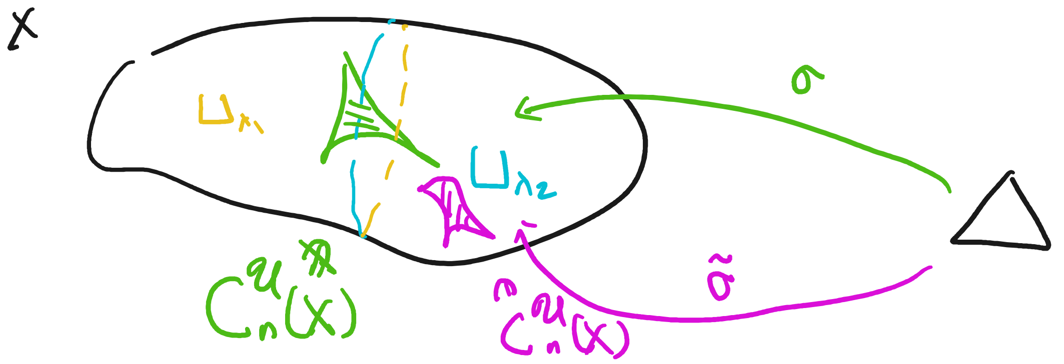
some $U_\lambda \in \mathcal{U}$



Since $\partial : C_n(X) \to C_{n-1}(X)$ takes $C_n^{\mathcal{U}}(X)$ to $C_{n-1}^{\mathcal{U}}(X)$, we obtain a subcomplex $(C_\bullet^{\mathcal{U}}(X), \partial)$

Its homology will be denoted by $H_n^{\mathcal{U}}(X)$

<u>Prop 2.21</u>

The inclusion $\iota : C_n^{\mathcal{U}}(X) \hookrightarrow C_n(X)$ is a chain homotopy equivalence.

In particular, $H_n^{\mathcal{U}}(X) \cong H_n(X) \quad \forall n$.

<u>sketch of pf</u> (p. 119 ~ 124)

$x, y \in \Delta^n \Rightarrow \overline{xy} \subseteq \Delta^n$ $\quad$ $0 \le t \le 1$, $tx + (1-t)y$ $\quad$ $\sigma : \Delta^n \to \Delta^N$, $\Delta^N \subseteq \mathbb{R}^{N+1}$

① Consider "a linear version of $C_n(\Delta^N)$"

$\sigma(tx + (1-t)y) = t\sigma(x) + (1-t)\sigma(y)$

" $LC_n(\Delta^N)$ "

and define the barycentric subdivision homo

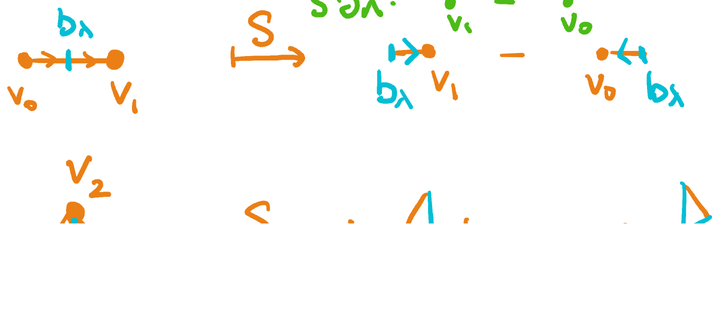
$$S : LC_n(\Delta^N) \longrightarrow LC_n(\Delta^N)$$

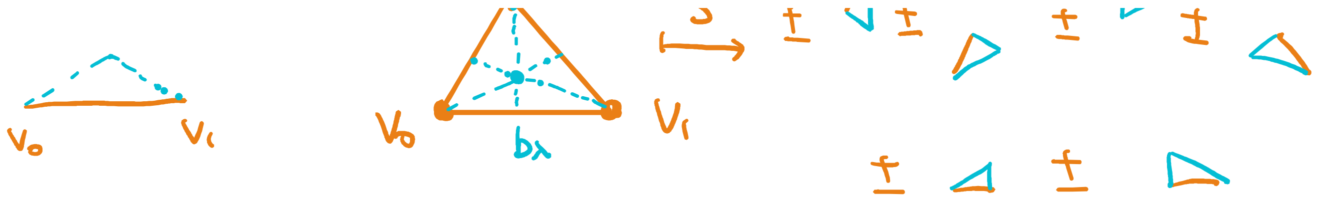
"add barycenter"

by induction ($S(\lambda) = b_\lambda(S\partial\lambda)$, $S = id$ on 0-simplexes)

$S\partial\lambda$: $v_1 - v_0$

picture of S:





② Define $T: C_n(\Delta^N) \to C_{n+1}(\Delta^N)$ inductively

$(T(\lambda) = b_\lambda(\lambda - T\partial\lambda))$ s.t.

$$\partial T + T\partial = id - S$$

So $S = id$ on homologies

--- 10/28 ---

③ For $\sigma \in C_n(X)$, $\sigma: \Delta^n \to X$

$$\sigma: \Delta^n \xrightarrow{id} \Delta^n \xrightarrow{\sigma} X$$

apply S, T to

and get S, T on $C_n(X)$.

④ Apply S, T many times so that each small singular simplex in the formula maps into $int(U_\lambda)$ for some $U_\lambda \in \mathcal{U}$. ∎

Proof of Excision Thm 2.20:

Let $\mathcal{U} = \{A, B\}$ and $\iota: C_n^{\mathcal{U}}(X) \to C_n(X)$.

In the proof of Prop 2.21, the author constructed a chain map $\rho: C(X) \to C^{\mathcal{U}}(X)$ ← "by iterating S"

and a chain homotopy $D: C_n(X) \to C_{n+1}(X)$ s.t.

(i) $\partial D + D\partial = id - \iota\rho$

(ii) $\rho\iota = id$

(iii) $\rho(C(A)) \subseteq C(A)$, $D(C(A)) \subseteq C(A)$

So the maps D, ρ descend to $C^{\mathcal{U}}(X)/C(A)$ and $C(X)/C(A)$ s.t. (i) and (ii) hold.

$\Rightarrow$ The inclusion

$$\boxed{C^{\mathcal{U}}(X)/C(A) \hookrightarrow C(X)/C(A)}$$

induces iso of homologies

Also note that

$$\boxed{C(B)/C(A \cap B) \longrightarrow C^{\mathcal{U}}(X)/C(A)}$$

also induces iso of homologies because both are the free abelian gp generated by the singular simplexes $\sigma: \Delta^n \to B$ s.t. $\sigma(\Delta^n) \nsubseteq A$

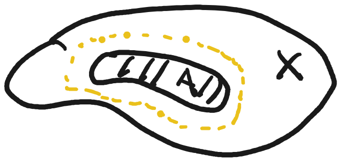
So we have

$$H_n(B, A \cap B) \cong H_n(C^{\mathcal{U}}(X)/C(A)) \cong H_n(X, A)$$

#

<u>Recall</u>

In <u>Thm 2.13</u>, we consider a space X together with a nonempty closed subsp A that is a deformation retract of some nbd in X



Such a pair (X,A) is called a good pair in the book

Prop 2.22

For good pairs (X,A), the quotient map

$$q : (X, A) \longrightarrow (X/A \,,\, A/A)$$

induces isomorphisms

$$q_* : H_n(X,A) \xrightarrow{\cong} H_n(X/A, \underset{\text{a pt}}{A/A}) \cong \tilde{H}_n(X/A) \quad \forall n$$

pf

Let V be a nbd of A in X that deformation retracts onto A. We have a commutative diagram:

$$\begin{tikzcd}
H_n(X,A) \arrow[r, "\phi", "\cong \text{ by } ①"'] \arrow[d, "q_*"'] \arrow[dr, phantom, "\circlearrowleft"] & H_n(X,V) \arrow[d, "q_*"] \arrow[dr, phantom, "\circlearrowleft"] & H_n(X\setminus A, V\setminus A) \arrow[l, "\tilde{\Phi}"', "\cong"] \arrow[d, "\tilde{q}_*"] \\
H_n(X/A, A/A) \arrow[r, "\psi \cong"'] & H_n(X/A, V/A) & H_n(X/A \setminus A/A, V/A \setminus A/A) \arrow[l, "\tilde{\Psi}", "\cong"']
\end{tikzcd}$$

$\leftarrow$ iso by Excision Thm

Note:

① (X, V, A) induces a long exact seq   by Remark last week

$$\cdots \longrightarrow H_n(V,A) \longrightarrow H_n(X,A) \xrightarrow[\cong]{\phi} H_n(X,V),$$

$\cdots \to H_n(V,A) \to \cdots \to H_{n-1}(V,A) \to \cdots$

$H_n(V,A) = 0$, $H_{n-1}(V,A) = 0$

because $(V,A) \simeq (A,A)$

(Prop 2.19)

So Φ is an iso.

② Similarly, Ψ is an iso.

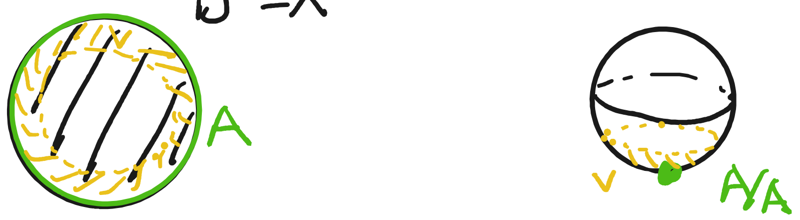
③ By Excision Thm, $\tilde{\Phi}$, $\tilde{\Psi}$ are iso.

④ $\tilde{q}_*$ is an iso because it's induced by the homeomorphism

$$(X \setminus A,\ V \setminus A) \xrightarrow{\cong} (X/A \setminus A/A,\ V/A \setminus A/A)$$

Therefore, $q_* = \Psi^{-1} \circ \tilde{\Psi} \circ \tilde{q}_* \circ \tilde{\Phi}^{-1} \circ \Phi$ is an iso. #

e.g. $D^2 = X$ $\qquad$ X/A



Summary of the proof of Thm 2.13:

Recall that for a good pair (X,A), we wanted to prove $\exists$ an exact seq.

$$\boxed{\cdots \to \tilde{H}_n(A) \to \tilde{H}_n(X) \to \tilde{H}_n(X/A) \to \tilde{H}_{n-1}(A) \to \cdots}$$

Step 1: Prove short exact seq of cxes induces a long exact seq of homologies

$\Rightarrow \exists$ long exact seq. of relative homologies:

$$\cdots \to \tilde{H}_n(A) \to \tilde{H}_n(X) \to \tilde{H}_n(X,A) \to \tilde{H}_{n-1}(A) \to \cdots$$

<u>Step2</u>: Prove <u>Excision Thm 2.20</u> and use it to show

$$\tilde{H}_n(X,A) \cong \tilde{H}_n(X/A) \quad (\underline{\text{Prop } 2.22})$$

for good pair (X,A)

<u>Cor 2.14</u>

$$\tilde{H}_k(S^n) \cong \begin{cases} \mathbb{Z}, & k=n \\ 0, & k \neq n \end{cases}$$

(Apply <u>Thm 2.13</u> to $(X,A) = (D^n, \partial D^n)$)

<u>Thm 2.26</u>

If nonempty open subsets $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ are homeomorphic, then $m = n$

pf

For $x \in U$, applying <u>Excision Thm</u> to

$$Z = U^c, \quad A = \mathbb{R}^m \setminus \{x\}, \quad X = \mathbb{R}^m,$$

we have

$$H_k(U, U \setminus \{x\}) \cong H_k(\mathbb{R}^m, \mathbb{R}^m \setminus \{x\})$$

By long exact seq.,

$$\cdots \to \tilde{H}_k(\mathbb{R}^m \setminus \{x\}) \to \underbrace{\tilde{H}_k(\mathbb{R}^m)}_{=0} \to \tilde{H}_k(\mathbb{R}^m, \mathbb{R}^m \setminus \{x\})$$

$$\xrightarrow{\cong} \tilde{H}_{k-1}(\mathbb{R}^m \setminus \{x\}) \to \underbrace{\tilde{H}_{k-1}(\mathbb{R}^m)}_{=0} \to \cdots$$

$$\cong \tilde{H}(S^{m-1}) \cong \{ \mathbb{Z} \quad k=m$$

$H_{k-1}(S \quad) = \begin{cases} \mathbb{Z}, & k = m \\ 0, & k \neq m \end{cases}$

So $H_k(U, U \setminus \{x\}) \cong \begin{cases} \mathbb{Z}, & k = m \\ 0, & k \neq m \end{cases}$ for $x \in U$.

Similarly,

$H_k(V, V \setminus \{y\}) \cong \begin{cases} \mathbb{Z}, & k = n \\ 0, & k \neq n \end{cases}$ for $y \in V$.

A homeomorphism $h: U \to V$ induces

$$(U, U \setminus \{x\}) \xrightarrow{\cong} (V, V \setminus \{h(x)\})$$

$$\Rightarrow \quad H_k(U, U \setminus \{x\}) \cong H_k(V, V \setminus \{h(x)\}) \quad \forall k$$

$$\Rightarrow \quad m = n \qquad \#$$

Naturality (p. 127)

Prop ①

For a map $f: (X, A) \to (Y, B)$, the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(A) \arrow[r] \arrow[d, "f_*"'] \arrow[dr, phantom, "\circlearrowleft"] & H_n(X) \arrow[r] \arrow[d, "f_*"'] \arrow[dr, phantom, "\circlearrowleft"] & H_n(X, A) \arrow[r, "\partial"] \arrow[d, "f_*"] \arrow[dr, phantom, "\circlearrowleft"] & H_{n-1}(A) \arrow[r] \arrow[d, "f_*"] & \cdots \\
\cdots \arrow[r] & H_n(B) \arrow[r] & H_n(Y) \arrow[r] & H_n(Y, B) \arrow[r, "\partial"'] & H_{n-1}(B) \arrow[r] & \cdots
\end{tikzcd}$$

commutes

Prop ② (alg. ver.)

Suppose $\alpha: (A, \partial^A) \to (A', \partial^{A'})$, $\beta: (B, \partial^B) \to (B', \partial^{B'})$

$\gamma: (C, \partial^C) \to (C', \partial^{C'})$ ...

$\gamma : (C, \partial) \to (C', \partial')$ form a morphism of short exact seq of cx" i.e. all the maps commute (see diagram, p127).

Then the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(A) \arrow[r] \arrow[d, "\alpha_*"] & H_n(B) \arrow[r] \arrow[d, "\beta_*"] & H_n(C) \arrow[r] \arrow[d, "\gamma_*"] & H_{n-1}(A) \arrow[r] \arrow[d, "\alpha_*"] & \cdots \\
\cdots \arrow[r] & H_n(A') \arrow[r] & H_n(B') \arrow[r] & H_n(C') \arrow[r] & H_{n-1}(A') \arrow[r] & \cdots
\end{tikzcd}$$

commutes

pf: exer. (diagram chasing)

<u>pf of Prop ①</u>:

$f : (X, A) \to (Y, B)$ induces a " mor of short exact seq. of cx"

$$\begin{tikzcd}
0 \arrow[r] & C(A) \arrow[r] \arrow[d, "f_\#"] & C(X) \arrow[r] \arrow[d, "f_\#"] & \underset{\overset{\parallel}{C(X)/C(A)}}{C(X, A)} \arrow[r] \arrow[d, "f_\#"] & 0 \\
0 \arrow[r] & C(B) \arrow[r] & C(Y) \arrow[r] & C(Y, B) \arrow[r] & 0
\end{tikzcd}$$

So <u>Prop ②</u> $\Rightarrow$ <u>Prop ①</u> $\square$

<u>Prop ③</u>

For a map $f : (X, A) \to (Y, B)$ of good pairs, the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \tilde{H}_n(A) \arrow[r] \arrow[d, "f_*"] \arrow[dr, phantom, "\circlearrowleft"] & \tilde{H}_n(X) \arrow[r] \arrow[d, "f_*"] \arrow[dr, phantom, "\circlearrowleft"] & \tilde{H}_n(X/A) \arrow[r] \arrow[d, "f_*"] \arrow[dr, phantom, "\circlearrowleft"] & \tilde{H}_{n-1}(A) \arrow[r] \arrow[d, "f_*"] & \cdots \\
\cdots \arrow[r] & \tilde{H}_n(B) \arrow[r] & \tilde{H}_n(Y) \arrow[r] & \tilde{H}_n(B/\cdot) \arrow[r] & \tilde{H}_{n-1}(B) \arrow[r] & \cdots
\end{tikzcd}$$

commutes

pf

By Prop ①, it follows from the commutivity of the diagram

$$\tilde{H}_n(A/A) = 0$$

$$\begin{tikzcd}
\tilde{H}_n(X/A) \arrow[r, "\cong"] \arrow[d, "f_*"'] \arrow[dr, phantom, "\text{by Prop ①}"] & H_n(X/A, A/A) \arrow[d, "f_*"] \arrow[dr, phantom, "\because\, fg = gf"] & H_n(X,A) \arrow[l, "\cong\ g_*"'] \arrow[d, "f_*"] \\
\tilde{H}_n(Y/B) \arrow[r] & H_n(Y/B, B/B) & H_n(Y,B) \arrow[l, "\cong"', "g_*"]
\end{tikzcd}$$

#

Eilenberg–Steenrod axioms

From Wikipedia, the free encyclopedia

In mathematics, specifically in algebraic topology, the **Eilenberg–Steenrod axioms** are properties that homology theories of topological spaces have in common. The quintessential example of a homology theory satisfying the axioms is singular homology, developed by Samuel Eilenberg and Norman Steenrod.

One can define a homology theory as a sequence of functors satisfying the Eilenberg–Steenrod axioms. The axiomatic approach, which was developed in 1945, allows one to prove results, such as the Mayer–Vietoris sequence, that are common to all homology theories satisfying the axioms.[1]

If one omits the dimension axiom (described below), then the remaining axioms define what is called an extraordinary homology theory. Extraordinary cohomology theories first arose in K-theory and cobordism.

Contents [hide]

Formal definition [edit source]

$H_n(X,A)$

The Eilenberg–Steenrod axioms apply to a sequence of functors H_n from the category of pairs (X, A) of topological spaces to the category of abelian groups, together with a natural transformation $\partial\colon H_i(X,A) \to \boxed{H_{i-1}(A)}$ called the **boundary map** (here $H_{i-1}(A)$ is a shorthand for $H_{i-1}(A,\emptyset)$. The axioms are:

$= H_{i-1}(A,\emptyset)$

1. **Homotopy**: Homotopic maps induce the same map in homology. That is, if $g\colon (X,A) \to (Y,B)$ is homotopic to $h\colon (X,A) \to (Y,B)$, then their induced homomorphisms are the same.
2. **Excision**: If (X, A) is a pair and *U* is a subset of *A* such that the closure of *U* is contained in the interior of *A*, then the inclusion map $i\colon (X \setminus U, A \setminus U) \to (X, A)$ induces an isomorphism in homology.
3. **Dimension**: Let *P* be the one-point space; then $H_n(P) = 0$ for all $n \neq 0$.
4. **Additivity**: If $X = \coprod_\alpha X_\alpha$, the disjoint union of a family of topological spaces X_α, then $H_n(X) \cong \bigoplus_\alpha H_n(X_\alpha)$.
5. **Exactness**: Each pair *(X, A)* induces a long exact sequence in homology, via the inclusions $i\colon A \to X$ and $j\colon X \to (X,A)$: $\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$.

If *P* is the one point space, then $H_0(P)$ is called the **coefficient group**. For example, singular homology (taken with integer coefficients, as is most common) has as coefficients the integers.

Consequences [edit source]

Some facts about homology groups can be derived directly from the axioms, such as the fact that homotopically equivalent spaces have isomorphic homology groups.

The homology of some relatively simple spaces, such as *n*-spheres, can be calculated directly from the axioms. From this it can be easily shown that the (*n* – 1)-sphere is not a retract of the *n*-disk. This is used in a proof of the Brouwer fixed point theorem.

Dimension axiom [edit source]

A "homology-like" theory satisfying all of the Eilenberg–Steenrod axioms except the dimension axiom is called an **extraordinary homology theory** (dually, **extraordinary cohomology theory**). Important examples of these were found in the 1950s, such as topological K-theory and cobordism theory, which are extraordinary *co*homology theories, and come with homology theories dual to them.