Algebraic Topology 10/21

Correction:

Hatcher's deformation retraction
= strong deformation retraction in previous lecture

From now on, we follow the def in book, i.e.
a deformation retraction of X onto A is
a homotopy $F: X \times I \to X$ s.t. $f_0 = F(-,0)$, $f_1 = F(-,1)$

① $f_0 = id_X$ ② $f_1(X) = A$

③ $F(a,t) = a \quad \forall a \in A^{\subset X}, \ \forall t \in I$

Recall

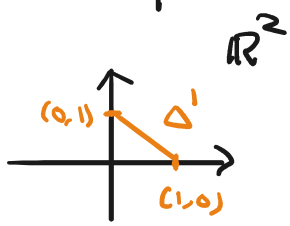
Let X be a topological space.

$$\Delta^n = \{ (t_0, \cdots, t_n) \in \mathbb{R}^n \mid t_0 + \cdots + t_n = 1, \ t_i \geq 0 \}$$

$C_n(X) :=$ free abelian group generated by all the continuous maps

$$\Delta^n \longrightarrow X$$

Mostly NOT finitely generated



e.g. $X = \mathbb{R}$, $n = 1$, $\Delta^1 \cong [0,1]$

$$C_1(\mathbb{R}) \cong \mathbb{Z} \cdot \underline{\{ \sigma : [0,1] \to \mathbb{R} \mid \sigma \text{ is continuous} \}}$$

generators ← uncountably

Boundary map

$\partial_n : C_n(X) \to C_{n-1}(X)$, $\partial_n \circ \partial_{n+1} = 0$

$\Rightarrow$ we have $H_n(X) := \ker(\partial_n) / \operatorname{im}(\partial_{n+1})$ ← singular homology

Remark

Difficult to compute directly from definition

$\Rightarrow$ Need some thms

e.g.

Thm 2.13

$\neq \phi$, closed, some condition

$A \subseteq X$, $\exists$ exact seq.

$$(*) \quad \cdots \to \tilde{H}_n(A) \to \tilde{H}_n(X) \to \tilde{H}_n(X/A) \to \tilde{H}_{n-1}(A) \to \cdots$$

Remark

$(*)$ can be applied to compute $H_*(S^n)$, and more

Proof of Thm 2.13 (outline):

Step1 (last week): ← we are here

long exact seq of "relative homology" $H_n(X,A)$

Step2:

relationship between $H_n(X,A)$ and $\tilde{H}_n(X/A)$

Recall (Thm 2.16)

A short exact seq of chain complexes

$$0 \to (A, \partial^A) \to (B, \partial^B) \to (C, \partial^C) \to 0$$

induces a long exact seq of homology groups

$$\cdots \to H_n(A) \to H_n(B) \to H_n(C) \to H_{n-1}(A) \to \cdots$$

Relative homology group:

Let $A \subseteq X$

$\Rightarrow$ $C_n(A)$ is a subgp of $C_n(X)$

generated by

$\sigma : \Delta^n \xrightarrow{\text{cts}} A \subseteq X$ $\Rightarrow \sigma$ is also a singular simplex in X

Let $C_n(X, A) := C_n(X) / C_n(A)$

Since $\partial_n^X (C_n(A)) \subseteq C_{n-1}(A)$, ∂_n^X induces a gp homo

$$\partial_n^{X,A} : C_n(X, A) = C_n(X)/C_n(A) \to C_{n-1}(X)/C_{n-1}(A) = C_{n-1}(X, A)$$

and $\partial_n^{X,A} \circ \partial_{n+1}^{X,A} = 0$

$\Rightarrow$ we have a chain complex

$$\cdots \to C_n(X, A) \xrightarrow{\partial_n^{X,A}} C_{n-1}(X,A) \to \cdots$$

The associated homology group

$$H_n(X,A) = \ker(\partial_n^{X,A}) / \operatorname{im}(\partial_{n+1}^{X,A})$$

is called the relative homology group

Note that we the short exact seq of cxes:

$$0 \to (C(A), \partial^A) \to (C(X), \partial^X) \to (C(X,A), \partial^{X,A}) \to 0$$

Remark

If $H \leq G$ are abelian gps, then

$$0 \to H \to G \to G/H \to 0$$

is a short exact seq.

Therefore, by Thm 2.16, we have

Thm

Let $A \subset X$. Then we have the long exact seq

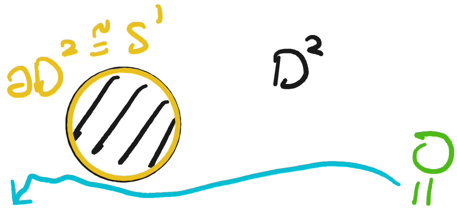
$$\cdots \to H_n(A) \to H_n(X) \to H_n(X,A) \to H_{n-1}(A) \to H_{n-1}(X) \to \cdots$$

and

$$\cdots \to \tilde{H}_n(A) \to \tilde{H}_n(X) \to \tilde{H}_n(X,A) \to \tilde{H}_{n-1}(A) \to \cdots$$

where $\tilde{H}_n(X,A) = \begin{cases} H_n(X,A) & \text{if } A \neq \phi \\ \tilde{H}_n(X) & \text{if } A = \phi \end{cases}$

Example 2.17



For $(X,A) = (D^n, \partial D^n)$,

$\exists$ exact seq.

$$\cdots \to \underbrace{\tilde{H}_i(D^n)}_{=0} \to \tilde{H}_i(D^n, \partial D^n) \xrightarrow[\text{1-1}]{\text{onto}} \tilde{H}_{i-1}(\partial D^n) \to \underbrace{\tilde{H}_{i-1}(D^n)}_{=0} \to \cdots$$

$$\Rightarrow \quad H_i(D^n, \partial D^n) \cong \tilde{H}_{i-1}(\partial D^n) \cong \tilde{H}_{i-1}(S^{n-1}) \quad \forall i$$

Example 2.18

Let $x_0 \in X$. $(X,A) = (X, \{x_0\})$

$\Rightarrow \exists$ exact seq.

$$\cdots \to \underbrace{\tilde{H}_i(\{x_0\})}_{=0} \to \tilde{H}_i(X) \xrightarrow[\text{1-1}]{\text{onto}} \tilde{H}_i(X, \{x_0\}) \to \underbrace{\tilde{H}_{i-1}(\{x_0\})}_{=0} \to \cdots$$

$$\Rightarrow \quad H_i(X, \{x_0\}) \cong \tilde{H}_i(X) \qquad \forall i.$$

We write

$$f: (X,A) \longrightarrow (Y,B)$$

if $f: X \to Y$ is a continuous map s.t. $f(A) \subseteq B$.

$\Rightarrow \quad f_\#(C_n(A)) \subseteq C_n(B)$

$\Rightarrow$ we have a chain map

$$f_\# : C_n(X,A) = C_n(X)/C_n(A) \to C_n(Y)/C_n(B) = C_n(Y,B)$$

$$[\textstyle\sum_i n_i \sigma_i] \longmapsto [\textstyle\sum_i n_i (f \circ \sigma_i)]$$

$\Rightarrow$ we have

$$f_* : H_n(X,A) \longrightarrow H_n(Y,B) \qquad \forall n.$$

Prop 2.19

If two maps $f, g : (X,A) \to (Y,B)$ are homotopic through maps $(X,A) \longrightarrow (Y,B)$

(i.e. $\exists$ homotopy $H : X \times I \to Y$ s.t. $H(A \times I) \subseteq B$)

then

$$f_* = g_* : H_n(X,A) \longrightarrow H_n(Y,B) \qquad \forall n$$

pf

Note the chain homotopy P in the proof of Thm 2.10 has the property $P(C_n(A)) \subseteq C_n(B)$

$\Rightarrow$ P induces a chain homotopy between

$$f_\#, g_\# : C(X,A) \longrightarrow C(Y,B). \quad \Rightarrow \text{ok}$$

$\#$

Cor

$\Rightarrow$ $(A,A) \hookrightarrow (V,A)$ is homotopy equi.

If $A \subset V \subset X$ is a deformation retraction onto A, then

$$H_n(V,A) \cong H_n(A,A) = 0$$

Remark

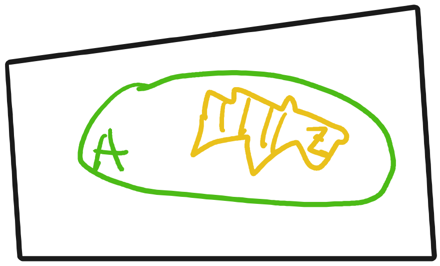
Let $B \subseteq A \subseteq X$. Then we have a short exact seq. of cxes:

$$0 \to (C(A,B), \partial^{A,B}) \to (C(X,B), \partial^{X,B}) \to (C(X,A), \partial^{X,A}) \to 0$$

So we have a long exact seq.

$$\cdots \to H_n(A,B) \to H_n(X,B) \to H_n(X,A) \to H_{n-1}(A,B) \to \cdots$$

(need this for proving Thm 2.13)



Excision

Thm 2.20 (Excision Thm)

closure   interior

Given subspaces $Z \subseteq A \subseteq X$ s.t. $cl(Z) \subseteq int(A)$,
the inclusion

$$(X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$$

$Z = X \setminus B$

induces isomorphisms

$$H_n(X \setminus Z, A \setminus Z) \xrightarrow{\cong} H_n(X, A) \quad \forall n$$

Equivalently, for subsps $A, B \subseteq X$ with the property

$$int(A) \cup int(B) = X$$

the inclusion

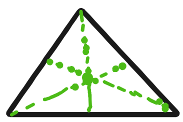
$B = X \setminus Z$

$$(B, A \cap B) \hookrightarrow (X, A)$$

induces isomorphisms

$$H_n(B, A \cap B) \xrightarrow{\cong} H_n(X, A)$$

重心的 Barycentric subdivision



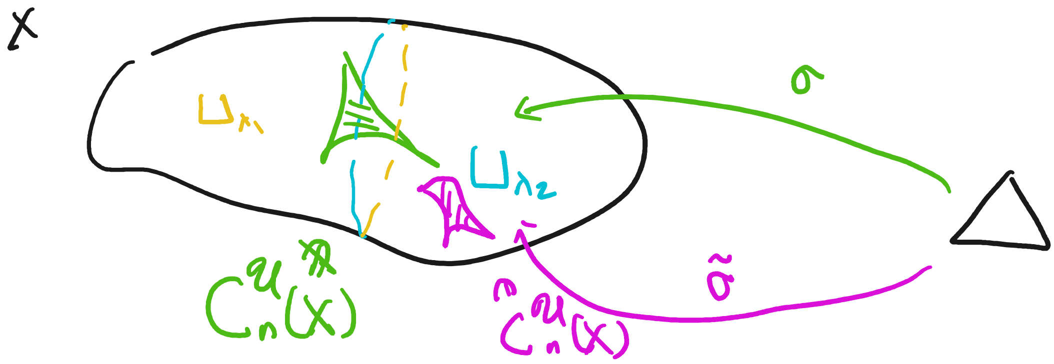
For a space X, let $\mathcal{U} = \{U_\lambda\}$, $U_\lambda \subseteq X$ s.t.

$$\bigcup_\lambda int(U_\lambda) = X$$

and let

$$C_n^{\mathcal{U}}(X) := \left\{ \sum n_i \sigma_i \in C_n(X) \;\middle|\; \sigma_i(\Delta^n) \subseteq U_\lambda \text{ for } \right\}$$

some $U_\lambda \in \mathcal{U}$ ∪



Since $\partial : C_n(X) \to C_{n-1}(X)$ takes $C_n^{\mathcal{U}}(X)$ to $C_{n-1}^{\mathcal{U}}(X)$,
we obtain a subcomplex $(C_\bullet^{\mathcal{U}}(X), \partial)$
Its homology will be denoted by $H_n^{\mathcal{U}}(X)$

<u>Prop 2.21</u>
The inclusion $\iota : C_n^{\mathcal{U}}(X) \hookrightarrow C_n(X)$ is a chain homotopy equivalence.
In particular, $H_n^{\mathcal{U}}(X) \cong H_n(X) \quad \forall n$.

<u>sketch of pf</u> (p. 119 ~ 124)

$x, y \in \Delta^n \Rightarrow \overline{xy} \subset \Delta^n$ $\sigma : \Delta^n \to \Delta^N \subset \mathbb{R}^{N+1}$

$0 \le t \le 1$ $tx + (1-t)y$

① Consider "a linear version of $C_n(\Delta^N)$"

$\sigma(tx+(1-t)y) = t\sigma(x) + (1-t)\sigma(y)$

" $LC_n(\Delta^N)$ "

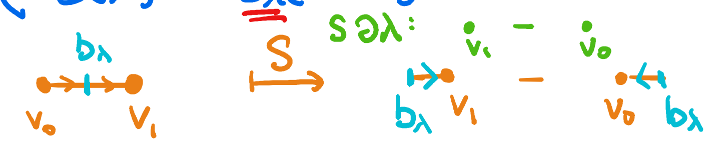
and define the barycentric subdivision homo

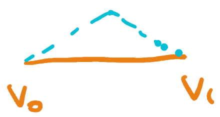
$$S : LC_n(\Delta^N) \longrightarrow LC_n(\Delta^N)$$

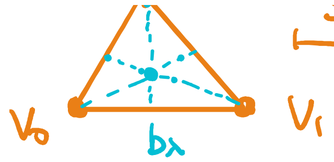
"add barycenter"

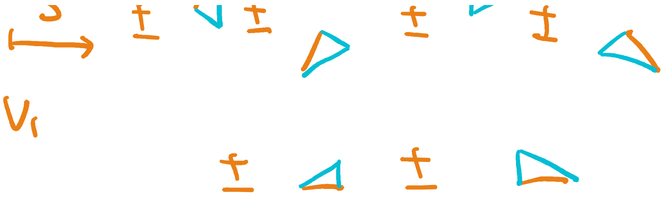
by induction ($S(\lambda) = b_\lambda(S\partial\lambda)$, $S = id$ on 0-simplexes)

picture of S:









② Define $T: C_n(\Delta^N) \to C_{n+1}(\Delta^N)$ inductively

$(\, T(\lambda) = b_\lambda(\lambda - T\partial\lambda) \,)$ s.t.

$$\partial T + T\partial = id - S$$

So $S = id$ on homologies