Algebraic Topology 10/14

__Recall__

__Thm 2.13__

Suppose $\overset{\not\phi}{A} \subseteq X$ satisfy a condition. Then we have an exact seq i.e. "ker = im"

$$\cdots \to \tilde{H}_n(A) \to \tilde{H}_n(X) \to \tilde{H}_n(X/A) \to \tilde{H}_{n-1}(A) \to \cdots$$

__Cor 2.14__

$$H_i(S^n) \cong \begin{cases} \mathbb{Z} & \text{if } i = 0, n \\ 0 & \text{if } i \neq 0, n \end{cases}$$

__Cor__

$$\mathbb{R}^n \not\cong \mathbb{R}^m \quad \text{if} \quad n \neq m$$

__pf__

Suppose $\phi : \mathbb{R}^n \to \mathbb{R}^m$ is a homeomorphism

Then

$$\phi|_{\mathbb{R}^n \setminus \{\vec{0}\}} : \underbrace{\mathbb{R}^n \setminus \{\vec{0}\}}_{\simeq S^{n-1}} \to \underbrace{\mathbb{R}^m \setminus \{\phi(\vec{0})\}}_{\simeq S^{m-1}}$$

$$\Rightarrow \underbrace{H_i(S^{n-1})}_{\cong \begin{cases} \mathbb{Z} & \text{if } i = n-1 \text{ or } 0 \\ 0 & \text{otherwise} \end{cases}} \cong \underbrace{H_i(S^{m-1})}_{\cong \begin{cases} \mathbb{Z} & \text{if } i = m-1 \text{ or } 0 \\ 0 & \text{otherwise} \end{cases}} \quad \forall i$$

$$\Rightarrow n = m \qquad \#$$

More homological alg: exact seq

Let (A, ∂^A), (B, ∂^B), (C, ∂^C) be chain complexes

$i : (A, \partial^A) \to (B, \partial^B)$, $\quad j : (B, \partial^B) \to (C, \partial^C)$

be chain maps. We say

$$0 \to (A, \partial^A) \to (B, \partial^B) \to (C, \partial^C) \to 0$$

is a short exact sequence of chain complexes if

$$0 \to A_n \xrightarrow{i_n} B_n \xrightarrow{j_n} C_n \to 0$$

is a short exact seq for each n

Thm 2.16

$\left\{ \begin{array}{l} j \circ i = 0 \\ \operatorname{im} i = \ker j \end{array} \right.$ $\quad i : 1\text{-}1 \quad \Uparrow \quad j : \text{onto}$

A short exact seq of cxes $\underbrace{0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0}_{\text{exact}}$ induces a long exact seq of homology gps

(*) $$\cdots \to H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{j_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \to$$
$$\xrightarrow{i_*} H_{n-1}(B) \to \cdots$$

where $\partial : H_n(C) \to H_{n-1}(A)$ is defined by "diagram chasing"

pf

① ∂ is well-def :

- If $b' \in B_n$ is another choice s.t. $j_n(b') = c$,

  then $b' - b \in \ker(j_n) = \operatorname{im}(i_n)$

  $\Rightarrow \exists!\ x \in A_n$ s.t. $i_n(x) = b' - b$

  $\Rightarrow i_{n-1}(a + \partial_n^A(x)) = \partial_n^B(b) + \partial_n^B(b' - b) = \partial_n^B(b')$

  $\Rightarrow \partial([c]) = [a + \partial^A(x)] = [a]$

$\partial([c]) = [a + \partial_n(x)] = [a]$

- If $c' \in \ker(\partial_n^C) \subseteq C_n$ s.t. $[c'] = [c]$, then

$c' = c + \partial_{n+1}^C(z)$ for some $z \in C_{n+1}$

$\exists y \in B_{n+1}$ s.t. $j_{n+1}(y) = z \Rightarrow j_n(b + \partial_{n+1}^B(y)) = c + \partial_{n+1}^C(z) = c'$

<u>Note</u> $\partial_n^B(b + \partial_{n+1}^B(y)) = \partial_n^B(b) + \underline{\partial\partial(y)}_{=0} = \partial_n^B(b)$

$\Rightarrow \partial([c']) = [a] = \partial([c])$

② ∂ is a gp homo: clear from def.

③ <u>exactness</u>:

⊛ is "chain ⇐ complex"

- $\operatorname{im} i_* \subseteq \ker j_*$: because $j_* \circ i_* = (j \circ i)_* = 0$
- $\operatorname{im} j_* \subseteq \ker \partial$: $\forall [b] \in H_n(B)$,

  $\partial(j_*([b])) = [a]$ s.t. $i_{n-1}(a) = \partial_n(b) = 0 \Rightarrow a = 0$
- $\operatorname{im} \partial \subseteq \ker i_*$: $\forall [c] \in H_n(C)$,

  $i_*(\partial([c])) = i_*([a]) = [\partial_n^B(b)] = 0$

- $\ker j_* \subseteq \operatorname{im} i_*$:

Let $[b] \in H_n(B)$ s.t. $j_*[b] = [j_n(b)] = 0$

i.e. $j_n(b) = \partial_{n+1}^C(z)$ for some $z \in C_{n+1}$

Let $y \in B_{n+1}$ s.t. $j_{n+1}(y) = z \Rightarrow j(\partial_{n+1}^B(y)) = \partial_{n+1}^C(z) = j_n(b)$

$\Rightarrow j_n(b - \partial_{n+1}^B(y)) = 0$

$\Rightarrow \exists x \in A_n$ s.t. $i_n(x) = b - \partial_{n+1}^B(y)$

$\Rightarrow i_*([x]) = [b - \partial_{n+1}^B(y)] = [b]$

$\Rightarrow [b] \in \operatorname{im} i_*$

$\Rightarrow \quad [b] \in \operatorname{im} i_*$

- $\ker \partial \subseteq \operatorname{im} j_*$:

  $\forall\ [c] \in \ker \partial \subseteq H_n(C) \Rightarrow \partial([c]) = [a] = 0$; $a = \partial_n^A(x)$ $\exists x \in A_n$

  $\Rightarrow \partial_n^B(i_n(x)) = i_{n-1}(\partial_n^A(x)) = i_{n-1}(a) = \partial_n^B(b)$

  $\Rightarrow \partial_n^B(b - i_n(x)) = 0 \Rightarrow [b - i_n(x)] \in H_n(B)$

  and $j_*([b - i_n(x)]) = [j_n(b) - \boxed{j_n i_n}^{\,0}(x)] = [j_n(b)] = [c]$

  $\Rightarrow [c] \in \operatorname{im}(j_*)$

- $\ker i_* \subseteq \operatorname{im} \partial$:

  Let $[a] \in \ker i_*$: $i_*([a]) = 0 \Rightarrow i_{n-1}(a) = \partial_n^B(b)$, $\exists b \in B_n$

  Let $c = j_n(b) \Rightarrow \partial_n^C(c) = \partial_n^C j_n(b) = j_{n-1}\partial_n^B(b)$

  $= j_{n-1} i_{n-1}(a) = 0 \Rightarrow \exists [c]$

  and $\partial([c]) = [a] \Rightarrow [a] \in \operatorname{im} \partial$

#

<u>Relative homology group</u>

Let $A \subseteq X \Rightarrow C_n(A)$ is a subgroup of $C_n(X)$

$C_n(A) \ni \sum n_i \sigma_i \longmapsto C_n(X)$

$\sigma_i : \Delta^n \to A \subseteq X$

Let

$$C_n(X, A) := C_n(X) / C_n(A)$$

Since $\partial_n^X(C_n(A)) \subseteq C_{n-1}(A)$, we have

$$\partial_n^{X,A} : C_n(X,A) \longrightarrow C_{n-1}(X,A)$$

$$C_n(X)/C_n(A) \ni [\sigma] \longmapsto [\partial_n^X(\sigma)] \in C_{n-1}(X)/C_{n-1}(A)$$

$(\partial^X)^2 = 0 \qquad (\partial^{X,A})^2$

$(\partial^X)^2 \Rightarrow (\partial^{X,A})^2 = 0$

$\Rightarrow$ We have a chain complex

$$\cdots \to C_n(X,A) \xrightarrow{\partial_n^{X,A}} C_{n-1}(X,A) \to \cdots$$

The associated homology group

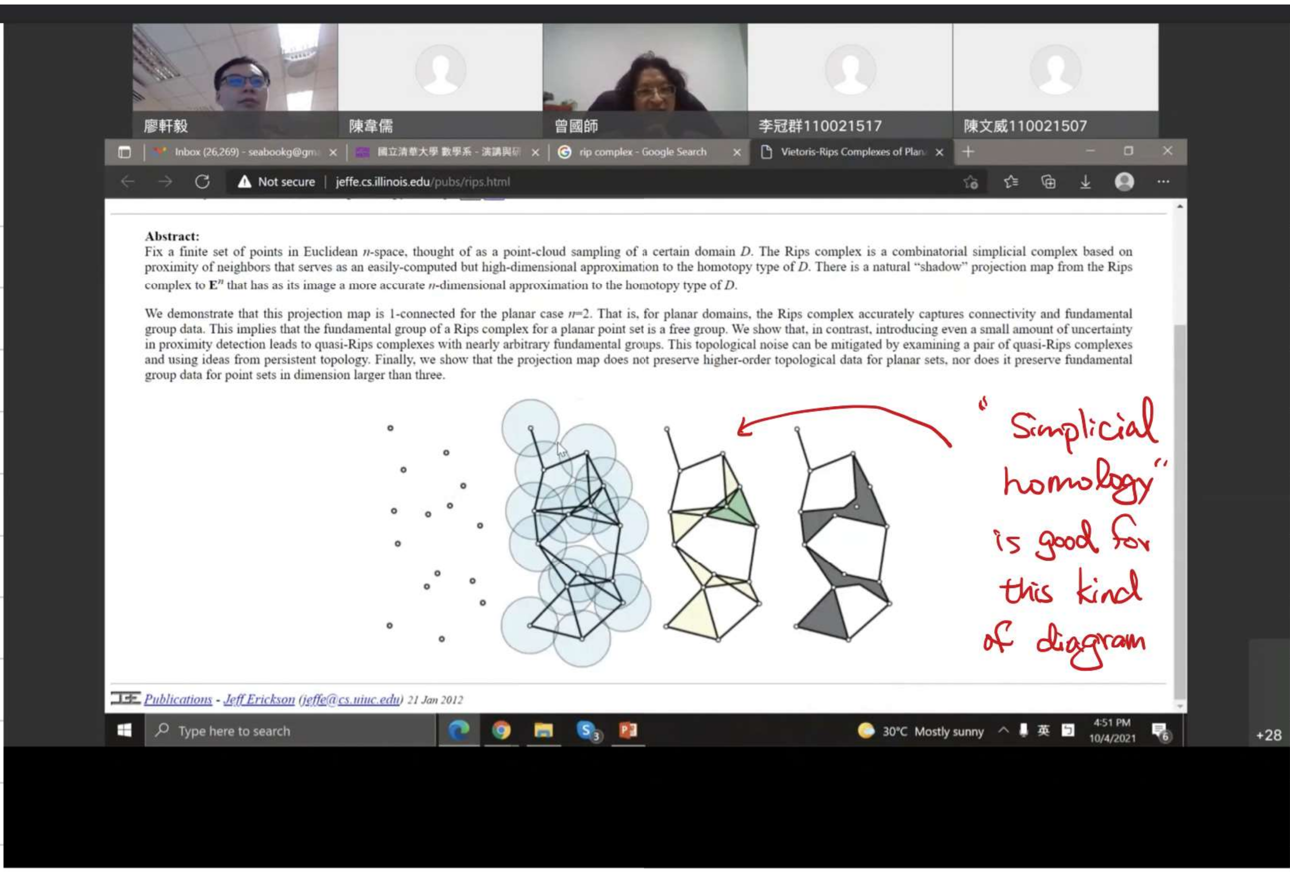
$$H_n(X,A) := \ker(\partial_n^{X,A}) / \operatorname{im}(\partial_{n+1}^{X,A})$$

is called the <u>relative homology group</u>.

$$0 \to (C_\bullet(A), \partial^A) \to (C_\bullet(X), \partial^X) \to \left(\underbrace{C_\bullet(X)/C_\bullet(A)}_{= C_\bullet(X,A)}, \partial^{X,A} \right) \to 0$$

is a short exact seq of cxes

$\overset{\text{Thm 2.16}}{\Rightarrow} \exists$ a long exact seq.

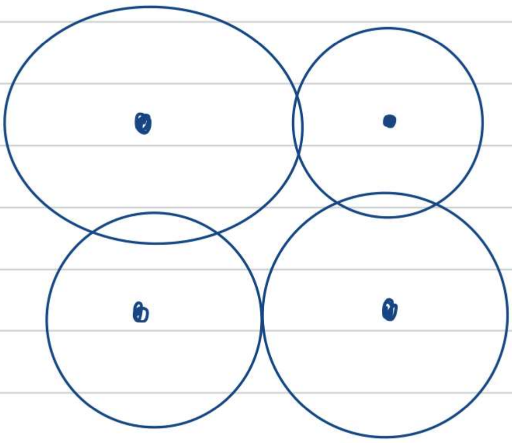


Abstract:
Fix a finite set of points in Euclidean n-space, thought of as a point-cloud sampling of a certain domain D. The Rips complex is a combinatorial simplicial complex based on proximity of neighbors that serves as an easily-computed but high-dimensional approximation to the homotopy type of D. There is a natural "shadow" projection map from the Rips complex to $\mathbf{E}^n$ that has as its image a more accurate n-dimensional approximation to the homotopy type of D.

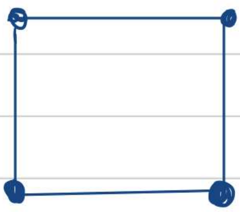
We demonstrate that this projection map is 1-connected for the planar case n=2. That is, for planar domains, the Rips complex accurately captures connectivity and fundamental group data. This implies that the fundamental group of a Rips complex for a planar point set is a free group. We show that, in contrast, introducing even a small amount of uncertainty in proximity detection leads to quasi-Rips complexes with nearly arbitrary fundamental groups. This topological noise can be mitigated by examining a pair of quasi-Rips complexes and using ideas from persistent topology. Finally, we show that the projection map does not preserve higher-order topological data for planar sets, nor does it preserve fundamental group data for point sets in dimension larger than three.

Publications - Jeff Erickson (jeffe@cs.uiuc.edu) 21 Jan 2012

①



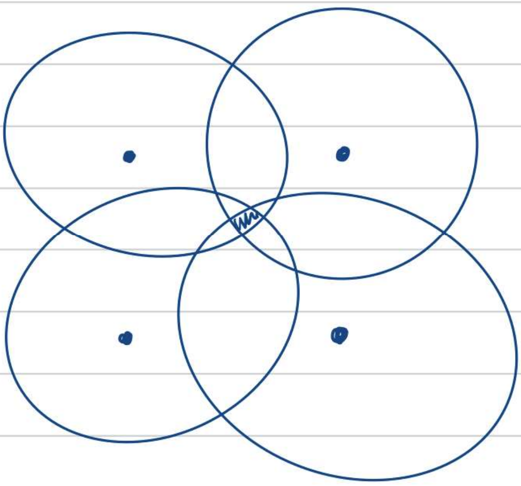
$\rightsquigarrow$ $X \sim$



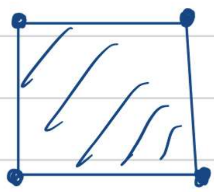
$\cong S^1$

$\Rightarrow \tilde{H}_1(X) \cong \mathbb{Z}$

②



$\rightsquigarrow$ $X' \sim$



$\sim \mathbb{R}^2$

$\Rightarrow \tilde{H}_1(X') = 0 \ncong \mathbb{Z}$

$$\partial : H_n(C) \longrightarrow H_{n-1}(A)$$

$\ker i_* \subseteq \operatorname{im} \partial$:

$\forall\ [a] \in \ker i_*$

$i_*[a] = 0$ in $H_{n-1}(B)$

$i_{n-1}(a) \in \operatorname{im}(\partial_n^B)$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & A_{n+1} \arrow[r, "\partial_{n+1}^A"] \arrow[d, "i_{n+1}"'] & A_n \arrow[r, "\partial_n^A"] \arrow[d, "i_n"'] & A_{n-1} \arrow[r] \arrow[d, "i_{n-1}"] & \cdots \\
\cdots \arrow[r] & B_{n+1} \arrow[r, "\partial_{n+1}^B"] \arrow[d, "j_{n+1}"'] & B_n \arrow[r, "\partial_n^B"] \arrow[d, "j_n"'] & B_{n-1} \arrow[r] \arrow[d, "j_{n-1}"] & \cdots \\
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}^C"'] \arrow[d] & C_n \arrow[r, "\partial_n^C"'] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

$a \in A_{n-1}$

$\exists\, b \mapsto i_{n-1}(a)$

$j_n(b) = c \mapsto j_{n-1} i_{n-1}(a) = 0$

$\Rightarrow\ [c] \in H_n(C) = \ker(\partial_n^C) / \operatorname{im}(\partial_{n+1}^C)$

$\partial([c]) = [a] \in \operatorname{im}(\partial)$