Ch2. Homology

π_1 is useful for study spaces of low dim

e.g. we used π_1 to prove $\mathbb{R}^2 \not\cong \mathbb{R}^n$ for $n \neq 2$

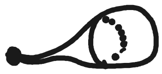
But π_1 is NOT enough to prove $\mathbb{R}^n \not\cong \mathbb{R}^m$ $\quad n \neq m$, $n, m > 2$

Q: what about high-dim spaces?

Remark

$\exists$ high-dim version of π_1:

$\pi_1 =$  / homotopy

higher π_1 (e.g. π_2) :  / homotopy

But computation of π_n are very difficult

Even $\pi_n(S^m)$ are highly nontrivial.

"difficult things" category of spaces $\longrightarrow$ category of groups "easier things"

- π_1 — NOT good for high dim sp
- π_n — too difficult
- (Co)homology : ① works for high-dim sp
  ② more managable than π_n

Remark

∃ many types of (co)homology such as

simplicial, singular (co)homology — for general topo. sp.

need "triagulization"

will focus on this

Čech, sheaf cohomology — usual in Alg. Geo.

de Rham cohomology — good for C^∞ manifolds

Lie alg. coh, Hochschild (co)homology, K-theory, etc.

$\langle x, f \rangle = f(x)$

$V \ni x$

$\mathrm{Hom}(V, \mathbb{R}) \ni f$ ← dual sp

$V \times \mathrm{Hom}(V, \mathbb{R}) \to \mathbb{R}$

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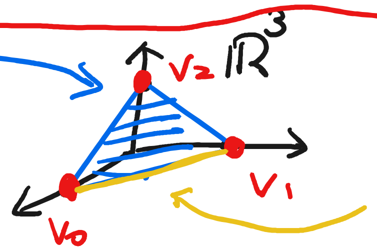
§2.1 Singular homology

The standard n-simplex Δ^n is

$$\boxed{\Delta^n := \{ (t_0, t_1, \cdots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} t_i = 1,\ t_i \geq 0 \ \forall i \}}$$

Δ^2



$[v_0, v_1, \hat{v_2}] = [v_0, v_1]$ a face

In textbook, a simplex with vertices $v_0, \cdots, v_n$ is denoted $[v_0, v_1, \cdots, v_n]$

$$\Delta^n = [v_0, v_1, \cdots, v_n]$$

where $v_i = (0, \cdots, 0, 1, 0, \cdots, 0)$ — $(i+1)$-th component

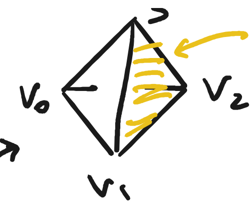
The simplex $[v_0, \cdots, \hat{v_i}, \cdots, v_n] = [v_0, \cdots, v_{i-1}, v_{i+1}, \cdots, v_n]$ — delete v_i

is called a *face* of $[v_0, \cdots, v_n]$

3-simplex :

solid

$[v_0, v_1, v_2, v_3]$



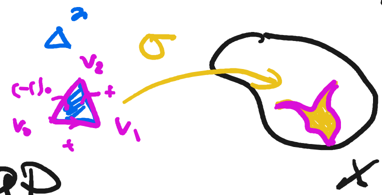
a face

$[v_1, v_2, v_3] = [\hat{v_0}, v_1, v_2, v_3]$

Let X be a topo. space.

A <u>singular n-simplex</u> in X is a continuous map

$$\sigma : \Delta^n \longrightarrow X$$



Let $C_n(X)$ be the free abelian gp generated by {singular simplexes in X}, i.e.

$$C_n(X) := \{ n_1\sigma_1 + \cdots + n_k\sigma_k \mid n_1, \cdots, n_k \in \mathbb{Z},\ \sigma_1, \cdots, \sigma_k : \Delta^n \to X \}$$

$\in \mathbb{Z} \leftarrow$ coefficient $\leftarrow$ "can be changed"

An element $\sum_{i=1}^{k} n_i \sigma_i \in C_n(X)$ is called a <u>(singular) n-chain</u>

$\sigma : \Delta^n \to X$

The <u>boundary map</u> $\partial_n : C_n(X) \to C_{n-1}(X)$ is defined by $\partial_n(\sum_j n_j \sigma_j) = \sum_j n_j \partial_n(\sigma_j)$, where

$$\partial_n(\sigma) := \sum_i (-1)^i \sigma|_{[v_0, \cdots, \hat{v_i}, \cdots, v_n]}$$

Here, $\sigma|_{[v_0, \cdots, \hat{v_i}, \cdots, v_n]} : \Delta^{n-1} \longrightarrow X$ is

$\subseteq \mathbb{R}^n$ $= \{ (t_0, \cdots, t_{n-1}) \in \mathbb{R}^n \mid t_0 + \cdots + t_{n-1} = 1 \}$

$$\sigma|_{[v_0, \cdots, \hat{v_i}, \cdots, v_n]}(t_0, \cdots, t_{n-1})$$
$$= \sigma(t_0, \cdots, t_{i-1}, 0, t_i, \cdots, t_{n-1}) \in X$$

Thus, we have

$$\cdots \to C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \to \cdots$$

singular chain complex

$$\cdots \xrightarrow{\partial_3} C_1(X) \xrightarrow{\partial_1} C_0(X) \to 0$$

<u>Lemma</u> (c.f. <u>Lemma 2.1</u>)

$$\partial_n \circ \partial_{n+1} = 0 \quad \forall n \geq 0$$

pf: computation (exer.)

$$\partial_1(\partial_2(\sigma)) = ? \qquad \sigma : \Delta^2 \to X, \quad \Delta^2 = [v_0, v_1, v_2]$$

$$\partial_2(\sigma) = \sigma|_{[v_1, v_2]} - \sigma|_{[v_0, v_2]} + \sigma|_{[v_0, v_1]}$$

$$\downarrow \partial_1 \qquad\qquad \downarrow \partial_1$$

$$\partial_1(\partial_2(\sigma)) = \cancel{\sigma(v_2)} - \cancel{\sigma(v_1)} - (\cancel{\sigma(v_2)} - \sigma(v_0)) + (\cancel{\sigma(v_1)} - \sigma(v_0)) = 0$$

Therefore,

$$\operatorname{im}(\partial_{n+1}) \trianglelefteq \ker(\partial_n)$$

The quotient group

$$H_n(X) := \ker(\partial_n) / \operatorname{im}(\partial_{n+1})$$

is called the (n-th) <u>singular homology group</u>

An element in $\ker(\partial_n)$ is called a (singular) <u>n-cycle</u>

" $\operatorname{im}(\partial_{n+1})$ " <u>n-boundary</u>

<u>Remark</u>

The data

$$\cdots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots$$

with the property $\partial_n \circ \partial_{n+1} = 0 \ \forall n$ (write $\partial \circ \partial = 0$ in short)

is called a chain complex

is called a <u>chain complex</u>

<u>Example (Prop 2.8)</u>

Let $X = *$ be a point. Then $\exists!$ continuous map $\Delta^n \longrightarrow X = * = $ contant map $= e_n$

$\Rightarrow C_n(X) = \mathbb{Z}\cdot\{e_n\} \cong \mathbb{Z}$

$n \cdot e_n \in C_n(X)$

$\Delta^{n+1} \to X = e_{n-1}$

<u>Note</u>: $\partial_n(e_n) = \sum_{i=0}^{n} (-1)^i \boxed{e_n|_{[v_0, \cdots, \hat{v_i}, \cdots, v_n]}}$

$= e_{n-1} - e_{n-1} + e_{n-1} - \cdots$

$= \begin{cases} 0 & \text{if } n \text{ is odd} \\ e_{n-1} & \text{if } n \text{ is even} \end{cases}$

So the singular chain complex is iso to

$\cdots \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \cdots \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$

$C_0(X)$

$\Rightarrow H_n(*) \cong \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases}$ #

<u>Remark (Prop 2.6)</u>

Let X be a space. If X_α are the path-connected components, then

$$H_n(X) \cong \bigoplus_\alpha H_n(X_\alpha)$$

pf: exer.

Therefore, we usually assume X is path-connected when we study generral theory of $H_n(X)$.

<u>Prop 2.7</u>

If X is nonempty and path-connected, then

$$H_0(X) \cong \mathbb{Z} \qquad \left(\Rightarrow H_0(\textstyle\coprod_\alpha X_\alpha) = \bigoplus_\alpha \mathbb{Z}\right)$$

pf

Note that since $\Delta^0 =$ a point $\to X$

$$C_0(X) = \left\{ \sum_{\text{finite}} n_i x_i \;\middle|\; x_i \in X \right\}$$

And we have the group homo (X is nonempty $\Rightarrow \varepsilon$ is onto)

$$\varepsilon : C_0(X) \longrightarrow \mathbb{Z} : \varepsilon\left(\sum n_i x_i\right) := \underline{\sum_i n_i} \in \mathbb{Z}$$

① For $\sigma : \Delta^1 \to X$, $\sigma(v_1) - \sigma(v_0)$

$$\varepsilon\, \partial_1(\sigma) = \varepsilon\Big(\underbrace{\sigma|_{[v_1]} - \sigma|_{[v_0]}}_{\sigma(v_1) - \sigma(v_0)}\Big) = 1 - 1 = 0$$

$$\Rightarrow \operatorname{im} \partial_1 \subseteq \ker \varepsilon$$

② $\forall \sum_{i=1}^{k} n_i x_i \in \ker \varepsilon$, we have $\underline{\sum_i n_i = 0}$

$$\Rightarrow \sum_{i=1}^{k} n_i x_i = n_1(x_1 - x_2) + (n_1+n_2)(x_2 - x_3)$$

$$+ (n_1+n_2+n_3)(x_3 - x_4) + \cdots$$

$$+ \boxed{(n_1 + n_2 + \cdots + n_k)}_{=0} \cdot x_k$$

$$= \sum_{i=1}^{k-1} (n_1 + \cdots + n_i)(x_i - x_{i+1})$$

Since X is path-conected, $\exists\, \sigma_i : \Delta^1 \longrightarrow X$ s.t. ($I \cong \Delta^1$)

$\sigma_i(1,0) = x_{i+1}$ $\quad \sigma_i(0,1) = x_i$ $\qquad$ $x_{i+1} \xrightarrow{\sigma_i} x_i$

$$\Rightarrow \partial_1(\sigma_i) = x_i - x_{i+1}$$

$$\Rightarrow \sum_{i=1}^{k} n_i x_i = \sum_{i=1}^{k-1} (n_1 + \cdots + n_i)\, \partial(\sigma_i) \in \operatorname{im}(\partial_1)$$

So, by ① + ②,

$\ker \varepsilon = \operatorname{im} \partial_1$

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$$\Rightarrow \quad \mathbb{Z} = \operatorname{im}(\varepsilon) \cong C_0(X)/\ker(\varepsilon) = C_0(X)/\operatorname{im}\partial_1 = H_0(X) \quad \#$$

<u>Def</u>

The <u>reduced homology group</u> $\tilde{H}_n(X)$ of X is

$$\tilde{H}_n(X) = \begin{cases} H_n(X) & \text{if } n \geq 1 \\ \ker(\varepsilon)/\operatorname{im}(\partial_1) & \text{if } n = 0 \end{cases} \qquad (\tilde{H}_0(X) \oplus \mathbb{Z} \cong H_0(X))$$

Some homological algebra

A <u>chain complex</u> is a seq. of abelian groups C_n together with a seq gp homo $\partial_n : C_n \to C_{n-1}$ s.t.

$$\partial_n \circ \partial_{n+1} = 0 \quad \forall n$$

Or equivalently, $C = \bigoplus_n C_n$, $\partial : C_\bullet \to C_{\bullet-1}$ s.t. $\partial^2 = 0$

The (n-th) <u>homology</u> of $(C_\bullet, \partial)$ is

$$H_n(C, \partial) := \ker(\partial_n)/\operatorname{im}(\partial_{n+1})$$

A <u>chain map</u> $\Phi : (C, \partial^C) \to (D, \partial^D)$ is a seq. of gp homo $\Phi_n : C_n \to D_n$ s.t.

$$\boxed{\Phi_{n-1} \circ \partial^C_n = \partial^D_n \circ \Phi_n} \quad \forall n$$

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial^C_{n+1}"] \arrow[d, "\Phi_{n+1}"'] \arrow[dr, phantom, "\circlearrowleft"] & C_n \arrow[r, "\partial^C_n"] \arrow[d, "\Phi_n"] & C_{n-1} \arrow[r] \arrow[d, "\Phi_{n-1}"] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r, "\partial^D_{n+1}"'] & D_n \arrow[r, "\partial^D_n"'] & D_{n-1} \arrow[r, "\partial^D_{n-1}"'] & \cdots
\end{tikzcd}$$

Commutative diagram

or we say this diagram <u>commutes</u>

<u>Prop</u>

A chain map $\Phi : (C, \partial) \longrightarrow (D, \partial)$ induces homomorphisms $\Phi_* : H_n(C, \partial^C) \longrightarrow H_n(D, \partial^D)$ $\forall n$

pf: exer.

Def.

We say two chain maps $\Phi, \Psi : (C, \partial^C) \to (D, \partial^D)$ are <u>(chain) homotopic</u> if $\exists$ seq. of homo $h_n : C_n \longrightarrow D_{n+1}$ s.t.

$$\boxed{\Phi_n - \Psi_n = \partial^D_{n+1} \circ h_n + h_{n-1} \circ \partial^C_n}$$

Such $h = (h_n)_n : C \to D$ is called a <u>chain homotopy</u>

$$\begin{tikzcd}
\arrow[r] & C_{n+1} \arrow[r] \arrow[d, "\Phi_{n+1}"'] & C_n \arrow[r, "\partial^C_n"] \arrow[dl, "h_n"'] \arrow[d, shift right, "\Phi_n"'] \arrow[d, shift left, "\Psi_n"] & C_{n-1} \arrow[r] \arrow[dl, "h_{n-1}"'] \arrow[d, "\Phi_{n-1}"] & {} \\
\arrow[r] & D_{n+1} \arrow[r] & D_n \arrow[r, "\partial^D_n"'] & D_{n-1} \arrow[r] & {}
\end{tikzcd}$$

Prop

If the chain map $\Phi, \Psi : (C, \partial^C) \to (D, \partial^D)$ are homotopic, then the induced maps

$$\Phi_* = \Psi_* : H_n(C) \longrightarrow H_n(D)$$

are equal $\forall n$

pf

Suppose $h = (h_n)$ is a chain homotopy.

$\forall [x] \in H_n(C) = \underset{\in x}{\underline{\ker(\partial^C_n)}} / \mathrm{im}(\partial^C_{n+1})$, $\partial^C_n(x) = 0$,

$[x] = x + \mathrm{im}(\partial^C_{n+1})$ $x \in$

we have

$\Phi_n(x) - \Psi_n(x)$

$= \partial^D(h_n(x)) + h_{n-1}(\boxed{\partial^C(x)})$ $\qquad 0 = \partial^C(x)$ $\qquad 0 + \mathrm{im}(\partial^D_{n+1})$

$$- \partial_{n+1}(h_n(x)) + h_{n-1}(\partial_n \cdots)$$

$$= \partial^D_{n+1}(h_n(x)) \in \mathrm{im}(\partial^D_{n+1}) \cancel{= 0 \text{ in } H_n(D)}$$

$$\Rightarrow [\Phi_n(x)] = [\Psi_n(x)] \text{ in } H_n(D) = \ker(\partial^D_n) / \mathrm{im}(\partial^D_{n+1}) \ \#$$

$$\Phi_n(x) + \mathrm{im}(\partial^D_{n+1}) = \Psi_n(x) + \mathrm{im}(\partial^D_{n+1}) = \{ y + \mathrm{im}(\partial^D_{n+1}) \}$$

$$\Leftrightarrow \Phi_n(x) - \Psi_n(x) \in \mathrm{im}(\partial^D_{n+1})$$