Algebraic Topology 9/23

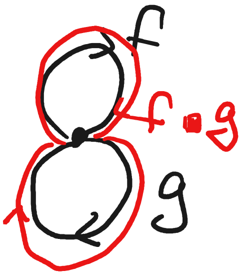
Recall :

Let X be a topological space, $x_0 \in X$.

$$\pi_1(X, x_0) = \{ f \mid f : \overset{[0,1]}{\overset{\shortparallel}{I}} \to X \text{ continuous}, \ f(0) = f(1) \} \Big/ \text{homotopy of paths}$$

fundamental group

$[f][g] = [f \cdot g]$,



If

$$\begin{tikzcd}(X, x_0) \arrow[r, bend left, "f"] & (Y, y_0) \arrow[r, bend left, "g"] & (Z, z_0)\end{tikzcd}$$,

$$\rightsquigarrow \quad \pi_1(X, x_0) \xrightarrow{f_*} \pi_1(Y, y_0) \xrightarrow{g_*} \pi_1(Z, z_0)$$

and $(g \circ f)_* = g_* \circ f_*$

$(id_X)_* = id_{\pi_1(X, x_0)}$

Fundamental group of S^1 :

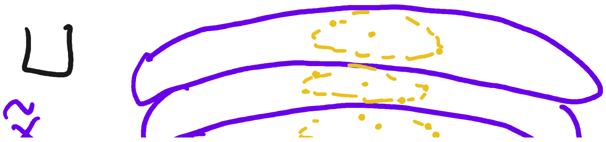
$\pi_1(\bigcirc) = ?$

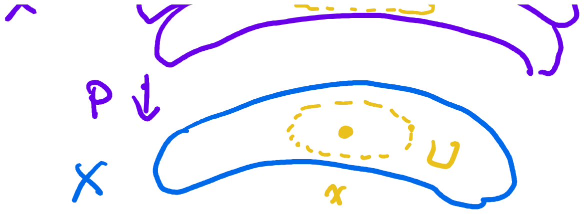
Use covering space to compute $\pi_1(S^1)$

(Recall: $\pi_1(●) = 0$, $\pi_1(\mathbb{R}^n) = 0$)

Def

Given a space X, a covering space of X consists of a space $\tilde{X}$ and a map $p : \tilde{X} \to X$ s.t. $\forall x \in X$, $\exists$ open neighborhood U of x in X s.t.





⊛ $$p^{-1}(U) = \coprod_{\lambda} U_\lambda \subseteq \tilde{X}$$

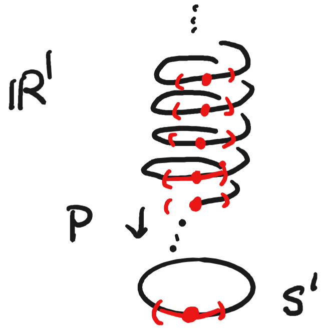
and $p|_{U_\lambda} : U_\lambda \to U$ is a homeomorphism for each λ

<u>Example</u>

① Let $p : \mathbb{R}^1 \to S^1 = \{(x, y) \mid x^2 + y^2 = 1\} \overset{\mathbb{R}^2}{\underset{\cup}{}}\!\!\subseteq \mathbb{R}^2$ be the map

$$p(s) := (\cos 2\pi s\ ,\ \sin 2\pi s)$$

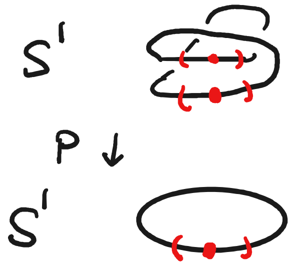
Then $(\mathbb{R}^1, p)$ is a covering space of S^1.



② Let $p : S^1 \to S^1 = \{z \in \mathbb{C} \mid |z|^2 = 1\}$

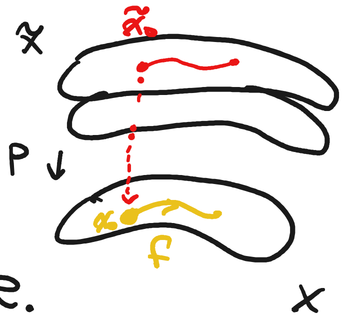
$$p(z) := z^2$$

Then (S^1, p) is a Covering space of S^1.



<u>Two properties about covering spaces</u>

(homopoty / path lifting property)



Let $p : \tilde{X} \to X$ be a covering space.

(a) For each path $f : I \to X$, $f(0) = x_0 \in X$, and each $\tilde{x}_0 \in p^{-1}(x_0)$, $\exists!$ lifted path $\tilde{f} : I \to \tilde{X}$ st

$f : I \longrightarrow X$ s.t.

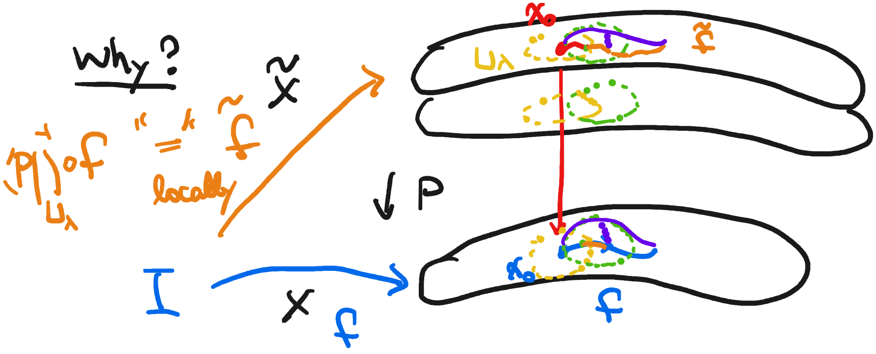
"we usually say "$\tilde f$ is a lift of f""

$$\underbrace{p\circ \tilde f = f}, \qquad \tilde f(0) = \tilde x_0$$

(b) For each homotopy $F : I\times I \longrightarrow X$ of paths starting at x_0, and each $\tilde x_0 \in p^{-1}(x_0)$, $\exists !$ lifted homotopy $\tilde F : I\times I \to \tilde X$ s.t.

$$p\circ \tilde F = F, \qquad \tilde F(0,t) \equiv \tilde x_0$$

Why?

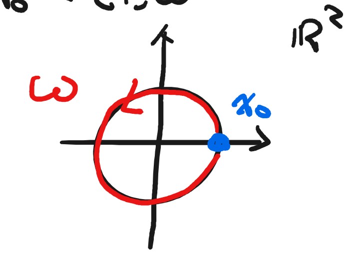


See §1.1 for details "property (a), (b), $\underline{\underline{(c)}}$"

Thm 1.7

$S^1 = \{x^2+y^2=1\} \subseteq \mathbb{R}^2$

$x_0 = (1,0)$



$$\pi_1(S^1) \cong \pi_1(S^1, x_0) \cong \mathbb{Z}$$

is generated by $[\omega]$, where

$$\omega(s) := (\cos 2\pi s, \sin 2\pi s)$$

is a loop based at x_0

pf

Let $\omega_n : I \to S^1$ be the loop ↙ 轉 n 圈 的 loop

$$\omega_n(s) := (\cos 2\pi n s, \sin 2\pi n s)$$

Step1: A loop based at x_0 is homotopic to ω_n for some n

Let $f: I \to S$ be a loop, $f(0) = f(1) = x_0$

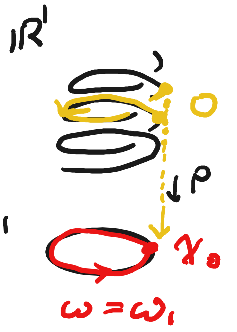
<u>Recall</u>

$p: \mathbb{R}^1 \to S^1$, $p(s) = (\cos 2\pi s, \sin 2\pi s)$

is a covering sp. $p(\underline{0}) = (1, 0) = x_0$

By (a), $\exists!\ \tilde{f}: I \to \mathbb{R}^1$ s.t.

$$p \circ \tilde{f} = f, \qquad \tilde{f}(0) = 0$$



Since $p(\tilde{f}(1)) = f(1) = x_0 = (1, 0)$, we have

$\tilde{f}(1) = n$ for some $n \in \mathbb{Z} \overset{\cdot}{=} p^{-1}(x_0) \subseteq \mathbb{R}^1$

Let $\tilde{\omega}_n : I \longrightarrow \mathbb{R}^1$, $\tilde{\omega}_n(s) = n \cdot s$

$\Rightarrow \quad p \circ \tilde{\omega}_n = \omega_n, \quad \tilde{\omega}_n(1) = n \cdot 1 = n = \tilde{f}(1)$

(Also, $\tilde{\omega}_n(0) = 0 = \tilde{f}(0)$)

Furthermore, note that

$$\tilde{f} \simeq \tilde{\omega}_n \quad \text{in } \mathbb{R}^1$$

<u>Note:</u> We actually use "$\pi_1(\mathbb{R}^1) = 0$"

(homotopy: $F(s,t) := (1-t)\tilde{f}(s) + t\,\tilde{\omega}_n(s)$)

$$\Rightarrow \quad f = p \circ \tilde{f} \simeq p \circ \tilde{\omega}_n = \omega_n \quad \text{in } S^1$$

$p \circ F$ is a homotopy

i.e.

$$[f] = [\omega_n] \quad \text{in } \pi_1(S^1, x_0)$$

So $\pi_1(S^1, x_0) = \{ [\omega_n] \mid n \in \mathbb{Z} \}$

$\left(\begin{array}{l} [\omega_n] \overset{?}{\neq} [\omega_m] \\ \overset{?}{\cong} \mathbb{Z} \end{array} \right)$

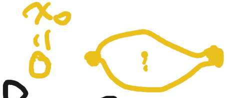
<u>Step2</u> $[\omega_n] \neq [\omega_m]$ in $\pi_1(S^1, x_0)$ if $n \neq m$

Suppose $\omega_n \simeq \omega_m$ and $H: I \times I \to S^1$ is a homotopy.

By (b), $\exists!$ path homotopy $\tilde{H}: I \times I \longrightarrow \mathbb{R}^1$ s.t.

$p\circ\tilde H = H$. $\tilde H(0,t) \equiv 0$ (Let $\tilde x_1 = \tilde H(1,t)$)

$\tilde x_0$ $\overset{\shortparallel}{0}$



$\tilde x_1$ $\overset{\uparrow}{\text{indep of } t}$

$\Rightarrow$ $\tilde H(-,0) : I \to \mathbb R$ is a lift of ω_n

$\tilde H(-,1) : I \to \mathbb R$ is a lifte of ω_m

same end : $\tilde H(1,0) = \tilde H(1,1) = \tilde x_1$

By uniqueness of (a),

$$\tilde H(-,0) = \tilde\omega_n$$
$$\tilde H(-,1) = \tilde\omega_m$$

So $n = \underset{\overset{\shortparallel}{n\cdot 1}}{\tilde\omega_n(1)} = \tilde H(1,0) = \tilde x_1 = \tilde H(1,1)$

$= \tilde\omega_m(1) = m$

$$\boxed{\begin{array}{l}\text{So by step 1+2.}\\ \pi_1(S^1) \longleftarrow \mathbb Z\\ [\omega_n] \longleftarrow\!\shortmid\ n\\ \text{is a bijection}\end{array}}$$

<u>Step3</u> $[\omega_n][\omega_m] = [\omega_{n+m}]$

Let $\tilde\omega(s) := \begin{cases} n\cdot 2s, & 0 \le s \le \frac12 \\ n + m(2s-1), & \frac12 \le s \le 1 \end{cases}$ $\leftarrow$ path in $\mathbb R$

$\tilde\omega(1) = n+m$

$\Rightarrow$ $p\circ\tilde\omega = \omega_n \blacksquare \omega_m$

and $\tilde\omega \simeq \tilde\omega_{n+m}$ in $\mathbb R$

$\Rightarrow$ $\omega_n \blacksquare \omega_m = p\circ\tilde\omega \simeq p\circ\tilde\omega_{n+m} = \omega_{n+m}$ #

<u>Applications:</u> can prove a few famous thms

<u>Thm 1.8</u> (Fundamental Thm of Algebra)

Every nonconstant polynomial with coefficients in $\mathbb C$ has a root in $\mathbb C$

<u>pf</u>

Assume $p(z) = z^n + a_1 z^{n-1} + \cdots + a_n$ has NO roots in $\mathbb C$

$\Rightarrow \forall r > 0,$

$$f_r(s) := \frac{p(re^{2\pi i s})\,/\,p(r)}{|p(re^{2\pi i s})\,/\,p(r)|} : I \to S^1 \subseteq \mathbb{C}$$

↻ fix r

defines a loop in S^1 based at 1.

Note:

① $\forall r, \quad f_r(s) \simeq f_0(s) \equiv 1 = \omega_0$ in S^1

② If $r > \max\{1, |a_1| + \cdots + |a_n|\}$, then

$f_r(s) \simeq \omega_n(s)$ is S^1

because

$$H(s,t) := \frac{p_t(re^{2\pi i s})/p_t(r)}{|p_t(re^{2\pi i s})/p_t(r)|}$$ is a homotopy in S^1

where $p_t(z) = z^n + t(a_1 z^{n-1} + \cdots + a_n)$

check: $p_t(re^{2\pi i s}) \neq 0$

So $\omega_0 \simeq f_r \simeq \omega_n \overset{\text{Thm 1.7}}{\Longrightarrow} 0 = n$, i.e., p is a constant #

some large $r > \max\{\dots\}$

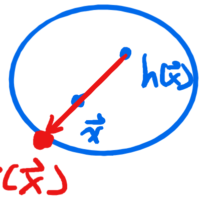
Thm 1.9 (A fixed point thm)

Any continuous map $h : D^2 = \{x^2 + y^2 \le 1\} \to D^2$ $\left(\overset{\mathbb{R}^2}{\cup\!\!\text{I}}\right)$ has a fixed point. (i.e. $h(\vec{x}) = \vec{x}$ has a sol in D^2)

pf

Assume $h(\vec{x}) \neq \vec{x} \quad \forall \vec{x} \in D^2$.

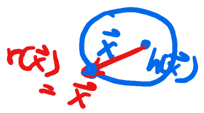
Define $r : D^2 \to S^1$ by



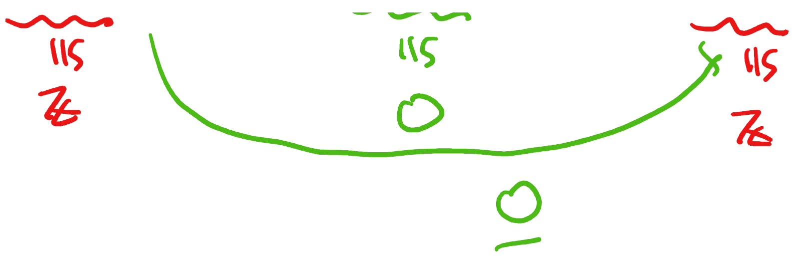
exer: r is continuous if $h(\vec{x}) \neq \vec{x} \ \forall \vec{x} \in D^2$

Note: (Let $i : S^1 \to D^2$ be the inclusion)

$r \circ i = id_{S^1}$



$$\Rightarrow \begin{tikzcd} \pi_1(S^1) \arrow[r, "i_*"] \arrow[rr, bend left, "id"] & \pi_1(D^2) \arrow[r, "r_*"] & \pi_1(S^1) \end{tikzcd}$$



$(\longrightarrow \longleftarrow)$

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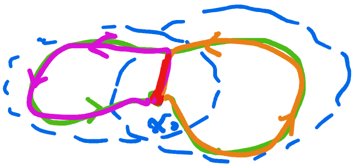
Fundamental group of S^n, $n \geq 2$

Lemma 1.15

Let X be a space, A_α be open subsets in X.



$X = \bigcup_\alpha A_\alpha$

Suppose

(i) $x_0 \in \bigcap_\alpha A_\alpha$

(ii) A_α are path-connected ∀ α

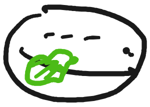
(iii) each intersection $A_\alpha \cap A_\beta$ is path-connected

Then every loop in X based at x_0 is homotopic to a product of loops each of which is contained in a single A_α

cf. §1.2 Van Kampen's thm

Prop 1.14

$\pi_1(S^n) = 0$ if $n \geq 2$



pf $\{x_1^2 + \cdots + x_{n+1}^2 = 1\} \subseteq \mathbb{R}^{n+1}$

$\quad ||$

$S^n = A_1 \cup A_2$

$A_2 = S^n \setminus \{(-1, 0, \cdots, 0)\}$

$A_1 = S^n \setminus \{(1, 0, \cdots, 0)\}$

$A_1 \cap A_2$ is path-connected

$\Rightarrow A_1 \cong A_2 \cong \mathbb{R}^n \Rightarrow \underline{\pi_1(A_1) = \pi_2(A_2) = 0}$

∀ f loop in S^n based at $x_0 = (0, \cdots, 0, 1)$.

by Lemma 1.15, ∃ f_1 loop in A_1, f_2 loop in A_2 s.t.

$$f \simeq \underbrace{f_1}_{\simeq \; c_{x_0}} \cdot \underbrace{f_2}_{\simeq \; c_{x_0}} \simeq c_{x_0}$$

#

Cor 1.16

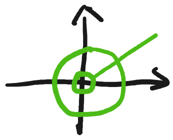
$\mathbb{R}^n$ is NOT homeomorphic to $\mathbb{R}^2$ for $n \neq 2$

pf

Suppose $f : \mathbb{R}^2 \to \mathbb{R}^n$ is a homeomorphism

$$\Rightarrow f|_{\mathbb{R}^2\setminus\{\vec{0}\}} : \underbrace{\mathbb{R}^2\setminus\{\vec{0}\}}_{\cong\, S^1 \times \mathbb{R}^1} \to \underbrace{\mathbb{R}^n\setminus\{f(\vec{0})\}}_{\cong\, S^n \times \mathbb{R}^1}$$

still a homeomorphism $\Rightarrow$ iso π_1



$$\pi_1(S^1\times\mathbb{R}^1) = \pi_1(S^1)\times\pi_1(\mathbb{R}^1) \cong \mathbb{Z}$$

, $\pi_1 = 0$ if $n \neq 2$

$\pi_1 = 0$

$\rightarrow\leftarrow$

#

Covering spaces v.s. fundamental group (§1.3)

If X is a reasonable space, then $\exists!$ a universal cover $\tilde{X}$ of X

see i.e. a simply-connected covering sp. of X

$$\begin{tikzcd}
 & \tilde{X} \arrow[d] \arrow[r, leftrightarrow] & (p_*\pi_1(\tilde{X})=)0 \arrow[d, phantom, "\rotatebox{-90}{$\leq$}"] \\
\text{some covering sp} \arrow[r] & X' \arrow[d] \arrow[r, leftrightarrow] & (p_*\pi_1(X')=)H \arrow[d, phantom, "\rotatebox{-90}{$\leq$}"] \\
 & X \arrow[r, leftrightarrow] & \pi_1(X)
\end{tikzcd}$$

Similar to Galois correspondence

" 1-1 correspondence "