Algebraic Topology (Fall 2021, NTHU, 9/16)

Need: abelian group, e.g. $\mathbb{Z}$, $\mathbb{Z}/n\mathbb{Z}$

linear algebra

Introduction

Main object to study: (topological) space

Examples of topological spaces:

metric space,  $= S^1$,  $= S^2$,

, $\mathbb{R}^n$, (finite-dim) vector space (over $\mathbb{R}$ or $\mathbb{C}$),

manifold, variety, scheme

Basic idea of main methods:

difficult things e.g. spaces, ring, ... $\xrightarrow{\text{some construction}}$ easier things e.g. group, vector sp, ... numbers

more operations here

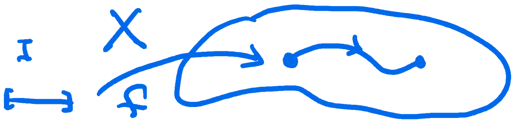
e.g. fundamental group, homology, cohomology

usually lose some information

usually category → functor → category

§1.1 Fundamental group ⇝ use "paths" in a space

Let X be a topo. sp.

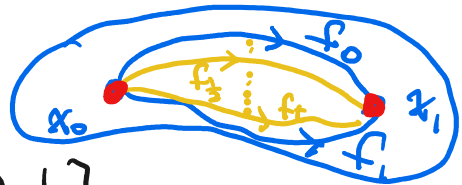


A **path** in X is a (continuous) map

$$f : I = [0,1] \longrightarrow X$$

A **homotopy of paths** in X is a continuous map $F : I \times I \longrightarrow X$ s.t.

- $F(0,t) = x_0, \ \forall t \in [0,1]$
- $F(1,t) = x_1, \ \forall t \in [0,1]$



Sometimes, we write $f_t(-) = F(-,t)$

If $\exists$ such a F, we say the paths

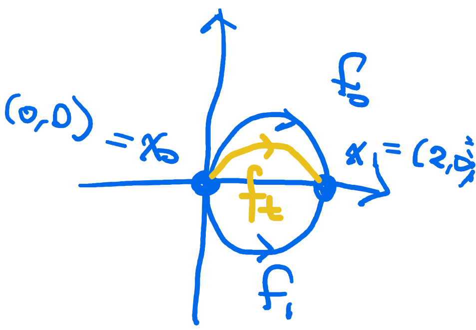
$$f_0 := F(-,0) \quad , \quad f_1 := F(-,1)$$

are **homotopic**, denoted $f_0 \simeq f_1$

Example 1.1

$X = \mathbb{R}^2$ (standard basis), $\quad f_0(s) = (2s-1, 1-(1-s)^2)$

$\qquad\qquad f_1(s) = (2s-1, -1+(1-s)^2)$

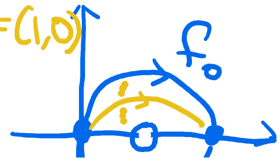


$$F(s,t) = (2s, \underline{(1-t)(1-(1-s)^2) + t(-1+(1-s)^2)})$$

$$[0,1] \times [0,1] \longrightarrow \mathbb{R}^2$$

← the line between $(1-t)(1-(1-s)^2) + t(-1+(1-s)^2)$

Note: if $\tilde{X} = \mathbb{R}^2 \setminus \{(1,0)\}$, $\ast\ F(\frac{1}{2},\frac{1}{2}) = (1,0)$



then F does NOT map into $\tilde{X}$

In fact, f_0 and f_1 are NOT homotopic in $\tilde{X}$ ← depends on where the paths are

Prop 1.2

The relation of homotopy on path is an equivalence relation

pf: exercise

Def

The equivalence class of a path f under homotopy is called the homotopy class of f, denoted $[f]$.

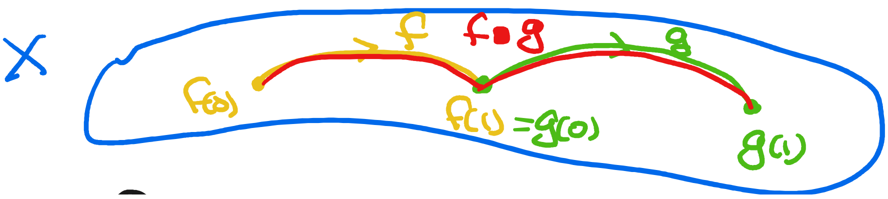
Given two paths $f, g : I \to X$ s.t.

$f(1) = g(0)$.

there is a composition path (or called product path) $f \cdot g : I \to X$ defined by

$$(f \cdot g)(s) := \begin{cases} f(2s), & 0 \le s \le \frac{1}{2}, \\ g(2s-1), & \frac{1}{2} \le s \le 1 \end{cases}$$

continuous



A path f is called loop if $f(0) = f(1)$ $\quad f(0)=f(1)$

Define $\pi_1(X, x_0) := \{ [f] \mid f: I \to X,\ f(0) = f(1) = x_0 \}$

— the fundamental group of X at the basepoint x_0

Prop 1.3

$\pi_1(X, x_0)$ with the product

$$[f][g] := [f \bullet g]$$

is a group

pf

① well-defined:

If $[f_0] = [f_1]$, $[g_0] = [g_1]$

i.e. $\exists F: I \times I \to X$,
$f_0 = F(-, 0)$, $f_1 = F(-, 1)$
$F(1, t) \equiv F(0, t) \equiv x_0$

i.e. $\exists G: I \times I \to X$,
s.t. $g_0 = G(-, 0)$, $g_1 = G(-, 1)$
$G(1, t) \equiv G(0, t) \equiv x_0$

then $[f_0][g_0] = [f_1][g_1]$

because

$$H: I \times I \to X,\quad H(s, t) := \begin{cases} F(2s, t), & 0 \le s \le \frac{1}{2} \\ G(2s-1, t), & \frac{1}{2} \le s \le 1 \end{cases}$$

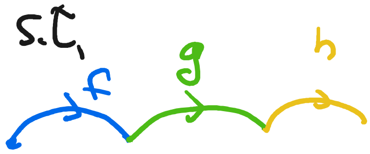
is a homotopy between $f_0 \bullet g_0$ and $f_1 \bullet g_1$

② associativity:

Lemma

Let $f, g, h: I \to X$ be paths s.t.

$f(1) = g(0)$, $g(1) = h(1)$



Then $\overset{[-|-|---] \to X}{(f \bullet g) \bullet h} \simeq \overset{[---|-|-] \to X}{f \bullet (g \bullet h)}$

pf:

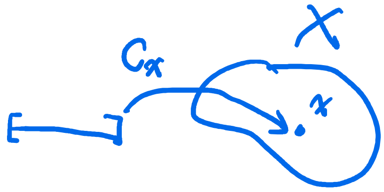
Define

$$H(s,t) := \begin{cases} f((1-t)4s + t \cdot 2s), & 0 \le s \le (1-t)\frac{1}{4} + t \cdot \frac{1}{2} \\ g((1-t)\cdot(4s-1) + t(4s-2)), & (1-t)\frac{1}{4} + t\cdot\frac{1}{2} \le s \le (1-t)\frac{1}{2} + t\frac{3}{4} \\ h((1-t)(2s-1) + t(4s-3)), & (1-t)\frac{1}{2} + t\cdot\frac{3}{4} \le s \le 1 \end{cases}$$

$\exists$ path for each t

exer: Check H is a homotopy between $(f \bullet g) \bullet h$ and $f \bullet (g \bullet h)$

③ identity:

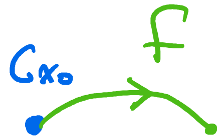


Lemma

Let $f : I \to X$ be any path.

$C_x : I \to X$ be the constant $C_x(s) = x$ $\forall s \in I$

If $x_0 = f(0)$, $x_1 = f(1)$, then (in π_1)



$$C_{x_0} \bullet f \simeq f$$

$$f \bullet C_{x_1} \simeq f$$

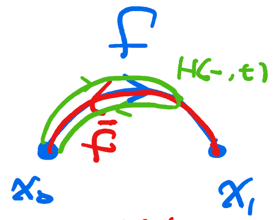
pf: similar to ②.

④ inverse:

Lemma

For a path $f : I \to X$, define

$$\bar{f}(s) := f(1-s)$$



Then $f \bullet \bar{f} \simeq C_{x_0}$ because $H(s,t) := \begin{cases} f(\overset{2s(1-t)}{\cancel{1-2s(1-t)}}) & 0 \le s \le \frac{1}{2} \\ f((2-2s)(1-t)) & \frac{1}{2} \le s \le 1 \end{cases}$

$\bar{f} \bullet f \simeq C_{x_1}$ is a homotopy similar reason

In particular, for $[f] \in \pi_1(X, x_0)$
$$[f]^{-1} = [\bar{f}] \qquad \#$$

<u>Remark</u>

The proof of <u>Prop 1.3</u> actually implies that the data

not necessarily loop

target

source

$$\begin{tikzcd} \Gamma_1 = \{ [f] \mid f : I \to X \} \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \\ \Gamma_0 = X \arrow[u, bend right, "\varepsilon"'] \end{tikzcd} \qquad \varepsilon(x) := [c_x]$$

$s([f]) := f(0)$

$t([f]) := f(1)$

forms a $\boxed{\text{groupoid}}$, called the <u>fundamental groupoid</u> of X

That is, a product $[f][g]$ is defined when $t([f]) = s([g])$ and it satisfies

- $([f][g])[h] = [f]([g][h])$ if $t([f]) = s([g])$, $t([g]) = s([h])$
- $[f]\,\varepsilon(t([f])) = [f] = \varepsilon(s([f]))\,[f]$
- $\forall [f]$, $\exists [f]^{-1} (= [\bar{f}])$ s.t.
  - $s([f]^{-1}) = t([f])$, $t([f]^{-1}) = s([f])$
  - $[f]^{-1} \cdot [f] = \varepsilon(t([f]))$, $[f] \cdot [f]^{-1} = \varepsilon(s([f]))$

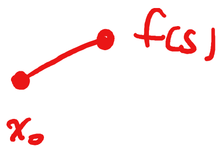
<u>Example 1.4</u>

$$D^2 = \{ x^2 + y^2 \le 1 \} \subseteq \mathbb{R}^2$$

For any loop $f : I \to D^2$, the map

let $x_0 = f(0) = f(1)$

$F(s,t) := (1-t)f(s) + t x_0$



is a homotopy between f and C_{x_0}

$\Rightarrow \quad [f] = [\underbrace{C_{x_0}}] \quad$ in $\pi_1(D^2, x_0)$ $\leftarrow$ identity

trivial group

$\Rightarrow \quad \pi_1(D^2, x_0) = 0$

Similarly, $\pi_1\left(\begin{array}{c}\text{any convex}\\ \text{subset in } \mathbb{R}^n\end{array}, \underline{\underline{x_0}}\right) = 0$

In particular, $\pi_1(\mathbb{R}^n) = 0$

Prop 1.5

Let $x_0, x_1 \in X$. Suppose $\exists$ path $h : I \to X$ s.t.

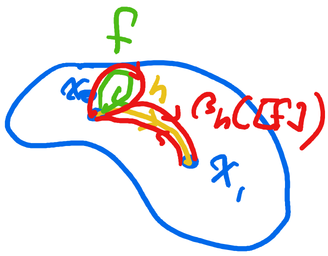
$h(0) = x_0 \quad , \quad h(1) = x_1$

Then $\quad \pi_1(X, x_0) \cong \pi_1(X, x_1)$.

In particular, if X is path-connected, then

$\pi_1(X, x_0) \cong \pi_1(X, x_1) \quad \forall\, x_0, x_1 \in X$

pf



The map

$\beta_h : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1)$:

$\beta_h([f]) := [h \bullet f \bullet \bar{h}]$

is a group isomorphism

#

Def

A space X is <u>simply-connected</u> if

X is path-connected and $\pi_1(X) = 0$

Prop 1.12

If X, Y are path-connected, then

$$\pi_1(\underbrace{X \times Y}_{\text{product space}}) \cong \pi_1(X) \times \pi_1(Y)$$

(e.g. $\mathbb{R}\times\mathbb{R} = \mathbb{R}^2$)

(e.g. $\pi_1(D^2 \times \mathbb{R}) = 0$)

pf

$$\begin{tikzcd} X \times Y \arrow[r, "pr_Y"] \arrow[d, "pr_X"] & Y \\ X & \end{tikzcd}$$

You can check:

$$\pi_1(X \times Y) \longrightarrow \pi_1(X) \times \pi_1(Y)$$
$$[f] \longmapsto ([pr_X \circ f], [pr_Y \circ f])$$

is a group iso. $\square$

Induced homomorphisms

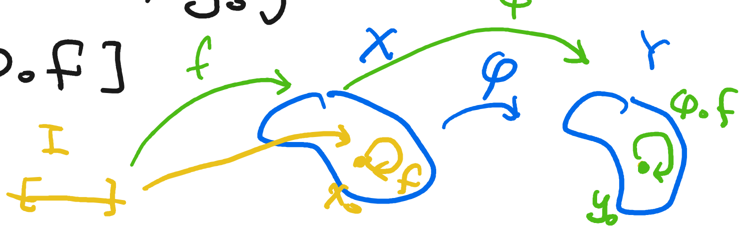
Let $\varphi : X \to Y$ be a continuous map s.t.

$\varphi(x_0) = y_0$

We also write $\varphi : (X, x_0) \longrightarrow (Y, y_0)$

Then we have the induced group homomorphism

$$\varphi_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$$
$$\varphi_*([f]) := [\varphi \circ f]$$



exer: φ_* is a group homomorphism

Moreover, if $\psi : Z \to X$ is another continuous map

- $(\varphi \circ \psi)_* = \varphi_* \circ \psi_*$
- $(id_X)_* = id_{\pi_1(X, x_0)}$

X, Y: path-connected

$$Z \xrightarrow{\psi} X \xrightarrow{\varphi} Y$$
$$\Downarrow$$
$$\pi_1(Z, z_0) \xrightarrow{\psi_*} \pi_1(X, x_0) \xrightarrow{\varphi_*} \pi_1(Y, y_0)$$

That is,

$\Rightarrow$ $X \cong Y$ $\Rightarrow$ $\pi_1(X) \cong \pi_1(Y)$

homoemorphic

<u>Prop</u>

π_1 is a functor from the category of pointed spaces to the category of groups