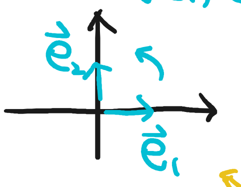
Algebraic Topology 1/6

Orientation

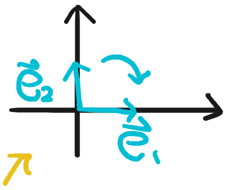
Let $\mathcal{B}(\mathbb{R}^n)$ be the set of ordered bases in $\mathbb{R}^n$.
We say $(\vec{v}_1, \cdots, \vec{v}_n)$, $(\vec{w}_1, \cdots, \vec{w}_n) \in \mathcal{B}(\mathbb{R}^n)$ have the same orientation (equivalence relation "$\sim$") if the matrix of changing coordinates has $\det > 0$

e.g. $\mathbb{R}^2$

$\{\vec{e}_1, \vec{e}_2\}$ $\xrightarrow{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$ $\{\vec{e}_2, \vec{e}_1\}$



$\det\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -1 < 0$



different orientations

Note that $\mathcal{B}(\mathbb{R}^n)/\sim$ has 2 classes.

A class in $\mathcal{B}(\mathbb{R}^n)/\sim$ is called an orientation

Recall that a homeomorphism $\Delta^n \to D^n$ induces a generator of $H_n(D^n, \partial D^n)$

inclusions are resp. homotopy equi $\xrightarrow{\cong} H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{\vec{0}\}) \cong \mathbb{Z}$

So for any $\sigma : \Delta^n \hookrightarrow \mathbb{R}^n$, $x \in \mathrm{int}(\mathrm{im}(\sigma))$, the class

$$[\sigma] \in H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) \xleftarrow[\sigma_*]{\cong} H_n(\Delta^n, \partial\Delta^n) \ni [\mathrm{id}] \qquad \mathrm{id} : \Delta^n \to \Delta^n$$

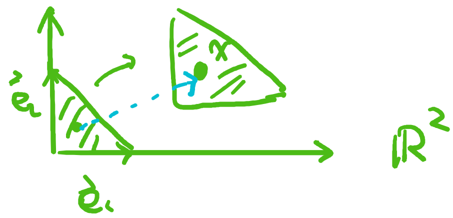
is a generator

Therefore, a choice of ordered basis $\beta = (\vec{v}_1, \cdots, \vec{v}_n)$ in $\mathbb{R}^n$ $\subset \mathbb{R}^{n+1}$

$\rightsquigarrow \sigma_\beta : \Delta^n \hookrightarrow \mathbb{R}^n : (t_0, \cdots, t_n) \mapsto \sum_{i=1} t_i \vec{v}_i + (x - \frac{\sum v_i}{n+1})$ ← check

$\rightsquigarrow$ we have a generator

$$[\sigma_\beta] \in H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\})$$



exer

$\beta \sim \gamma \Leftrightarrow [\sigma_\beta] = [\sigma_\gamma]$

Def

dim $M = n$

A local orientation of a mfd M at $x \in M$ is a choice of generator μ_x of $H_n(M, M \setminus \{x\})$

$\cong$ excision

$H_n(U, U \setminus \{x\})$

$\cong$

$H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\})$

$\cong$

$\mathbb{Z}$

An orientation of M is a function $M \ni x \longmapsto \mu_x \in H_n(M, M \setminus \{x\})$ s.t. $\forall x \in M$ $\exists$ nbd $\overset{\mathbb{R}^n \cong}{U} \subseteq M$ containing an open ball $B(x; r) = B \subseteq \mathbb{R}^n$ s.t. $\exists$ generator $\mu_B \in H_n(M, M \setminus B) \cong H_n(\mathbb{R}^n, \mathbb{R}^n \setminus B) \cong \mathbb{Z}$ with the property

$$\begin{array}{ccc} H_n(M, M \setminus B) & \xrightarrow{\cong} & H_n(M, M \setminus \{y\}) \\ \cup\!\!| & & \cup\!\!| \\ \mu_B & \longmapsto & \mu_y \end{array} \quad \forall y \in B$$

If an orientation exists, then M is orientable

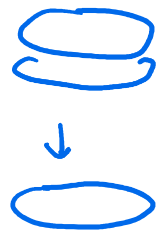
Example

- $\mathbb{R}^n$ is orientable
- Mobius band is NOT orientable

Remark (Prop 3.25)

$$\tilde{M} := \{\mu_x \mid \mu_x \text{ is a local orientation of } M \text{ at } x\}$$

$$\begin{array}{ccc} \tilde{M} & & \mu_x \\ \downarrow & & \downarrow \\ M & & x \end{array}$$



is a 2-fold covering space.

And M is orientable $\Leftrightarrow$ $\tilde{M}$ has 2 connected components.

Replacing the coeff gp by R (← will assume commutative with 1_R), one can get definitions of "R-orientation" and "R-orientable"

Thm 3.26

Let M be a closed (= compact, without boundaries; e.g.) connected n-mfd. Then

(a) If M is R-orientable, then the map
$$H_n(M;R) \longrightarrow H_n(M, M \setminus \{x\}; R) \cong R$$
is an iso $\forall x \in M$

(b) If M is NOT R-orientable, the map
$$H_n(M;R) \longrightarrow H_n(M, M\setminus\{x\}; R) \cong R$$
is 1-1, with image $\{r \in R \mid 2r = 0\}$ $\forall x \in M$

(c) $H_i(M;R) = 0 \quad \forall i > n$

$$\boxed{\begin{array}{l} \text{So} \\ \quad M \text{ is orientable} \Leftrightarrow H_n(M) \cong \mathbb{Z} \\ \quad \uparrow \qquad\qquad\qquad\quad (\text{otherwise } = 0) \\ \quad \text{connected, closed} \end{array}}$$

Def

An element of $H_n(M;R)$ whose image in $H_n(M, M-\{x\};R)$ is a generator $\forall x \in M$ is called a **fundamental class** for M (or orientation class)

μ_x

" $\|$ "

$[M]$

Cor (p236)

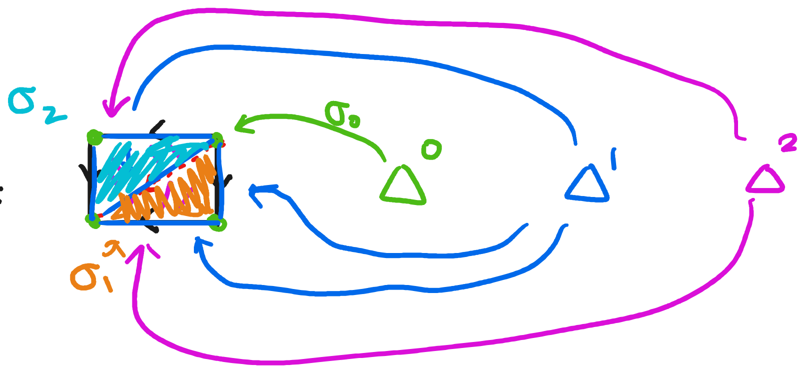
A fundamental class exists iff M is closed and R-orientable

Construction of fundamental class

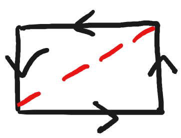
See Def on p.103 for the definition of Δ-complex

Example

① torus $T =$



② Klein bottle $K =$



Remark

- Any Δ-complex is a CW complex
- [illegible] in ① is still NOT a honest Δ

$\rightsquigarrow$ see def of simplicial complex

honest Δ

Suppose M has a structure of Δ-complex

Let $\sigma_1, \cdots, \sigma_m$ be n-simplexes of M.

<u>Prop</u> (p 228)

M is orientable iff one can chose $k_i = \pm 1$

s.t.

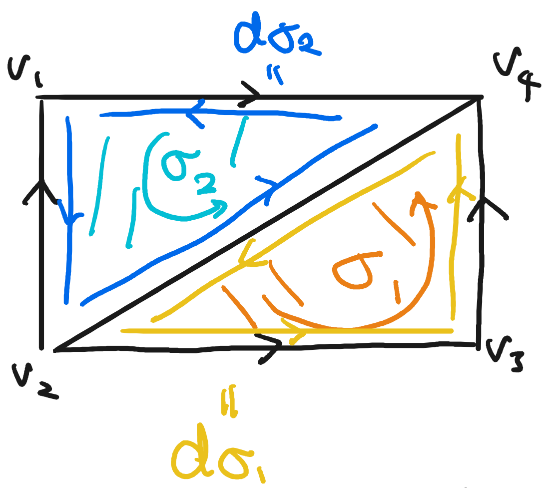
$$d\left(\sum_i k_i \sigma_i\right) = 0$$

in cellular homology.

In this case, $[\sum_i k_i \sigma_i]$ is a fundamental class for M

<u>Example</u>

① torus $T =$



$\sigma_2 = [v_1\ v_2\ v_4]$

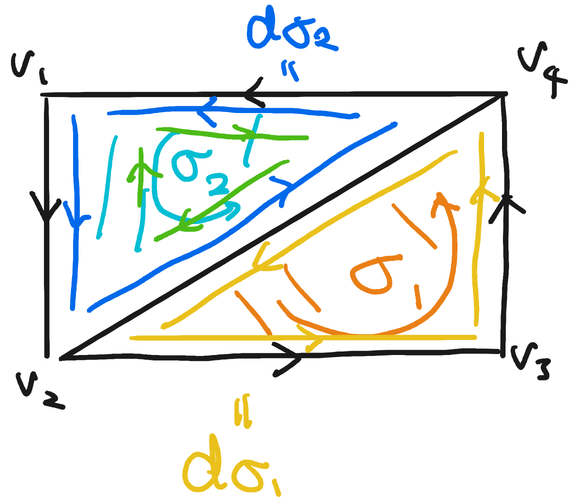
$\sigma_1 = [v_2\ v_3\ v_4]$

$\Rightarrow d(\sigma_1 + \sigma_2) = 0 \Rightarrow \underbrace{[\sigma_1 + \sigma_2]}_{} = [T]$ is a fundamental class

σ 1

② K = Klein bottle

boundaries do not cancel each other



Remark

- Any smooth mfd is triangulizable
- NOT every topological mfd is triangulizable

  see arXiv 1607.08163

Cap product

Def

The cap product is an R-bilinear map

$$\frown : \underbrace{C_k(X;R)}_{\sigma:\Delta^k\to X} \times \underbrace{C^\ell(X;R)}_{\varphi} \to \underbrace{C_{k-\ell}(X;R)}_{\Delta^{k-\ell}\to X}, \quad k \geq \ell$$

$\Delta^{k-\ell} \to X : (t_0, \cdots, t_{k-\ell}) \mapsto \sigma(0, \cdots, 0, t_0, \cdots, t_{k-\ell})$

$$\sigma \frown \varphi := \overbrace{\varphi(\underbrace{\sigma|_{[v_0,\cdots,v_\ell]}}_{\Delta^\ell \to X})}^{\in R} \cdot \sigma|_{[v_\ell,\cdots,v_k]}$$

$(t_0, \cdots, t_\ell) \mapsto \sigma(t_0, \cdots, t_\ell, 0, \cdots, 0)$

Lemma (p 240)

$$\partial(\sigma \frown \varphi) = (-1)^\ell (\partial\sigma \frown \varphi - \sigma \frown \delta\varphi)$$

So we have

$$H_k(X;R) \times H^\ell(X;R) \xrightarrow{\frown} H_{k-\ell}(X;R)$$

<u>Prop</u> (p 249)

For $\alpha \in C_{k+\ell}(X;R)$, $\varphi \in C^k(X;R)$, $\psi \in C^\ell(X;R)$,

$$\psi(\alpha \frown \varphi) = (\varphi \smile \psi)(\alpha) \qquad \left(\text{“} \psi \cdot (\varphi \cdot \alpha) = (\psi \cdot \varphi) \cdot \alpha \text{”} \right)$$

<u>Thm 3.30 (Poincaré duality)</u>

If M is a closed R-orientable n-mfd with fundamental class $[M] \in H_n(M;R)$, then the map

$$D : H^k(M;R) \to H_{n-k}(M;R),$$
$$D(\varphi) := [M] \frown \varphi$$

is an isomorphism $\forall k$

<u>Example</u>

	H_0	H_1	H_2
orientable → torus $= T$	$\mathbb{Z}$	$\mathbb{Z}^2$ same	$\mathbb{Z}$
NOT orientable → $\mathbb{RP}^2$	$\mathbb{Z}$	$\mathbb{Z}_2$ different	0
K = Klein bottle	$\mathbb{Z}$ different	$\mathbb{Z} \oplus \mathbb{Z}_2$	0

orientable

	H_0	H_1	$\cdots$			H_n
S^n	$\mathbb{Z}$	0	0	$\cdots$	0	$\mathbb{Z}$

same

	H_0	H_1	$\cdots$				H_{2n-2}	H_{2n-1}	H_{2n}
$\mathbb{CP}^n$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\cdots$	0	$\mathbb{Z}$	0	$\mathbb{Z}$

Ext

$\rightsquigarrow$

Hom

Tor

$\rightsquigarrow$

$\otimes$