Algebraic Topology 12/30 新年快樂!!

# Algebraic structures behind cohomology

**Def**

Let $R$ be a ring.

A (left) R-module $M$ consists of an abelian gp $(M,+)$ and an operation (scalar product) $\cdot : R\times M \to M$

s.t. $\forall r, s \in R$, $x, y \in M$,

(i) $r\cdot(x+y) = r\cdot x + r\cdot y$

(ii) $(r+s)\cdot x = r\cdot x + s\cdot x$

(iii) $(rs)\cdot x = r\cdot(s\cdot x)$

(iv) $1_R \cdot x = x$ (if $R$ has $1_R$)

e.g.

$d(f\cdot 1) = df \neq f\cdot d1$ ⇒ NOT $C^\infty(\mathbb{R}^n)$-linear but $\mathbb{R}$-linear

← "R-linear maps"

homomorphisms, submodules, generators, finitely generated, ker, quotient, ... are defined as usual

**Example**

① If $R = \Bbbk$ is a field, then

$\Bbbk$-module = $\Bbbk$ vector space

$\Bbbk/2\cdot\Bbbk = 0$

② $\mathbb{Z}$-module = abelian group

$\mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2 \neq 0$

$n\cdot x = \overbrace{x + x + \cdots + x}^{n}$, $(-n)\cdot x = \overbrace{(-x) + \cdots + (-x)}^{n}$

③ $C_n(X)$ is a $\mathbb{Z}$-module

$$C_n(X) = \left\{ \sum_i n_i \cdot \sigma_i \;\middle|\; \sigma_i : \Delta^n \to X \right\}$$

$$m \cdot \left(\sum_i n_i \cdot \sigma_i\right) = \sum_i m n_i \cdot \sigma_i$$

④ $C_n(X;R) = \left\{ \sum_i r_i \cdot \sigma_i \;\middle|\; \begin{array}{l} \sigma_i : \Delta^n \to X \\ r_i \in R \end{array} \right\}$

is an $R$-module

⑤ $H_n(X;R)$ is an $R$-module

($\partial$ : $R$-linear map ↗ quotient module

$$s \cdot \left[\sum_i r_i \sigma_i\right] = \left[\sum_i (s r_i) \cdot \sigma_i\right]$$

⑥ $C^n(X;R)$ and $H^n(X;R)$ are $R$-module:

For $\varphi \in C^n(X;R) = \mathrm{Hom}(C_n(X), R) = \mathrm{Hom}_R(C_n(X;R), R)$

$r \in R$

$$(r \cdot \varphi)(\sigma) = r \cdot \varphi(\sigma)$$

$$r \cdot [\varphi] = [r \cdot \varphi]$$

<u>Def</u>

An (associative) algebra $A$ over $R$

(or $R$-algebra) is an $R$-module

together with a binary operation

$$\bullet : A \times A \longrightarrow A$$

$r, s \in R$

which is bilinear:

$x, y, z \in A$

which is bilinear: $\quad x,y,z \in A$

$$(rx+sy)\cdot z = r\cdot(x\cdot z) + s\cdot(y\cdot z)$$
$$x\cdot(ry+sz) = r\cdot(x\cdot y) + s\cdot(x\cdot z)$$

and

$$r\cdot(x\cdot y) = (r\cdot x)\cdot y \quad \leftarrow \text{associativity}$$

Very often, an algebra is assumed to have identity element $1_A$ and alg mor keeps "1"

algebra morphism

"morphism of algebras"

<u>Example</u>

① $R$ is an $R$-alg

② The polynomial ring $R[x]$ is an $R$-alg

③ $(C^\bullet(X;R), \cup)$ is an $R$-alg.

④ $C^\infty(\mathbb{R}^n, \mathbb{R})$ is a $\mathbb{R}$-alg

<u>Def</u>

A graded ring ($\mathbb{Z}_+$-graded) (resp. graded algebra) is a ring (resp. alg) $A$ with a decomposition

$$A = \bigoplus_{k=0}^{\infty} A_k$$

of additive subgps (resp. $R$-submodules)

s.t. $\forall\, a \in A_k,\ b \in A_\ell$, $\quad a\cdot b \in A_{k+\ell}$

Notation:

Notation:

$a \in A_k \iff |a| = k =$ degree of $a$

Example

The polynomial ring $R[x_1, x_2, \cdots, x_k]$ is a graded alg.

$$(R[x_1, x_2, \cdots, x_n])_k = \left\{ \begin{array}{c} \text{homogeneous poly} \\ \text{of deg } k \end{array} \right\}$$

Def

A graded ring is said to be commutative (or graded commutative) if

$$a \cdot b = (-1)^{|a|\cdot|b|} b \cdot a$$

A differential graded algebra (= dga) (over $R$) is a graded $R$-alg $A$ together with an $R$-linear operator

$$d : A \to A$$

of degree one (i.e. $|d(a)| = \underbrace{1}_{\text{"}|d|\text{"}} + |a|$ )

s.t.

$$d(a \cdot b) = d(a) \cdot b + (-1)^{\overset{= |d|\cdot|a|}{|a|}} a \cdot d(b)$$

"Koszul convention"

and

$$d^2 = d \circ d = 0$$

**Prop**

If $(A, d)$ is a dga, then $\ker(d)/\mathrm{im}(d)$

is a graded ring

pf: See the remark after Lemma 3.6

**Example**

① The singular cochain complex

degree

$$(C^{\bullet}(X;R), \smile, \delta)$$

is a dga

② The singular coh $(H^{\bullet}(X;R), \smile)$ is a commutative graded ring if $R$ is commutative

③ The exterior algebra (generated by $V$) ($V$: vector sp over $\Bbbk$)

degree

"tensor alg"

$$\Lambda^{\bullet} V = T^{\bullet} V / \langle x \otimes y + y \otimes x \mid x, y \in V \rangle$$

$$T^{\bullet} V = \bigoplus_{k=0}^{\infty} \underbrace{V \otimes_{\Bbbk} \cdots \otimes_{\Bbbk} V}_{k}$$

$$x \otimes y \in V \otimes V$$

mean: $x \cdot y + y \cdot x = 0$

is a commutative graded algebra

④ The differential forms

$$(\Omega^\bullet(M), \wedge, d_{dR})$$

form a cdga i.e. commutative dga over $\mathbb{R}$

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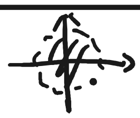
## §3.3 Poinare duality

### Orientation of manifold

A (topological) manifold of dimension $n$ is a Hausdorff (and 2nd countable) topological space which is locally homeomorphic to $\mathbb{R}^n$

**Problem of orientation:**

Recall: $U =$ 

$$\int_U f(x,y)\, dx \wedge dy = \int_U f(x,y)\, dx\, dy$$

Note: $f(x,y)\, dx \wedge dy = - f(x,y)\, dy \wedge dx$

$$\Rightarrow \int_U f(x,y)\, dx \wedge dy = - \int_U f(x,y)\, dy \wedge dx$$

$$\begin{array}{ccc} \| & & \| \\ \int_U f(x,y)\, dx\, dy & \overset{??}{=} & -\int_U f(x,y)\, dy\, dx \end{array}$$

**Orientation** $\approx$ "choice of order of basis"

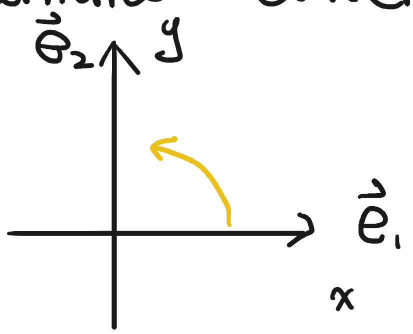
$\mathbb{R}^2$ $\quad \{\vec{e}_1 = (1,0),\ \vec{e}_2 = (0,1)\}$ is an ordered basis for $\mathbb{R}^2$

We say

$$\vec{e}_1, \vec{e}_2 \qquad \text{and} \qquad \vec{e}_2, \vec{e}_2$$

determine different orientations

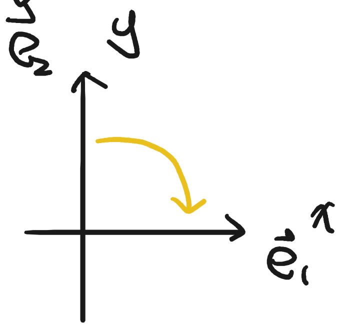
①




$$\int_U f\, dx \wedge dy$$

$$= \int_U f\, dx dy$$

②




$$\int_U f\, dx \wedge dy$$

$$= -\int_U f\, dy \wedge dx$$

$$= -\int_U f\, dy\, dx$$

$$= -\int_U f\, dx dy$$