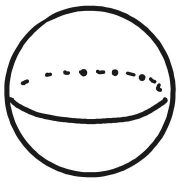
Special topic: de Rham thm

Geometry studies properties of spaces

e.g. earth $\approx$ 

think it as $\{x^2+y^2+z^2=1\} \subseteq \mathbb{R}^3$

$\rightsquigarrow$ Algebraic Geometry

经度 纬度

study it by "maps", longitude, latitude

$\rightsquigarrow$ Differential Geometry

$\rightsquigarrow$ manifold

Def

A differentiable (resp. topological) manifold of dimension n is a set M and a family of injective maps

$$\phi_\alpha : \underset{\text{open}}{U_\alpha} \subseteq \mathbb{R}^n \longrightarrow M$$

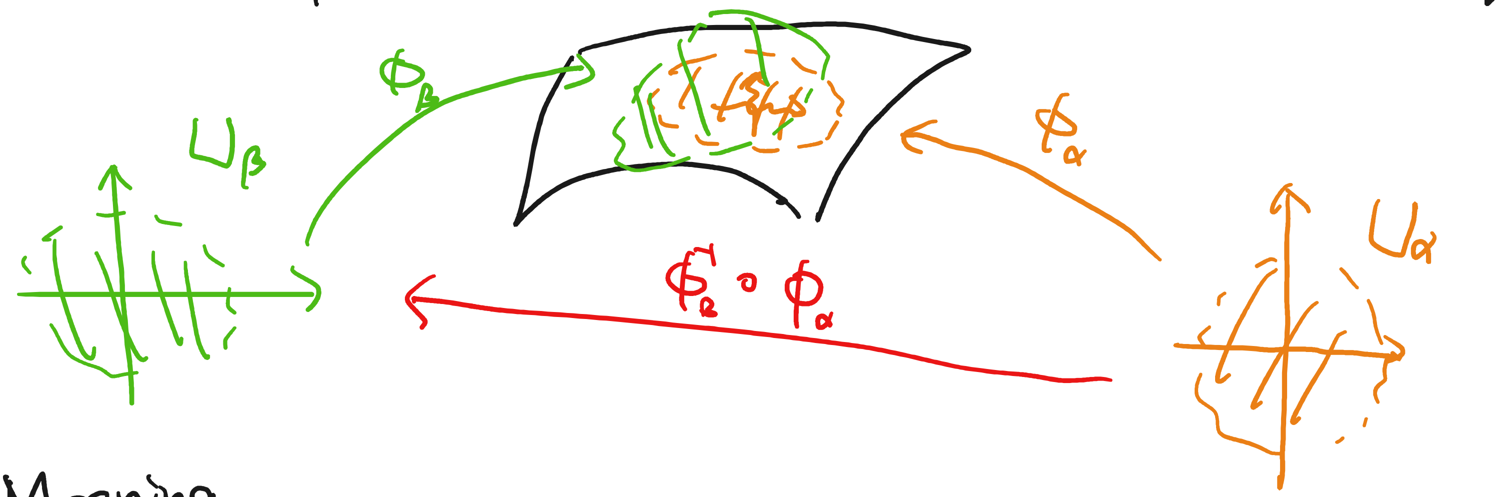
s.t.

(i) $\bigcup_\alpha \phi_\alpha(U_\alpha) = M$

(ii) $\forall\, \alpha, \beta$ with $\phi_\alpha(U_\alpha) \cap \phi_\beta(U_\beta) = W \neq \phi$,

the sets $\phi_\alpha^{-1}(W)$ and $\phi_\beta^{-1}(W)$ are open in $\mathbb{R}^n$ and the transition maps

$\phi_\beta^{-1} \circ \phi_\alpha$ are differentiable (resp. continuous)



Meaning

- (U_α, ϕ_α) is a (local) coordinate chart
- a family $\{(U_\alpha, \phi_\alpha)\}$ with $\bigcup_\alpha U_\alpha = M$ is called an atlas 地圖集

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Example of manifolds:

① open subsets in $\mathbb{R}^n$

② S^n

③ $\mathbb{Q}$ is NOT a mfd

Step 1:

Smooth singular homology

Let M be a smooth mfd or an open subset in $\mathbb{R}^n$.

$C_p^\infty(M)$ = free abelian group generated by ~~continuous~~ smooth maps

$\sigma : \Delta^p \longrightarrow M$

⇝ homology: $H_p^\infty(M)$

⇒ we have the inclusion

⇒ we have the inclusion

$$\iota : C_p^{\infty}(M) \hookrightarrow C_p(M)$$

which is a chain map

<u>Thm</u> (<u>Thm 18.7</u> in Lee's book " Introduction to Smooth Mfd")

"
[Lee]

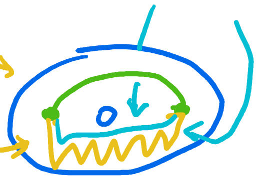
The induced map

$$\iota_* : H_p^{\infty}(M) \longrightarrow H_p(M)$$

is an iso $\forall p$ $\forall$ smooth mfd M.

"close enough"
homotopic

"far"
Not homotopic



<u>idea of pf</u>

Use " Whitney Approximation Thm" (<u>Thm</u> 6.26 in [Lee])

Let M, N be smooth mfds, and $F: N \to M$ is continuos

$\Rightarrow$ F is homotopic to a smooth map $N \to M$

to construct a " smoothing operator"

$$s : C_p(M) \longrightarrow C_p^{\infty}(M)$$

s.t.

- s is a chain map
- $s \circ \iota = id$
- $\iota \circ s$ is chain homotopic to id #

<u>Step 2</u>: <u>Integration</u>

Define a map $I : \underset{[\omega]}{H^p_{dR}(M)} \longrightarrow \underset{I([\omega])}{H^p(M;\mathbb{R})}$

$\cong$ ← universal coeff thm

$\mathrm{Hom}_{\mathbb{R}}(H_p(M;\mathbb{R}), \mathbb{R})$

$$\text{by Step1} \to \cong \ \mathrm{Hom}_{\mathbb{R}}(\underbrace{H_p^{\infty}(M;\mathbb{R})}_{\ni [c] = [\sum c_i \sigma_i]}, \mathbb{R})$$

$$I([\omega])([c]) = \int_c \omega = \sum c_i \int_{\sigma_i} \omega$$

<u>Stoke's Thm</u> (<u>Thm</u> 18.12 in [Lee]: $\int_{\partial c} \omega = \int_c d\omega$)

$\Rightarrow$ $\forall p$, $\forall$ smooth mfd M, $I : H^p_{dR}(M) \to H^p(M;\mathbb{R})$ is well-defined

<u>Step3</u>:

<u>De Rham Thm</u> (<u>Thm</u> 18.14 in [Lee])

I is an isomorphism

<u>sketch of pf</u>

We say M is "de Rham" if I is an iso

<u>Step1</u>: Prove every convex open subset of $\mathbb{R}^n$ is de Rham

<u>Step2</u>: Use Mayer-Vietoris seq. to prove M is de Rham

$$\begin{tikzcd}
\cdots & H^p_{dR}(U\cap V) \arrow[l] \arrow[d, "\cong"] & H^p_{dR}(U)\oplus H^p_{dR}(V) \arrow[l] \arrow[d, "\cong"] & H^p_{dR}(U\cup V) \arrow[l] \arrow[d, "I"] & H^{p-1}_{dR}(U\cap V) \arrow[l] \arrow[d, "\cong"] & \cdots \arrow[l] \arrow[d, "\cong"] \\
& H^p \cdots & \arrow[l] & \arrow[l] & \arrow[l] & \arrow[l]
\end{tikzcd}$$