Algebraic Topology 12/6

<u>Recall</u>

Let R be a ring. For $\varphi \in C^k(X;R)$, $\psi \in C^\ell(X;R)$

$$\varphi \cup \psi \in C^{k+\ell}(X;R)$$

$\forall\ \sigma : \Delta^{k+\ell} \to X$,

$$(\varphi \cup \psi)(\sigma) = \varphi(\sigma|_{[v_0,\dots,v_k]}) \cdot \psi(\sigma|_{[v_k,\dots,v_{k+\ell}]})$$

<u>Lemma 3.6</u>

$$\delta(\varphi \cup \psi) = (\delta\varphi) \cup \psi + (-1)^k \varphi \cup (\delta\psi)$$

$\Rightarrow$

$$\cup : H^k(X;R) \times H^\ell(X;R) \to H^{k+\ell}(X;R)$$

$[1_R]$ = identity wrt $\cup$

<u>Prop 3.10</u>

For a map $f : X \to Y$, the induced maps $f^* : H^n(Y;R) \to H^n(X;R)$ satisfy

$$f^*(\alpha \cup \beta) = f^*(\alpha) \cup f^*(\beta)$$

<u>pf</u>

This follows from the cochain formula

$$f^\#(\varphi) \cup f^\#(\psi) = f^\#(\varphi \cup \psi):$$

$$(f^\#\varphi \cup f^\#\psi)(\sigma) = (f^\#\varphi)(\sigma|_{[v_0,\dots,v_k]}) \cdot (f^\#\psi)(\sigma|_{[v_k,\dots,v_{k+\ell}]})$$

$$= \varphi(f \circ \sigma|_{[v_0,\dots,v_k]})\, \psi(f \circ \sigma|_{[v_k,\dots,v_{k+\ell}]})$$

$$= (\varphi \cup \psi)(f \circ \sigma)$$

$= f^{\#}(\varphi \smile \psi)(\sigma)$ #

<u>Thm 3.11</u>

Suppose R is commutative. Then $\forall\, \alpha \in H^k(X;R)$, $\beta \in H^\ell(X;R)$,

$$\alpha \smile \beta = (-1)^{k\ell}\, \beta \smile \alpha$$

pf

Suppose $\varphi \in C^k(X;R)$, $\psi \in C^\ell(X;R)$ s.t.

$$\alpha = [\varphi] \qquad \beta = [\psi]$$

Recall that for $\sigma : \Delta^{k+\ell} \to X$

$$(\varphi \smile \psi)(\sigma) = \varphi(\sigma|_{[v_0,\cdots,v_k]}) \cdot \psi(\sigma|_{[v_k,\cdots,v_{k+\ell}]})$$

$$(\psi \smile \varphi)(\sigma) = \psi(\sigma|_{[v_0,\cdots,v_\ell]}) \cdot \varphi(\sigma|_{[v_\ell,\cdots,v_{k+\ell}]})$$

$\because$ R is commutative

$$= \varphi(\sigma|_{[v_\ell,\cdots,v_{k+\ell}]}) \cdot \psi(\sigma|_{[v_0,\cdots,v_\ell]})$$

Let $P : C_n(X) \to C_n(X)$ be the operator

$$P(\sigma) := \varepsilon_n \cdot \bar{\sigma}$$

where $\bar{\sigma}, \sigma : \Delta^n \to X$, $\Delta^n \subseteq \mathbb{R}^{n+1}$

$$\bar{\sigma}(t_0,\cdots,t_n) := \sigma(t_n, t_{n-1}, \cdots, t_0)$$

i.e. $\bar{\sigma} := \sigma|_{[v_n, v_{n-1}, \cdots, v_0]}$

$$\varepsilon_n := (-1)^{1+2+\cdots+n} = (-1)^{\frac{n(n+1)}{2}}$$

the sign of the permutation $(n\ n-1\ \cdots\ 2\ 1\ 0)$

<u>Claim</u>

$P : C_n(X) \to C_n(X)$ is ① a chain map

$\rho: C_n(X) \to C_n(X)$ is a chain map

② chain homotopic to id

If the claim is true, then

- $\rho^* = id : H^n(X;R) \to H^n(X;R)$

- Since
$$\varepsilon_{k+\ell} = \underline{(-1)^{k\ell}}\, \varepsilon_k \varepsilon_\ell ,$$
$$= (-1)^{1+2+\cdots+(k+\ell)}$$
$$= (-1)^{(1+\cdots+k)+(k+1)+\cdots+(k+\ell)}$$
$$= (-1)^{1+\cdots+k}(-1)^{k\ell}(-1)^{1+\cdots+\ell}$$
we have
$$(\rho^*\varphi \cup \rho^*\psi)(\sigma)$$
$$= \varphi(\rho(\sigma|_{[v_0,\dots,v_k]}))\cdot \psi(\rho(\sigma|_{[v_k,\dots,v_{k+\ell}]}))$$
$$= \varepsilon_k \cdot \varepsilon_\ell\, \varphi(\sigma|_{[v_k,\dots,v_0]}) \cdot \psi(\sigma|_{[v_{k+\ell},\dots,v_k]})$$
$$\rho^*(\psi\cup\varphi)(\sigma)$$
$$= \varepsilon_{k+\ell}(\psi\cup\varphi)(\sigma|_{[v_{k+\ell},\dots,v_0]})$$
$$= \varepsilon_{k+\ell}\, \psi(\sigma|_{[v_{k+\ell},\dots,v_k]})\cdot \varphi(\sigma|_{[v_k,\dots,v_0]})$$
$$= (-1)^{k\ell}(\rho^*\varphi \cup \rho^*\psi)(\sigma)$$

$$\Rightarrow [\rho^*(\psi\cup\varphi)] = (-1)^{k\ell}[\rho^*\varphi \cup \rho^*\psi]$$
$$[\psi\cup\varphi] \qquad\qquad (-1)^{k\ell}[\rho^*\varphi]\cup[\rho^*\psi]$$
$$\beta \cup \alpha = (-1)^{k\ell}\,\alpha\cup\beta \qquad \#$$

<u>pf of Claim:</u>

① $\partial \rho = \rho \partial$: direct computation

② chain homotopy :

Define $P : C_n(X) \longrightarrow C_{n+1}(X)$ by $\quad \sigma : \Delta^n \to X$

$$P(\sigma) := \sum_{i=0}^{n} (-1)^i \, \varepsilon_{n-i} \, (\sigma\pi)\big|_{[v_0, \cdots, v_i, w_n, \cdots, w_i]}$$

Here,

$$(\sigma\pi)\big|_{[v_0, \cdots, v_i, w_n, \cdots, w_i]} \underline{(t_0, \cdots, t_{n+1})} \quad \in \mathbb{R}^{n+2} \qquad \Delta^{n+1} \subseteq \mathbb{R}^{n+2} \to X$$

$$= \sigma\pi\Big(\sum_{j=0}^{i} t_j v_j + \sum_{k=i+1}^{n+1} t_k w_{n+i+1-k} \Big)$$

$$= \sigma(t_0, t_1, \cdots, t_{i-1}, (t_i + t_{n+1}), t_n, t_{n-1}, \cdots, t_{i+1})$$

One can check that

$$\partial P + P\partial = \rho - \mathrm{id}$$

#

Conclusion

Let R be a ring. The coh $(H^{\bullet}(X;R), \cup)$ is a ring with the property

$$\cup : H^i(X;R) \times H^j(X;R) \longrightarrow H^{i+j}(X;R)$$ — "graded ring"

Furthermore, if R is commutative, then

$$\alpha \cup \beta = (-1)^{ij} \, \beta \cup \alpha$$

Such a ring is called a "Commutative graded ring" — explain later

Example ($R = \mathbb{Z}$, $X = S^n$, $n \geq 1$)

Recall: $H_k(S^n) \cong \begin{cases} \mathbb{Z}, & k = 0, n \\ 0, & k \neq 0, n \end{cases}$ is free $\forall k$

$$\Rightarrow \operatorname{Ext}(\underbrace{H_{k-1}(S^n)}_{\text{free}}, \mathbb{Z}) = 0$$

universal coeff thm $\Rightarrow$ $H^k(S^n;\mathbb{Z}) \cong \operatorname{Hom}(H_k(S^n), \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & k=0,n \\ 0, & k \neq 0, n \end{cases}$

$$H^\bullet(S^n;\mathbb{Z}) \cong \overset{0}{\underset{\substack{\uplus \\ a\cdot 1}}{\mathbb{Z}}} \oplus 0 \oplus 0 \oplus 0 \oplus \cdots \oplus \overset{n}{\underset{\substack{\uplus \\ b\cdot[\sigma]}}{\mathbb{Z}}} \oplus 0 \cdots$$

$1 \smile [\sigma] = [\sigma]$, $\underbrace{[\sigma] \smile [\sigma]}_{\substack{\| \\ 0}} \in H^{2n}(S^n;\mathbb{Z}) = \{0\}$

So, as rings,

$$H^\bullet(S^n;\mathbb{Z}) \cong (\mathbb{Z} \oplus \mathbb{Z}, \smile)$$

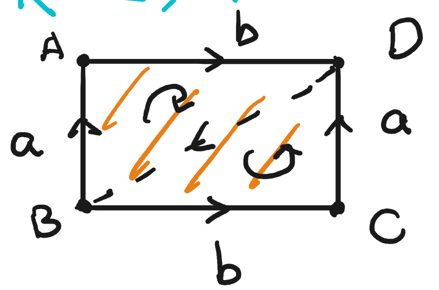
$$(a,b) \smile (c,d) = (ac, ad + bc)$$

$$\left(= (a\cdot 1 + b\cdot[\sigma])(c \cdot 1 + d\cdot[\sigma]) \right)$$

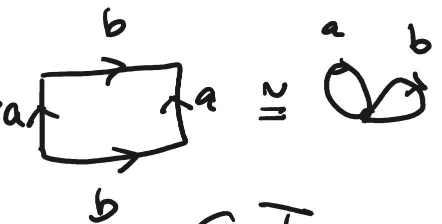
$\clubsuit$

Example 3.7 ($R = \mathbb{Z}$, $X = T =$ torus)

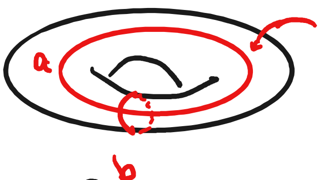
$X = T$ =



$S =$



$\subseteq T$



$H_0(T) \cong \mathbb{Z} \cong H^0(T)$

$H_1(T) \cong \mathbb{Z}^2 \cong H^1(T)$ ∂ of d_2 in cellular cx

$H_2(T) \cong \mathbb{Z} \cong H^2(T)$ deg $= 0$

Step 1: find generators of homology

$$\cdots \to 0 = H_2(S) \to H_2(T) \to \underbrace{H_2(T,S)}_{} \to H_1(S) \to \cdots$$

$$\tilde{H}_2(T/S) \overset{\text{IIS}}{\cong} H_2(D^2/S^1) \cong H_2(D^2, S^1)$$

$\Rightarrow$ $H_2(T)$ is generated by the class induced by

$$\underbrace{(BAD) - (BCD)}_{} \overset{\partial}{\longmapsto} 0$$

$\leftarrow$ generator

where

$(BAD): \Delta^2 \to T$ induced by $v_0 \mapsto B$, $v_1 \mapsto A$, $v_2 \mapsto D$

(BCD) is defined similarly

Let $z \in H_2(T)$ be $z = [(BAD) - (BCD)]$

$(\mathrm{Ext}(\text{free}, \mathbb{Z}) = 0)$

Step2: take dual

Since $H_k(T)$ are free, by universal coeff thm

$$H^0(T;\mathbb{Z}) \cong \mathrm{Hom}(H_0(T), \mathbb{Z}) \cong \mathbb{Z}$$

$$H^1(T;\mathbb{Z}) \underset{h}{\cong} \mathrm{Hom}(\underset{\substack{\text{IIS} \\ \mathbb{Z}\oplus\mathbb{Z}}}{H_1(T)}, \mathbb{Z}) \cong \mathbb{Z}^2$$

$\mathbb{Z}\oplus\mathbb{Z} \leftarrow$ generated by a, b

$$H^2(T;\mathbb{Z}) \underset{h}{\cong} \mathrm{Hom}(\underset{\substack{\text{IIS} \\ \mathbb{Z}}}{H_2(T)}, \mathbb{Z}) \cong \mathbb{Z}$$

generated by z

Let $\alpha, \beta \in H^1(T;\mathbb{Z})$ be the dual basis of a, b.

i.e.

$h(\alpha)(a) = 1$, $h(\alpha)(b) = 0$

$h(\beta)(a) = 0$, $h(\beta)(b) = 1$

zeta

$\zeta \in H^2(T;\mathbb{Z})$ be the dual basis of z

i.e. $h(\zeta)(z) = 1$

Step3 Compute $\smile$

$$h(\alpha \smile \beta)(z)$$

$$= h(\alpha\cup\beta)((BAD)) - h(\alpha\cup\beta)((BCD))$$

$$= \alpha(\underbrace{(BA)}_{a}) \cdot \beta(\underbrace{(AD)}_{b}) - \alpha(\underbrace{(BC)}_{b}) \cdot \beta(\underbrace{(CD)}_{a})$$

$$= 1 \cdot 1 - 0 \cdot 0 = 1$$

$$= h(\zeta)(Z)$$

$$\Rightarrow \quad \alpha \cup \beta = \zeta$$

Conclusion

$$H^\bullet(T;\mathbb{Z}) \cong \overset{0}{\mathbb{Z}} \oplus \overset{1}{\mathbb{Z}^2} \oplus \overset{2}{\mathbb{Z}}$$

generator: $\quad 1 \quad \alpha, \beta \quad \zeta$

with the relations

$$1 \cup x = x \cup 1 = x \qquad \forall x \in H^\bullet(T;\mathbb{Z})$$

$$\alpha \cup \beta = \zeta \Rightarrow \beta \cup \alpha = (-1)^{1\cdot 1} \alpha \cup \beta = -\zeta$$

$$\alpha \cup \alpha = 0 \leftarrow \alpha \cup \alpha = (-1)^{1\cdot 1} \alpha \cup \alpha = -\alpha \cup \alpha$$

$$\Rightarrow 2(\alpha \cup \alpha) = 0 \quad \text{in } H^2(T;\mathbb{Z}) \cong \mathbb{Z}$$

$$\beta \cup \beta = 0 \qquad \Rightarrow \alpha \cup \alpha = 0$$

$$H^3(T;\mathbb{Z}) \ni \alpha \cup \zeta = \zeta \cup \alpha = \beta \cup \zeta = \zeta \cup \beta = 0$$

$$H^4(T;\mathbb{Z}) \ni \zeta \cup \zeta = 0$$

exer.

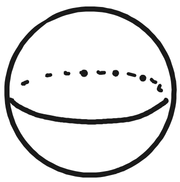
Compute the coh rings of closed surfaces

Example 3.7, Example 3.8

Manifold

Geometry studies properties of spaces

e.g. earth $\cong$



think it as $\{x^2+y^2+z^2=1\} \subseteq \mathbb{R}^3$

$\rightsquigarrow$ Algebraic Geometry

study it by "maps", longitude, latitude (经度, 纬度)

$\rightsquigarrow$ Differential Geometry

$\rightsquigarrow$ manifold

Def

A differentiable (resp. topological) manifold of dimension n is a set M and a family of injective maps

$$\phi_\alpha : U_\alpha \longrightarrow M$$

$U_\alpha \underset{\text{open}}{\subseteq} \mathbb{R}^n$

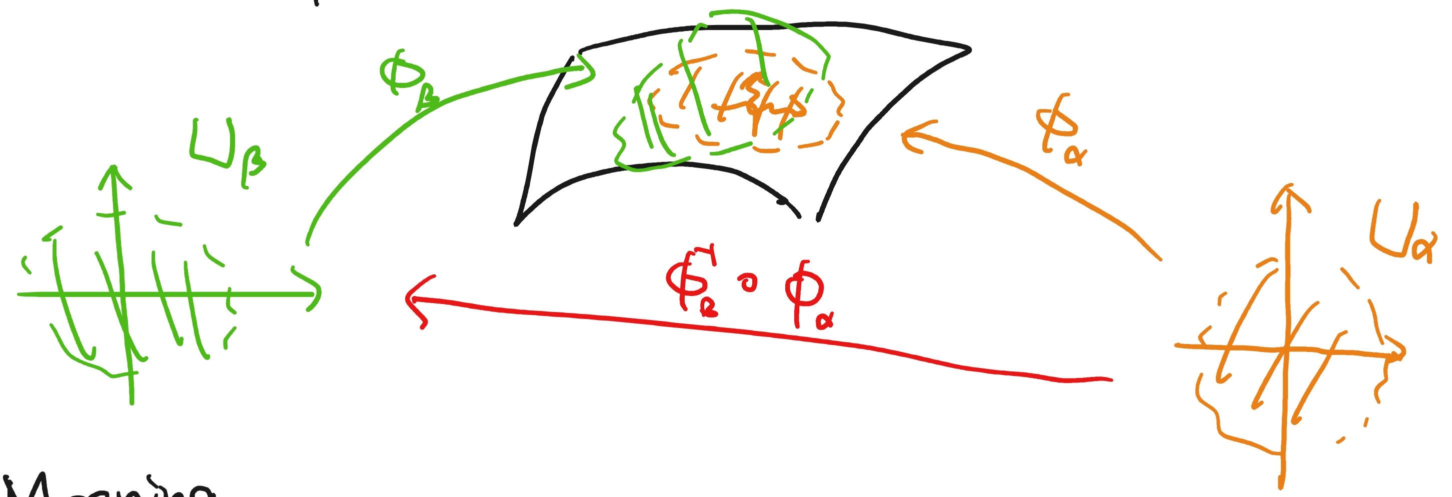
s.t.

(i) $\bigcup_\alpha \phi_\alpha(U_\alpha) = M$

(ii) $\forall\ \alpha, \beta$ with $\phi_\alpha(U_\alpha) \cap \phi_\beta(U_\beta) = W \neq \phi$,

the sets $\phi_\alpha^{-1}(W)$ and $\phi_\beta^{-1}(W)$ are open in $\mathbb{R}^n$ and the transition maps

$\phi_\beta^{-1} \circ \phi_\alpha$ are differentiable (resp. continuous)



<u>Meaning</u>

- (U_α, ϕ_α) is a (local) coordinate chart
- a family $\{(U_\alpha, \phi_\alpha)\}$ with $\bigcup_\alpha U_\alpha = M$ is called an <u>atlas</u> 地圖集