Algebraic Topology 12/9

② <u>Relative coh</u>:

Let $A \subseteq X$. $\rightsquigarrow$ $\left(C_n(X,A) = C_n(X)/C_n(A) \;,\; \partial \right)$

$\rightsquigarrow$ $\left(C^n(X,A;G) = \mathrm{Hom}(C_n(X,A), G) \;,\; \delta = \partial^* \right)$

$\rightsquigarrow$ $H^n(X,A;G) = \ker\delta / \mathrm{im}\,\delta$ at $C^n(X,A;G)$

$\rightsquigarrow$

$$\cdots \to H^n(X,A;G) \xrightarrow{j^*} H^n(X;G) \xrightarrow{i^*} H^n(A;G) \xrightarrow{\delta} H^{n+1}(X,A;G) \to \cdots$$

is exact.

<u>Prop</u> (p.200)

The connecting homomorphism $\delta : H^n(A;G) \to H^{n+1}(X,A;G)$ and $\partial : H_{n+1}(X,A) \to H_n(A)$ are dual to each other in the sense that

$$\begin{tikzcd}
H^n(A;G) \arrow[r, "\delta"] \arrow[d, "h"'] \arrow[dr, phantom, "\circlearrowleft"] & H^{n+1}(X,A;G) \arrow[d, "h"] \\
\mathrm{Hom}(H_n(A), G) \arrow[r, "\partial^*"'] & \mathrm{Hom}(H_{n+1}(X,A), G)
\end{tikzcd}$$

pf: exer

<u>Prop</u>

Since $C_n(X,A)$ are free, by <u>Thm 3.2</u>

$$0 \to \mathrm{Ext}(H_{n-1}(X,A),G) \to H^n(X,A;G) \xrightarrow{h} \mathrm{Hom}(H_n(X,A),G) \to 0$$

are split and exact

are split and exact

③ <u>Induced homomorphism</u>:

$$\sigma : \Delta^n \to X \xrightarrow{f} Y$$
$$\sigma \longmapsto f \circ \sigma$$

For $f : X \to Y$, we have $f_\# : C_n(X) \to C_n(Y)$

$$\varphi \longmapsto \varphi \circ f_\#$$

Its dual $f^\# : C^n(Y;G) \to C^n(X;G)$ is a cochain map

$\Rightarrow$ we have $f^* : H^n(Y;G) \to H^n(X;G)$

Similarly, for $f : (X,A) \to (Y,B)$, we have

$$f^* : H^n(Y,B;G) \to H^n(X,A;G)$$

<u>Prop</u>

$$(fg)^* = g^* \circ f^*, \quad id^* = id$$

<u>Remark</u>

A map $f : (X,A) \to (Y,B)$ induces a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}(H_{n-1}(X,A),G) \arrow[r] & H^n(X,A;G) \arrow[r] & \mathrm{Hom}(H_n(X,A),G) \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Ext}(H_{n-1}(Y,B),G) \arrow[r] \arrow[u, "(f_*)^*"] & H^n(Y,B;G) \arrow[r] \arrow[u, "f^*"] & \mathrm{Hom}(H_n(Y,B),G) \arrow[r] \arrow[u, "(f_*)^*"] & 0
\end{tikzcd}$$

④ <u>Homotopy invariance</u>:

Recall that if $f \simeq g : (X,A) \to (Y,B)$, then

(<u>Thm</u> 2.10) $\exists$ chain homotopy

$$P : C_n(X,A) \to C_{n+1}(Y,B)$$

of

s.t. $g_\# - f_\# = \partial P + P\partial$ $(\Rightarrow g_* = f_*)$

take dual

$\Rightarrow$ $P^*: \underset{\substack{\| \\ C^{n+1}(Y,B;G)}}{\mathrm{Hom}(C_{n+1}(Y,B),G)} \to \underset{\substack{\| \\ C^n(X,A;G)}}{\mathrm{Hom}(C_n(X,A),G)}$

φ^G

$(g_\# - f_\#)^*(\varphi) = \varphi\circ g_\# - \varphi\circ f_\# = \varphi\circ(g_\# - f_\#)$

s.t.

$$g^\# - f^\# = P^*\delta + \delta P^*$$

$= \varphi\circ(\partial P + P\partial)$

$= (\varphi\circ\partial)\circ P + (\varphi\circ P)\circ\partial$

$= (\delta\varphi)\circ P + (P^*\varphi)\circ\partial$

$= P^*(\delta\varphi) + \delta(P^*\varphi)$

$\Rightarrow$ Prop

If $f \simeq g : (X,A) \to (Y,B)$, then

$$f^* = g^* : H^n(Y,B;G) \to H^n(X,A;G)$$

⑤ Excision:

Thm

Suppose $Z \subseteq A \subseteq X$, $cl(Z) \subseteq int(A)$. Then the inclusion $i:(X-Z, A-Z) \longrightarrow (X,A)$ induces iso

$$i^*: H^n(X,A;G) \xrightarrow{\cong} H^n(X-Z, A-Z; G)$$

pf

By excision of homology,

$$i_*: H_n(X-Z, A-Z) \xrightarrow{\cong} H_n(X,A)$$

are iso.

$\Rightarrow$

$$\begin{array}{ccccccccc}
0 \to & \mathrm{Ext}(H_{n-1}(X-Z,A-Z),G) & \to & H^n(X-Z,A-Z;G) & \to & \mathrm{Hom}(H_n(X-Z,A-Z),G) & \to 0 \\
\| & \cong\uparrow (i_*)^* & \circlearrowleft & \uparrow i^* & \circlearrowleft & \cong\uparrow (i_*)^* & \| \\
0 \to & \mathrm{Ext}(H_{n-1}(X,A),G) & \longrightarrow & H^n(X,A;G) & \longrightarrow & \mathrm{Hom}(H_n(X,A),G) & \to 0
\end{array}$$

$\Rightarrow$ by five-lemma, i^* are iso #

(5) Cellular cohomology:

For a CW complex X, one has the cellular cochain complex $(H^n(X^n, X^{n-1}; G), d_n)$ defined as in the diagram

(*)

$$\begin{tikzcd}
 & & H^{n-1}(X^{n-1};G) \arrow[r] \arrow[dr, "\delta_{n-1}"] & H^{n-1}(X^{n-2};G) \overset{(*)}{=} 0 & & & \\
\cdots \arrow[r] & H^{n-1}(X^{n-1}, X^{n-2};G) \arrow[ur, "j_{n-1}"] \arrow[rr, "d_{n-1}"] & & H^n(X^n, X^{n-1};G) \arrow[dr, "j_n"] \arrow[rr, "d_n"] & & H^{n+1}(X^{n+1}, X^n;G) \arrow[r] & \cdots \\
 & & 0 = H^n(X^{n+1}, X^n;G) \arrow[r] & \underbrace{H^n(X^{n+1};G)}_{\cong H^n(X;G)} \arrow[r] & H^n(X^n;G) \arrow[ur, "\delta_n"] \arrow[r] & H^n(X^{n-1};G) = 0 &
\end{tikzcd}$$

Thm 3.5

(i) $H^n(X;G) \cong \ker d_n / \operatorname{im} d_{n-1}$

(ii) Furthermore, the cellular cochain complex $(H^n(X^n, X^{n-1}; G), d_n)$ is iso to the dual of the cellular chain complex

$H_k(X^n, X^{n-1}) \cong \begin{cases} \mathbb{Z}^{\#\text{ of } n \text{ cells}} & k=n \\ 0 & k \neq 0 \end{cases}$

pf

(i) By universal coeff. thm,

$$0 \to \underbrace{Ext(\overset{\text{free}}{H_{k-1}(X^n, X^{n-1})}, G)}_{=0} \to H^k(X^n, X^{n-1}; G) \xrightarrow{h} Hom(H_k(X^n, X^{n-1}), G) \to 0$$

$\Rightarrow H^k(X^n, X^{n-1}; G) = 0$ for $k \neq n$

$\overset{\text{long exact seq}}{\Rightarrow}$ $H^k(X^n;G) \cong H^k(X^{n-1};G) \quad \forall k \neq n, n-1$

$\Rightarrow$ if $k > n$, then $H^k(X^n;G) \cong H^k(X^{n-1};G) \cong \cdots \cong H^k(X^0;G) = 0$

(by pf of

$\Rightarrow$ (*)

By universal coeff thm, (assume $k \leq n+1$)

Lemma 2.34(c)

$$0 \to \underbrace{Ext(H_k(X, X^{n+1}), G)}_{=0} \to H^k(X, X^{n+1}; G) \to Hom(\boxed{H_k(X, X^{n+1})}^{=0}; G) \to 0$$

$\Rightarrow H^k(X, X^{n+1}; G) = 0$ if $k \leq n+1$

$\Rightarrow H^n(X; G) \cong H^n(X^{n+1}; G)$

So the diagram (A) implies

$$H^n(X;G) \cong H^n(X^{n+1};G) \cong \ker \delta_n \cong \check{j}_n(\ker \delta_n) / \ker \check{j}_n$$

$$= \ker d_n / \operatorname{im} \delta_{n-1} = \ker d_n / \operatorname{im} d_{n-1}$$

(ii) We have the diagram

cellular coboundary map

$$\begin{tikzcd}
H^k(X^k, X^{k-1}; G) \arrow[r, "\check{j}"] \arrow[d, "h", "\cong"'] \arrow[rr, bend left] & H^k(X^k; G) \arrow[r, "\delta"] \arrow[d, "h"] & H^{k+1}(X^{k+1}, X^k; G) \arrow[d, "h"', "\cong"] \\
Hom(H_k(X^k, X^{k-1}), G) \arrow[r] \arrow[rr, bend right] & Hom(H_k(X^k), G) \arrow[r, "\partial^*"] & Hom(H_{k+1}(X^{k+1}, X^k), G)
\end{tikzcd}$$

universal coeff thm

dual of cellular boundary map

$\Rightarrow$ (ii) is ok #

① Mayer-Vietoris seq:

If $X = int(A) \cup int(B)$, then we have the long exact seq.

$$\boxed{\cdots \to H^n(X;G) \to H^n(A;G) \oplus H^n(B;G) \to H^n(A \cap B; G) \to H^{n+1}(X;G) \to \cdots}$$

exer: Read p. 203, 204 for details and other versions

§3.2 Cup product

Let R be a ring. For cochain $\varphi \in C^k(X;R)$, $\psi \in C^\ell(X;R)$, the cup product $\varphi \smile \psi \in C^{k+\ell}(X;R)$ is the cochain whose value on $\sigma : \Delta^{k+\ell} \to X$ is given by

$$(\varphi \smile \psi)(\sigma) := \varphi(\sigma|_{[v_0,\cdots,v_k]}) \cdot \psi(\sigma|_{[v_k,\cdots,v_{k+\ell}]})$$

product in R

where $\sigma|_{[v_0,\cdots,v_k]} : \Delta^k \to X : (t_0,\cdots,t_k) \mapsto \sigma(t_0,\cdots,t_k,0,\cdots,0)$

$\mathbb{R}^{k+1} \supset \Delta^k$, $\sum t_i = 1,\ t_i \geq 0$

$\sigma|_{[v_k,\cdots,v_{k+\ell}]} : \Delta^\ell \to X : (t_0,\cdots,t_\ell) \mapsto \sigma(0,\cdots,0,t_0,\cdots,t_\ell)$

Lemma 3.6

$$\delta(\varphi \smile \psi) = (\delta\varphi) \smile \psi + (-1)^k \varphi \smile (\delta\psi)$$

for $\varphi \in C^k(X;R)$, $\psi \in C^\ell(X;R)$

pf

For $\sigma : \Delta^{k+\ell+1} \to X$, we have

$= \varphi(\partial\, \sigma|_{[\cdots]}) = \sum_{i=0}^{k+1} (-1)^i \varphi(\sigma|_{[v_0 \cdots \hat{v}_i \cdots]})$

$$(\delta\varphi \smile \psi)(\sigma) = (\delta\varphi)(\sigma|_{[v_0,\cdots,v_{k+1}]}) \cdot \psi(\sigma|_{[v_{k+1},\cdots,v_{k+\ell+1}]})$$

$$= \sum_{i=0}^{k+1} (-1)^i \varphi(\sigma|_{[v_0,\cdots,\hat{v}_i,\cdots,v_{k+1}]}) \cdot \psi(\sigma|_{[v_{k+1},\cdots,v_{k+\ell+1}]})$$

$$(-1)^k(\varphi \smile \delta\psi)(\sigma) = (-1)^k\Big(\varphi(\sigma|_{[v_0,\cdots,v_k]}) \cdot \sum_{i=0}^{\ell+1} (-1)^i \psi(\sigma|_{[v_k,\cdots,\hat{v}_{k+i},\cdots,v_{k+\ell+1}]})\Big)$$

$$= \sum_{i=k}^{k+\ell+1} (-1)^i \varphi(\sigma|_{[v_0,\cdots,v_k]}) \cdot \psi(\sigma|_{[v_k,\cdots,\hat{v}_i,\cdots,v_{k+\ell+1}]})$$

+)

$$(\delta\varphi\smile\psi)(\sigma) + (-1)^k(\varphi\smile\delta\psi)(\sigma) = \delta(\varphi\smile\psi)(\sigma)$$ #

Remark

$\smile$ is associative and distributive on $C^\bullet(X;R)$

Lemm 3.6 $\Rightarrow$ $(C^\bullet(X;R), \delta, \smile)$ is a "differential graded algebra"

Thus, one has a well-defined product

$$H^k(X;R)\times H^\ell(X;R) \xrightarrow{\smile} H^{k+\ell}(X;R) : [\varphi]\smile[\psi] := [\varphi\smile\psi]$$

- $\delta(\varphi\smile\psi) = \underbrace{\delta\varphi}_{=0}\smile\psi \pm \varphi\smile\underbrace{\delta\psi}_{=0} = 0$
- if $\delta\psi = 0$,

$$[\delta\xi\smile\psi] = \underbrace{[\delta(\xi\smile\psi)]}_{=0} \pm [\xi\smile\cancel{\underbrace{\delta\psi}_{=0}}] = 0$$

If R has an identity 1_R, then the class

$$1\in H^0(X;R) \quad \text{represented by} \quad \underset{0\text{-simplex}}{x\in X} \mapsto 1_R$$

is an identity for $\smile$