Algebraic Topology 12/2

Dual complex

Let $\cdots \to C_n \xrightarrow{\partial} C_{n-1} \to \cdots \to C_1 \xrightarrow{\partial} C_0 \to 0$

be a chain complex. and G be an abelian gp

Denote

$$C^*_n := \mathrm{Hom}(C_n, G) \ni f$$

$$C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{f} G, \quad f\circ\partial = \delta_n(f)$$

$$\delta = \delta_n = \partial^* : C^*_n \to C^*_{n+1} : f \mapsto f\circ\partial$$

Note: $(\alpha\circ\beta)^* = \beta^*\circ\alpha^*$, $id^* = id$, $0^* = 0$

Lemma

$$\delta\circ\delta = 0$$

pf $\delta^2(f) = f\circ\boxed{\partial\circ\partial}^{=0} = 0$ ⌘

We have the cochain complex (i.e. (D_n, δ), $\delta : D_n \to D_{n+1}$) $\delta^2 = 0$

$$\cdots \leftarrow C^*_{n+1} \xleftarrow{\delta_n} C^*_n \xleftarrow{\delta_{n-1}} C^*_{n-1} \leftarrow \cdots \leftarrow C^*_1 \leftarrow C^*_0 \leftarrow 0$$

"dual complex"

The group $H^n(C;G) := \ker(\delta_n)/\mathrm{im}(\delta_{n-1})$ is called the cohomology group of (C^*_n, δ)

Remark

One can ① take dual then take ker/im

② take ker/im then take dual

① $H^n(C;G) = \ker(\delta)/\mathrm{im}(\delta)$

② $\mathrm{Hom}(H_n(C,\partial), G)$

These two are different in general.
The relationship between them is a part of content of "universal coefficient thm"

Example ⊛

$G = \mathbb{Z}$

$$\begin{array}{ccccccc} & & C_1 & \partial & C_0 & & \\ & & \| & \| & \| & & \\ \text{Consider} \quad 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\times m} & \mathbb{Z} & \longrightarrow & 0 \end{array}$$

$\Rightarrow$ homology : $H_0 = \mathbb{Z}/m\mathbb{Z} = \mathbb{Z}_m$, $H_1 = 0$

dual of homology :

$\underline{\mathrm{Hom}(H_0, \mathbb{Z}) = \mathrm{Hom}(\mathbb{Z}_m, \mathbb{Z}) = 0}$

$\underline{\mathrm{Hom}(H_1, \mathbb{Z}) = 0}$

dual complex :

$$\begin{array}{ccccccc} & & f(mx) & \longleftarrow\!\shortmid & f(x) & & \\ & & \cap & & \cap & & \\ 0 & \longleftarrow & \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) & \xleftarrow{\partial^*} & \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) & \longleftarrow & 0 \\ & & \cong & & \cong & & \\ & & \mathbb{Z} \ni f(m\cdot 1) & \xleftarrow{\times m} & f(1) \in \mathbb{Z} & & \\ & & \| & & & & \\ & & m\cdot f(1) & & & & \end{array}$$

cohomology : $\underline{H^0 = 0}$, $\underline{H^1 = \mathbb{Z}_m}$

Universal coefficient theorem

Let H, G be ab. gps. Suppose $\{x_\lambda\}_{\lambda \in \Lambda}$ generates H.
Let

$F_0 =$ free ab gp generated by $\{x_\lambda\}_{\lambda \in \Lambda}$

$$F_1 = \ker(F_0 \twoheadrightarrow H)$$ ← also a free ab gp

Then the seq

this part is called a free resolution of H

exer: Read p193 for the defition of free resolution

$$\boxed{0 \to F_1 \xhookrightarrow{f_1} F_0} \xrightarrow{f_0} H \to 0$$

exer: $\ker(f_1^*) = \operatorname{im}(f_0^*)$ exact here

is an exact seq. Furthermore, we have

$$0 \leftarrow \underline{\operatorname{Hom}(F_1, G)} \xleftarrow{f_1^*} \operatorname{Hom}(F_0, G) \xleftarrow{f_0^*} \operatorname{Hom}(H, G) \leftarrow 0$$

which is a cochain complex, but <u>NOT exact</u> at $\operatorname{Hom}(F_1, G)$ in general. Define

$$\underline{\operatorname{Ext}(H,G) \; (= \operatorname{Ext}^1(H,G)) := \operatorname{Hom}(F_1,G) / \operatorname{im}(f_1^*)}$$

<u>Lemma</u> (<u>Lemma 3.1</u>, p194)

$\operatorname{Ext}(H,G)$ is independent of the choice of F_0, F_1

pf: skip <u>exer</u>: Read Lemma 3.1 and its proof

<u>Properties of Ext(H,G)</u> (p195)

<u>exer</u> Prove the properties

- $\operatorname{Ext}(H \oplus H', G) \cong \operatorname{Ext}(H,G) \oplus \operatorname{Ext}(H', G)$
- $\operatorname{Ext}(H,G) = 0$ if H is free
- $\operatorname{Ext}(\mathbb{Z}_n, G) \cong G/nG$

<u>Thm 3.2</u> (Universal coefficient theorem for cohomology)

Let C be a chain complex of <u>free</u> abelian groups.

Then the seq.

$$0 \to \operatorname{Ext}(H_{n-1}(C), G) \to \underline{H^n(C;G)} \xrightarrow{h} \operatorname{Hom}(H_n(C), G) \to 0$$

is a split exact seq. i.e. $\cong \operatorname{Ext}(H_{n-1}, G) \oplus \operatorname{Hom}(H_n, G)$

pf. p 191 ~ 195, skip here

Note: $0 \to A \hookrightarrow A \oplus B \twoheadrightarrow B \to 0$ is exact, $a \mapsto (a,0)$, $(a,b) \mapsto b$

(p.191) $[\varphi]$

$h: H^n(C;G) \longrightarrow \operatorname{Hom}(H_n(C), G)$ $\quad$ $[c] \in H_n(C)$, $c \in C_n$, $\varphi: C_n \to G$

$h([\varphi])([c]) := \varphi(c)$

Lemma h is well-defined

pf: • $\delta\varphi = 0 = \varphi \circ \partial \Rightarrow \varphi|_{\operatorname{im}\partial} = 0$

$\Rightarrow \varphi: H_n(C) = \ker\partial / \operatorname{im}\partial \to G$

• $(\delta\psi)(c) = \psi(\partial(c)) = 0$ #

Example ⊛

$C: \quad 0 \longrightarrow \mathbb{Z} \xrightarrow{\times m} \mathbb{Z} \to 0$,

$\Rightarrow H_0 = \mathbb{Z}_m, \quad H_1 = 0 \quad \Rightarrow \operatorname{Ext}(H_0, \mathbb{Z}) \cong \mathbb{Z}_m$ by last class

$0 \to \operatorname{Ext}(H_0, \mathbb{Z}) = \mathbb{Z}_m \to H^1(C;\mathbb{Z}) \to \operatorname{Hom}(H_1, \mathbb{Z}) = 0 \to 0$

Cohomology of spaces

Let X be a space, G be an ab gp. $C_n(X) = \left\{ \begin{matrix} \text{singular} \\ n\text{-chains} \end{matrix} \right\}$

We define the group $C^n(X;G)$ of singular n-cochains with coefficients in G to be $\operatorname{Hom}(C_n(X), G)$

The coboundary map $\delta: C^n(X;G) \to C^{n+1}(X;G)$ is the dual ∂^* of the boundary map, i.e.

$\forall\, \varphi \in C^n(X;G)$, $\sigma: \Delta^{n+1} \to X$

$$(\delta\varphi)(\sigma) = \sum_i (-1)^i \varphi(\sigma|_{[v_0 \cdots \hat{v}_i \cdots v_{n+1}]})$$

Since $\partial^2 = 0 \Rightarrow \delta^2 = 0$, we can define the cohomology group $H(X;G)$ with coefficient in G

to be $\ker\delta / \operatorname{im}\delta$ at $C^n(X;G)$ in

$$\cdots \leftarrow C^{n+1}(X;G) \xleftarrow{\delta} C^n(X;G) \xleftarrow{\delta} C^{n-1}(X;G) \leftarrow \cdots$$

Elements in $\ker\delta$ are cocycles and elements in $\operatorname{im}\delta$ are coboundaries

By Thm3.2, we have the following

Thm

The seq

$$0 \to \operatorname{Ext}(H_{n-1}(X), G) \to H^n(X;G) \to \operatorname{Hom}(H_n(X), G) \to 0$$

is split exact

Remark

- Since $\operatorname{Ext}(H_{-1}(X), G) = 0$, we have $H^0(X;G) \cong \operatorname{Hom}(H_0(X), G)$
- Since $H_0(X) \cong \bigoplus_{\text{path components}} \mathbb{Z}$ is free, we have $\operatorname{Ext}(H_0(X);G) = 0$

  $\Rightarrow H^1(X;G) \cong \operatorname{Hom}(H_1(X), G)$
- If F is a field, then

  $H^n(X;F) \cong \operatorname{Hom}_F(H_n(X;F), F)$

Most thms for singular homology have coh versions:

① Reduced cohomology group:

$\tilde{H}^n(X;G)$ can be defined by dualizing the augmented chain complex chain complex

$$\sum n_i x_i \longmapsto \sum n_i$$

$$\cdots \to C_1(X) \to C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

then taking ker/im. So we have

$$\tilde{H}^n(X;G) = H^n(X;G), \quad \forall n>0$$

$$\tilde{H}^0(X;G) \cong \mathrm{Hom}(\tilde{H}_0(X), G) \leftarrow \text{by universal coeff thm } (\oplus \operatorname{Ext}(\tilde{H}_{-1}, G)=0)$$

By the construction,

$$\tilde{H}^0(X;G) = \left\{ \begin{array}{l} \text{functions } X \to G \text{ that are} \\ \text{constant on path-components} \end{array} \right\} \Big/ \left\{ \begin{array}{l} X \to G \\ \text{constant on} \\ \text{all } X \end{array} \right\}$$

② Relative cohomology group and long exact seq of a pair (X,A):

We first dualize

$$C_n(X)/C_n(A) = C_n(X,A)$$

$$0 \to C_n(A) \xrightarrow{i} C_n(X) \xrightarrow{j} C_n(X,A) \to 0$$

by apply $\mathrm{Hom}(-,G)$ to get

$$(*) \quad 0 \leftarrow C^n(A;G) \xleftarrow{i^*} C^n(X;G) \xleftarrow{j^*} C^n(X,A;G) \leftarrow 0$$

where $C^n(X,A;G) := \mathrm{Hom}(C_n(X,A), G)$

$H^n(X,A;G)$ $=$ ker/im at

<u>Lemma</u>

$(*)$ is a short exact seq of cochain cxes

pf

<u>i^*, j^* are compatible with $\delta = \partial^*$</u> : direct computation

<u>i^* is onto</u>: Since $C_n(A)$ and $C_n(X)$ are free, we can extend $\alpha \in \mathrm{Hom}(C_n(A), G)$ to $\hat{\alpha} \in \mathrm{Hom}(C_n(X), G)$ by $\hat{\alpha}(\sigma) := 0 \ \forall \sigma: \Delta^n \to X$, $\sigma(\Delta^n) \not\subset A$

$\Rightarrow \quad i^*(\hat{\alpha}) = \alpha$

<u>$\ker i^* = \operatorname{im} j^*$</u> : $\beta \in \ker i^* \iff \beta(\sigma) = 0 \quad \forall \sigma \in C_n(A)$

$\iff \exists \gamma : C_n(X)/C_n(A) \to G$ s.t. $\beta = \gamma \circ j$

<u>j^* is 1-1</u> : true because j is onto $\quad \#$

Therefore, we have the long exact seq

(***) $\quad \cdots \to H^n(X, A; G) \xrightarrow{j^*} H^n(X; G) \xrightarrow{i^*} H^n(A; G) \xrightarrow{\delta} H^{n+1}(X, A; G) \to \cdots$

Similarly, one also has $\left(\tilde{H}^n(X, A; G) \overset{A \neq \phi}{:=} H^n(X, A; G) \ \forall n \right)$

$\cdots \to \tilde{H}^n(X, A; G) \xrightarrow{j^*} \tilde{H}^n(X; G) \xrightarrow{i^*} \tilde{H}^n(A; G) \xrightarrow{\delta} \tilde{H}^{n+1}(X, A; G) \to \cdots$

and $(B \subseteq A \subseteq X)$

$\cdots \to H^n(X, A; G) \to H^n(X, B; G) \to H^n(A, B; G) \to H^{n+1}(X, A; G) \to \cdots$

<u>Note</u> (***) $\Rightarrow \tilde{H}^n(X; G) \cong H^n(X, \{x_0\}; G) \qquad x_0 \in X$