$n=2$:

Let $\sigma : \Delta^2 \longrightarrow D^2$ be a homeomorphism.

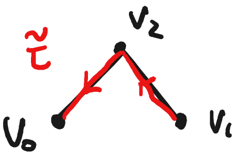
$$\begin{tikzcd}
H_2(D^2, S^1) \arrow[r, "\partial", "\cong"'] & \tilde{H}_1(S^1) \\
{[\sigma]} \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & {[\sum (-1)^i \sigma|_{[\cdots \hat{v}_i \cdots]}] = [\sigma|_{[v_1,v_2]} - \sigma|_{[v_0,v_2]} + \sigma|_{[v_0,v_1]}]} \arrow[u, phantom, sloped, "\in"]
\end{tikzcd}$$

$\uparrow$ is a generator $\Longleftarrow$ $[\tau]$ a generator $=$

Note

If we choose $\tilde{\tau} : \Delta^1 \longrightarrow S^1$, $\tilde{\tau}(t) = \begin{cases} \sigma|_{[v_1,v_2]}(2t), & 0 \leq t \leq \frac{1}{2} \\ \sigma|_{[v_0,v_2]}(2-2t), & \frac{1}{2} \leq t \leq 1 \end{cases}$

then $\tilde{\tau}^{-1} + (\sigma|_{[v_1,v_2]} - \sigma|_{[v_0,v_1]}) \in \operatorname{im}(\partial : C_2(S^1) \to C_1(S^1))$

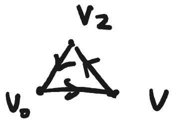


because $\exists\ \omega : \Delta^2 \to S^1$ s.t.

exer

$\omega|_{[v_1,v_2]} = \sigma|_{[v_1,v_2]}$, $\omega|_{[v_0,v_2]} = \sigma|_{[v_0,v_2]}$,

$\omega|_{[v_0,v_1]} = \tilde{\tau}^{-1}$

Similarly, if we choose $\tau : \Delta^1 \to$  $\xrightarrow{\sigma} S^1$,

then we have ① $\partial\tau = 0$

② $[\tau] = [\sigma|_{[v_1,v_2]} - \sigma|_{[v_0,v_2]} + \sigma|_{[v_0,v_1]}]$ in $H_1(S^1)$

$\uparrow$ this is a generator

11/25

Conclusion of last week discussion:

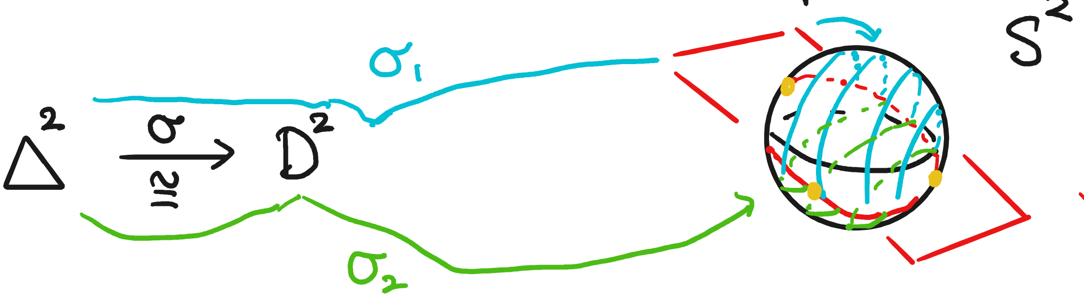
If $\sigma : \Delta^2 \longrightarrow D^2$ is a homeomorphism, then

① $[\sigma] \in H_2(D^2, S^1)$ is a generator

② $[\Delta^2 \xrightarrow{\sigma} D^2 \to D^2/S^1 \cong S^2] \in \tilde{H}_2(S^2)$ is a generator

Another generator of $H_2(S^2)$:

Let W be a 2-dim vec. subsp in $\mathbb{R}^3$, and $\sigma_1, \sigma_2 : \Delta^2 \to S^2$ be the maps



Denote

$$\Delta_1^2 = \sigma_1(\Delta^2), \qquad \Delta_2^2 = \sigma_2(\Delta^2)$$

Note that we have

by long exact seq of (S^2, Δ_2^2) by excision

$$\begin{tikzcd} \tilde{H}_2(S^2) \arrow[r, "\cong"] & H_2(S^2, \Delta_2^2) & H_2(\Delta_1^2, \partial\Delta_1^2) \arrow[l, "\cong"'] \\ {[\sigma_1 - \sigma_2]} \arrow[r, mapsto] & \square = \square & {[\sigma_1]} \arrow[l, mapsto] \end{tikzcd}$$

generator

Note: $\partial(\sigma_1 - \sigma_2) = 0$

So $[\sigma_1 - \sigma_2]$ is a generator of $\tilde{H}_2(S^2)$

general n:

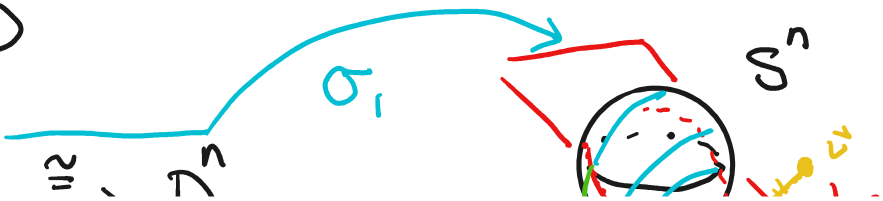
Inductively, one can show that

① a homeomorphism $\sigma : \Delta^n \to D^n$ induces a generator $[\sigma]$ of $H_n(D^n, S^{n-1})$

② Cut S^n by an n-dim vec subsp W in $\mathbb{R}^{n+1}$.

Each piece $\cong D^n$

We have



$\Delta^n \cong D^n$

$[\sigma_1 - \sigma_2]$ is a generator of $H_n(S^n)$

Remark

Let $r_W : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be the reflection along W

$\Rightarrow$ $\boxed{\sigma_1 = r_W \circ \sigma_2}$

Proof of Lemma

(i) A reflection $r_W : S^n \to S^n$ gives us a generator $[\sigma_1 - \sigma_2]$ of $H_n(S^n)$, and

$$(r_W)_*([\sigma_1 - \sigma_2]) = [r_W \circ \sigma_1 - r_W \circ \sigma_2]$$

$$= [\sigma_2 - \sigma_1] = -[\sigma_1 - \sigma_2]$$

$\Rightarrow$ $\deg(r_W) = -1$

(ii) $a : S^n \to S^n : \vec{v} \mapsto -\vec{v}$ can be written as

$$a = r_1 \circ r_2 \circ \cdots \circ r_{n+1}$$

where r_i = reflection along $\{x_i = 0\} \subseteq \mathbb{R}^{n+1}$

So

$$\deg(a) = \deg(r_1) \cdots \deg(r_{n+1}) = (-1)^{n+1}$$ ∎

Conclusion of Example 2.31:

The map

$$S^n \to \mathbb{R}P^n \longrightarrow \mathbb{R}P^n / \mathbb{R}P^{n-1} \cong S^n$$

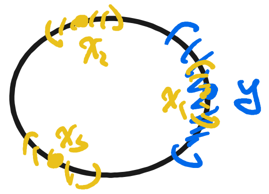
has degree $\underbrace{1 + (-1)^{n+1}}$

Example 2.32

Let $f : S^1 \subseteq \mathbb{C} \to S^1 : f(z) = z^k$

① $k=0$: $f =$ constant $\Rightarrow \deg(f) = 0$ (Property (b), p134)

② $k>0$: for $y \in S^1$, $f^{-1}(y)$ has k points $x_1, \dots, x_k$



map from  to 

is homotopic to rotation $\simeq$ id

which has $+1$

$$\Rightarrow \deg(f) = \sum_{i=1}^{k} \overset{+1}{\overset{\shortparallel}{\underbrace{\deg(f|_{x_i})}}} = k$$

③ $k<0$: $f(z) = (z^{-1})^{\underbrace{(-k)}^{>0}}$

The map $z \mapsto z^{-1}$ is the reflection along $\operatorname{Im}(z)=0$

$\Rightarrow \deg = -1$

$\Rightarrow \deg(f) = (-1)(-k) = k$

Conclusion

$$\deg(S^1 \to S^1 : z \mapsto z^k) = k$$

□

Remark

If $f : S^n \to S^n$ is differentiable, then $\deg(f)$ is

closely related to its differentials
See Milnor's book "Topology from the differentiable viewpoint"

Homology of $\mathbb{RP}^n$ (Example 2.42):

Recall from Example 0.4 that

$$\mathbb{RP}^n = e^0 \cup_{\varphi_1} e^1 \cup_{\varphi_2} \cdots \cup_{\varphi_n} e^n$$

$\varphi_k : \partial D^k \cong S^{k-1} \to \underbrace{\mathbb{RP}^{k-1}}_{\overset{\shortparallel}{e^0 \cup \cdots \cup e^{k-1}}}$ is the quotient map

By Example 2.31, the degree of

$S^{k-1} =$  $\xrightarrow{\varphi_k} \mathbb{RP}^{k-1} \longrightarrow \mathbb{RP}^{k-1}/\mathbb{RP}^{k-2} \cong S^{k-1}$

is
$$1 + (-1)^k = \begin{cases} 2 & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

So the cellular complex of $\mathbb{RP}^n$ is

$$\begin{cases} \cdots \to 0 \to \overset{e^n}{\mathbb{Z}} \xrightarrow{\times 2} \overset{e^{n-1}}{\mathbb{Z}} \xrightarrow{0} \cdots \xrightarrow{\times 2} \overset{e^1}{\mathbb{Z}} \xrightarrow{0} \overset{e^0}{\mathbb{Z}} & n \text{ even} \\ \cdots \to 0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\times 2} \cdots \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} & n \text{ odd} \end{cases}$$

$\Rightarrow$
$$H_k(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & \text{for } k=0,\ k=n \text{ odd} \\ \mathbb{Z}_2 & \text{for } k \text{ odd},\ 0<k<n \\ 0 & \text{otherwise} \end{cases}$$

exer:

Compute other examples in p.141 ~ 146

Euler characteristic

Recall that a finitely generated abelian group is isomorphic to

$$\mathbb{Z}^r \oplus G$$

for some number r, some finite group G

The number r is called the rank of this group

Def

Suppose a space X has finitely generated homology groups. Then the number

$$\chi(X) := \sum_n (-1)^n \operatorname{rank}(H_n(X))$$

is called the Euler characteristic of X

Thm 2.44

Let X be a finite CW complex, and

$$C_n := \# \text{ of } n\text{-cells}$$

$$\Rightarrow \quad \chi(X) = \sum_n (-1)^n C_n$$

finitely generated by assumption

Pf

C_{k-1} C_1 C_0

Let

$$0 \to \overset{\mathbb{Z}^{c_k}}{\overset{\cong}{\underline{C_k}}} \xrightarrow{d_k} \overset{\mathbb{Z}}{\overset{\cong}{C_{k-1}}} \to \cdots \to \overset{\mathbb{Z}^{-}}{\overset{\cong}{C_1}} \xrightarrow{d_1} \overset{\mathbb{Z}}{\overset{\cong}{C_0}} \xrightarrow{d_0} 0$$

be the corresponding cellular complex

Since

$$H_n(X) = \ker(d_n) / \operatorname{im}(d_{n+1})$$

we have

$$\operatorname{rank} H_n(X) = \operatorname{rank} \ker(d_n) - \underline{\operatorname{rank} \operatorname{im}(d_{n+1})}$$

By iso thm,

$$C_n / \ker(d_n) \cong \operatorname{im}(d_n)$$

$$\Rightarrow \quad \operatorname{rank} C_n = c_n = \underline{\operatorname{rank} \operatorname{im}(d_n)} + \operatorname{rank} \ker(d_n)$$

$$\Rightarrow \quad \operatorname{rank} H_n(X) = \operatorname{rank} \ker(d_n) + \operatorname{rank} \ker(d_{n+1}) - c_{n+1}$$

$$\Rightarrow \quad \sum_n (-1)^n \operatorname{rank} H_n(X)$$

$$= (\underline{\overset{= c_0}{\operatorname{rank} \ker(d_0)}} + \cancel{\operatorname{rank} \ker(d_1)} - c_1)$$

$$- (\cancel{\operatorname{rank} \ker(d_1)} + \operatorname{rank} \cancel{\ker(d_2)} - c_2)$$

$$\cdots$$

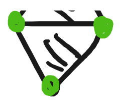
$$= \sum_n (-1)^n c_n$$

#

<u>Example</u>

$c_0 = 4 = V$ = vertices

$\Rightarrow \chi = c_0 - c_1 + c_2$

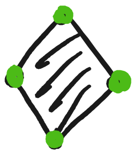


$C_1 = 5 = e = $ edges

$C_2 = 2 = f = $ faces

$= 4 - 5 + 2 = \boxed{1}$

$= v - e + f$



$C_0 = 4$

$C_1 = 4$

$C_2 = 1$

$\Rightarrow \chi = C_0 - C_1 + C_2$

$= 4 - 4 + 1 = \boxed{1}$

② See Example 2.36 & 2.37

M_g = orientable closed surface of genus g

N_g = nonorientable " "

$\Rightarrow \chi(M_g) = 2-2g \qquad \chi(N_g) = 2-g$.

Mayer-Vietoris seq (another seq for computation)

Version I (p 149)

Suppose $A, B \subseteq X$ are subspaces s.t. $X = \text{int}(A) \cup \text{int}(B)$

Then

$$\circledast \quad \cdots \to H_n(A\cap B) \xrightarrow{\Phi} H_n(A) \oplus H_n(B) \xrightarrow{\Psi} H_n(X) \to H_{n-1}(A\cap B) \to \cdots$$

is exact, where $\Phi([x]) = ([x], -[x])$, $\Psi([x],[y]) = [x+y]$

pf: exer

Version II (p 149)

Suppose $X = A \cup B$ s.t. A and B are deformation retracts of nbds U and V, resp, with $U \cap V$ deformation retracting onto $A \cup B$. Then $\circledast$ is still exact

sketch of pf

Sketch of pf

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(A\cap B) \arrow[r] \arrow[d, "\cong"] & H_n(A)\oplus H_n(B) \arrow[r] \arrow[d, "\cong"] & \text{"}H_n(A+B)\text{"} \arrow[r] \arrow[d] & H_{n-1}(A\cap B) \arrow[r] \arrow[d, "\cong"] & H_{n-1}(A)\oplus H_{n-1}(B) \arrow[r] \arrow[d, "\cong"] & \cdots \\
\cdots \arrow[r] & H_n(U\cap V) \arrow[r] & H_n(U)\oplus H_n(V) \arrow[r] & \boxed{\text{"}H_n(U+V)\text{"}} \arrow[r] \arrow[d, phantom, sloped, "\cong"] & H_{n-1}(U\cap V) \arrow[r] & H_{n-1}(U)\oplus H_{n-1}(V) \arrow[r] & \cdots \\
& & & H_n(X) & & &
\end{tikzcd}$$

Five lemma (p129)

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\alpha"'] & B \arrow[r] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"'] & D \arrow[r] \arrow[d, "\delta"'] & E \arrow[d, "\varepsilon"'] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E'
\end{tikzcd}$$

① Commutative

Suppose ② the 2 rows are exact

③ $\alpha, \beta, \delta, \varepsilon$ are iso

Then γ is also an iso.

Example 2.46

Take $X = S^n \subseteq \mathbb{R}^{n+1}$, $A = \{x_{n+1} \geq 0\}$, $B = \{x_{n+1} \leq 0\}$

$\Rightarrow$ $X = A \cup B$

$\Rightarrow$

$$\begin{tikzcd}
& 0 \arrow[d, equal] & & & 0 \arrow[d, equal] & \\
\cdots \arrow[r] & \tilde H_k(A)\oplus \tilde H_k(B) \arrow[r] & \tilde H_k(X) \arrow[r, "\cong"] \arrow[d, equal] & \tilde H_{k-1}(A\cap B) \arrow[r] \arrow[d, equal] & \tilde H_{k-1}(A)\oplus \tilde H_{k-1}(B) \arrow[r] & \cdots \\
& & \tilde H_k(S^n) & \tilde H_{k-1}(S^{n-1}) & &
\end{tikzcd}$$

is exact

$\Rightarrow$ $\tilde H_k(S^n) \cong \tilde H_{k-1}(S^{n-1})$ *

exer : Example 2.47

Homology with coefficients

Let G be an abelian gp (or a ring)

$$C_n(X;G) := \{\sum n_i \sigma_i \mid n_i \in G,\ \sigma_i : \Delta^n \to X\}$$

$$\partial(\sum n_i\sigma_i) = \sum_{ij} (-1)^i n_i \sigma_i|_{[v_0, \cdots \hat{v}_j \cdots v_n]}$$

The resulting homology groups $H_n(X;G)$ are called homology groups with coefficients in G

By the same constructions and arguments, we have

- relative homology with coeff in G : $H_n(X,A;G)$
- reduced " " : $\tilde{H}_n(X;G)$
- long exact seq of relative homologies
- excision thm
- Mayer-Vietoris seq.
- cellular homology

exer (Lemma 2.49)

Prove that

① $H_n(X;G) \cong H^{CW}_n(X;G)$

② $d_n(\sum_\alpha n_\alpha e^n_\alpha) = \sum_{\alpha,\beta} d_{\alpha\beta} n_\alpha e^{n-1}_\beta$

---

§3.1 Cohomology

Cohomology is a kind of dual notion of homology

A big difference between homology and coh:

Cohomology is equiped with a natural product, called cup product, which is useful in many topics such as characterictic classes

Remark

There is also a product in homology, called cross produst, but it's useless.

Next: "dual complex" "universal coefficient thm"