Algebraic Topology 11/18

Recall (computation method)

Step 1 Find a CW structure (key 1) of the space X

Step 2 Compute the boundary map of cellular complex by degrees of the attaching maps (key 2):

$$d_n(e^n_\alpha) = \sum_\beta d_{\alpha\beta}\, e^{n-1}_\beta$$

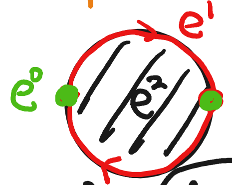
$$d_{\alpha\beta} = \text{degree of } \partial D^n_\alpha \cong S^{n-1}_\alpha \to X^{n-1} \to X^{n-1}/X^{n-1} \setminus e_\beta \;\; \cong S^{n-1}_\beta$$

Step 3 Compute cellular homology

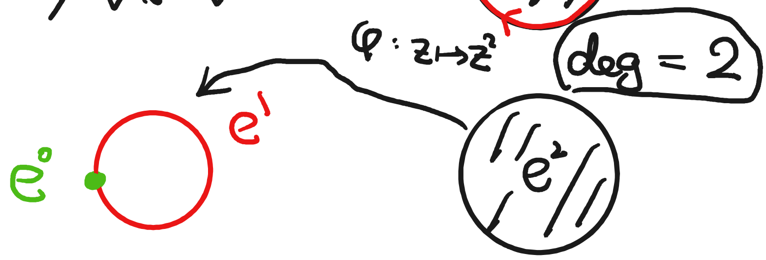
Example 2.37 (nonorientable closed surface N_g)

① $N_1 = \mathbb{RP}^2$ (See Example 0.4)

$$\cong S^2/_{\vec{v}\sim -\vec{v}} \cong$$



$$\cong e^0 \cup e^1 \cup_\varphi e^2$$



cellular complex of $\mathbb{RP}^2$:

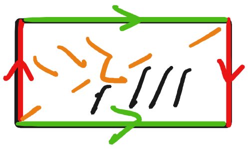
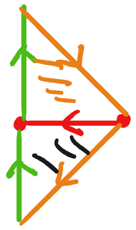
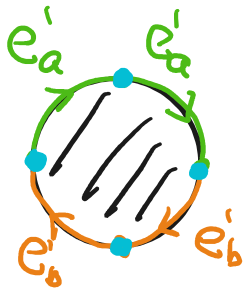
$$0 \to \underset{2}{\mathbb{Z}} \xrightarrow{\times 2} \underset{1}{\mathbb{Z}} \xrightarrow{0} \underset{0}{\mathbb{Z}} \to 0$$

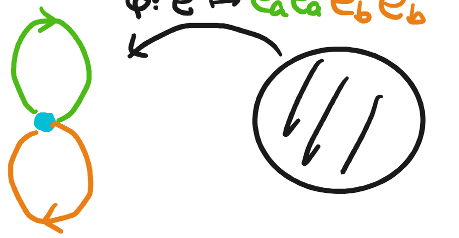
$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \to 0$$

$\Rightarrow H_0(\mathbb{RP}^2) \cong \mathbb{Z}, \qquad H_1(\mathbb{RP}^2) \cong \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2$

$H_k(\mathbb{RP}^2) = 0$ if $k \neq 0, 1$ #

② $N_2 \cong$ Klein bottle $= K$

$\cong$  $\cong$  $\cong$ 

$\cong e^0 \cup e^1_a \cup e^1_b \cup e^2$ 

Cellular complex:

$$0 \to \underset{e^2}{\mathbb{Z}} \longrightarrow \underset{e^1_a \quad e^1_b}{\mathbb{Z} \oplus \mathbb{Z}} \xrightarrow{0} \underset{e^0}{\mathbb{Z}} \to 0$$

$$1 \longmapsto (2, 2)$$

$(x+y, y) \leftarrow\!\shortmid (x, y)$

$\Rightarrow H_0(K) \cong \mathbb{Z}, \quad H_1(K) \cong \mathbb{Z} \oplus \mathbb{Z} / \langle (2,2) \rangle \cong \mathbb{Z} \oplus \mathbb{Z}_2$

$[(a,b)] \longmapsto (a-b, b)$

$(1, 0)$

$H_n(K) = 0, \ n \neq 0, 1$ #

exer:

$H_k(N_g) = ?$ $\quad N_g =$ nonorientable closed surface of genus g. See Example 2.37

Further reading:

Classification of closed surfaces $\begin{cases} \text{orientable, genus } g, \ g \geq 0 \\ \text{nonorientable, genus } g, \ g \geq 1 \end{cases}$

Google ↗

Next goal : Compute $H_k(\mathbb{R}P^n)$

Computation of degree

Let $f: S^n \to S^n$, $n>0$. Suppose $\exists\, y \in S^n$ s.t.

$$f^{-1}(y) = \{x_1, \cdots, x_m\} \quad \text{is a finite set.}$$

Let $U_1, \cdots, U_m$ be disjoint nbds of $x_1, \cdots, x_m$ and V be nbd of y s.t. $f(U_i) \subseteq V \quad \forall\, i = 1, \cdots, m.$

$\Rightarrow \quad f(U_i - \{x_i\}) \subseteq V - \{y\}$ and

by excision $\to \cong$

$$\begin{tikzcd}
 & H_n(U_i, U_i - \{x_i\}) \arrow[r, "f_*"] \arrow[d, "k_i"] \arrow[dl, "\cong"'] & H_n(V, V - \{y\}) \arrow[d, "\cong"] \\
\circledast \; H_n(S^n, S^n - \{x_i\}) & H_n(S^n, S^n - f^{-1}(y)) \arrow[l, "p_i"'] \arrow[r, "f_*"] & H_n(S^n, S^n - \{y\}) \\
 & H_n(S^n) \arrow[u, "j"] \arrow[ul, "\cong"] \arrow[r, "f_*"] & H_n(S^n) \arrow[u, "\cong"']
\end{tikzcd}$$

$\cong \leftarrow$ excision

by long exact seq of $(S^n, S^n - \{x_i\}) \to \cong$

$\cong \leftarrow$ by long exact seq of $(S^n, S^n - \{y\})$

Thus, $H_n(U_i, U_i - \{x_i\}) \cong H_n(S^n) \cong \mathbb{Z} \cong H_n(V, V - \{y\})$

and $\exists!\, d \in \mathbb{Z}$ s.t. $f_*(x) = d \cdot x$ under these iso's

This number d is called the local degree of f at x_i, written $\deg f|_{x_i}$

Prop 2.30

Let $f : S^n \to S^n$, $n>0$, Suppose $\exists\, y \in S^n$ s.t.

$$f^{-1}(y) = \{x_1, \cdots, x_m\} \quad \text{is finite.}$$

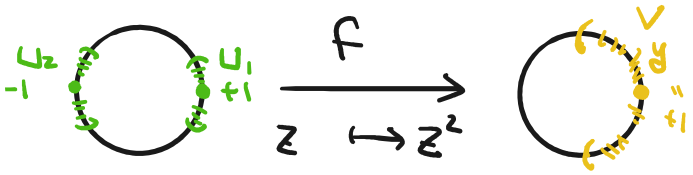
Then
$$\deg f = \sum_{i=1}^{m} \deg f|_{x_i}$$

pf : by studying $\circledast$. omit here.

Example

$f: S^1 \to S^1 \subseteq \mathbb{C} : z \mapsto z^2$

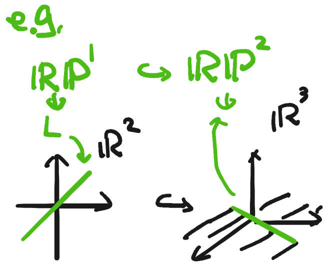
$y = +1, \quad f^{-1}(1) = \{\pm 1\}$



$\because$ " $f|_{U_i} \simeq id$ " $\therefore \quad \deg f|_{x_i} = 1$

$\Rightarrow \quad \deg f = \deg f|_{x_1} + \deg f|_{x_2} = 1 + 1 = 2$

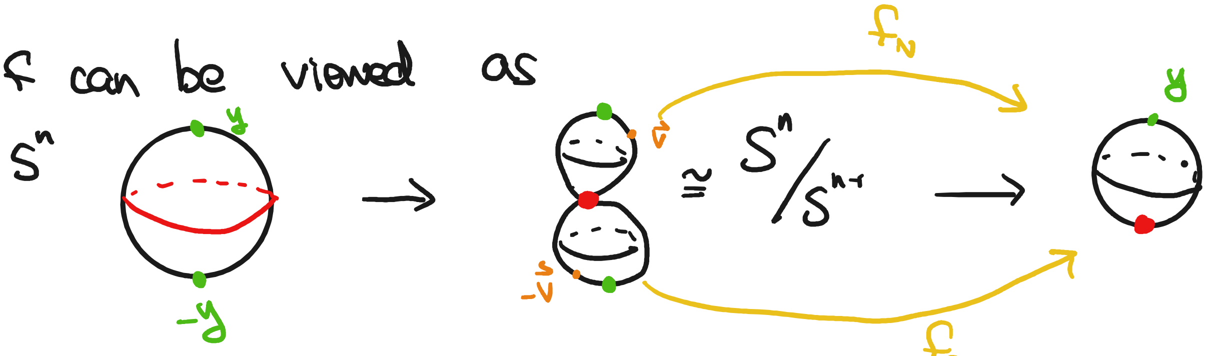
Example 2.31 Recall $\mathbb{RP}^n \cong e^0 \cup e^1 \cup \cdots \cup e^n$



Let $f : S^n \to \mathbb{RP}^n \cong S^n/_{\vec{v} \sim -\vec{v}} \to \mathbb{RP}^n/\mathbb{RP}^{n-1} \cong S^n$

$$S^n \xrightarrow{f} S^n$$

f can be viewed as



$\deg f = \deg f|_y + \deg f|_{-y}$

$= \deg f_N + \deg f_S = 1 + \deg(a)$

Note: " $f_N \simeq id$ " " $f_S \simeq a$ "

where $a: S^n \to S^n : \vec{v} \mapsto -\vec{v}$ $\qquad H^n(S^n)$

Q: $\deg(a) = ?$

Lemma

(i) (Property (e), p134) A reflection of S^n (w.r.t. an n-dim subsp in $\mathbb{R}^{n+1}$) has deg -1

(ii) (Property (f), p134)

$$\deg(a) = (-1)^{n+1}$$

Generator of $H_n(S^n)$ (Example 2.23, p125):

Recall: Since $(D^n, S^{n-1} \cong \partial D^n)$ is a good pair, we have

$$\cdots \to \tilde{H}_n(D^n) = 0 \to H_n(D^n, S^{n-1}) \xrightarrow{\cong} \tilde{H}_{n-1}(S^{n-1}) \to \tilde{H}_{n-1}(D^n) = 0 \to \cdots$$

$$H_n(D^n, S^{n-1}) \cong H_n(D^n/S^{n-1}, S^{n-1}/S^{n-1}) \cong \tilde{H}_n(S^n)$$

$$\tilde{H}_{n-1}(S^{n-1}) \cong \tilde{H}_n(S^n)$$

n=0: 2 points, x_1, x_{-1}

$$\tilde{H}_0(S^0) = \langle [x_1 - x_{-1}] \rangle \cong \mathbb{Z}$$

where x_i is the constant map $\Delta^0 \to S^0$ whose image is x_i

n=1: "a path, $\sigma(0) = x_{\pm 1}$, $\sigma(1) = x_{\mp 1}$" "$[x_{-1}, x_1]$"

Let $\sigma : \Delta^1 \to D^1$ be a homeomorphism

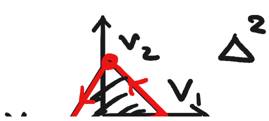
$$[\sigma] \longleftrightarrow [\sigma(1) - \sigma(0)] = \pm[x_1 - x_{-1}]$$

$$H_1(D^1, S^0) \xrightarrow{\cong} \tilde{H}_0(S^0) \quad \text{a generator}$$

$$\downarrow \cong$$

$$H_1(D^1/S^0, S^0/S^0) \cong \tilde{H}_1(S^1)$$

$[\omega]$ "$\omega(x) = e^{2\pi i \sigma(x)}$"



$\underline{n=2}$:

Let $\sigma : \Delta^2 \to D^2$ be a homeomorphism.

$$\begin{array}{ccc} H_2(D^2, S^1) & \xrightarrow[\cong]{\partial} & \tilde{H}_1(S^1) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ [\sigma] & \longmapsto & \left[\sum (-1)^i \sigma|_{[\cdots \hat{v}_i \cdots]}\right] = \left[\sigma|_{[v_1,v_2]} - \sigma|_{[v_0,v_2]} + \sigma|_{[v_0,v_1]}\right] \end{array}$$

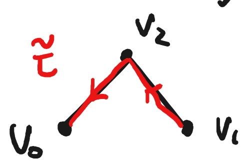
$[\sigma]$ is a generator $\Longleftarrow$ $[\tau]$ a generator $=$

Note

If we choose $\tilde{\tau} : \Delta^1 \to S^1$, $\tilde{\tau}(t) = \begin{cases} \sigma|_{[v_1,v_2]}(2t), & 0 \le t \le \frac{1}{2} \\ \sigma|_{[v_0,v_2]}(2-2t), & \frac{1}{2} \le t \le 1 \end{cases}$

then $\tilde{\tau}^{-1} + (\sigma|_{[v_1,v_2]} - \sigma|_{[v_0,v_1]}) \in \operatorname{im}(\partial : C_2(S^1) \to C_1(S^1))$

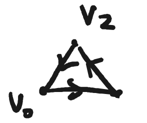
because $\exists\ \omega : \Delta^2 \to S^1$ s.t.



exer

$\omega|_{[v_1,v_2]} = \sigma|_{[v_1,v_2]}$, $\omega|_{[v_0,v_2]} = \sigma|_{[v_0,v_2]}$,

$\omega|_{[v_0,v_1]} = \tilde{\tau}^{-1}$

Similarly, if we choose $\tau : \Delta^1 \to$  $\xrightarrow{\sigma} S^1$,

then we have ① $\partial\tau = 0$

② $[\tau] = [\sigma|_{[v_1,v_2]} - \sigma|_{[v_0,v_1]} + \sigma|_{[v_0,v_1]}]$ in $H_1(S^1)$

this is a generator