

Revised on 2023-11-14

1 Chapter 2: Regular Surfaces.

Remark 1.1 *You may have to read the Appendix of this chapter in p.118 of the textbook first, if you have forgotten some stuff in Advanced Calculus.*

1.1 The First Fundamental Form; Area (this is Section 2-5 of the textbook).

Let $S \subset \mathbb{R}^3$ be a regular surface. The natural inner product of \mathbb{R}^3 induces on each tangent plane $T_p S \subset \mathbb{R}^3$ an inner product, denoted as $\langle \cdot, \cdot \rangle_p$. For any $w_1, w_2 \in T_p S$, then $\langle w_1, w_2 \rangle_p$ is equal to the inner product of w_1 and w_2 viewed as vectors in \mathbb{R}^3 .

Definition 1.2 *The map*

$$I_p(w) = \langle w, w \rangle_p = |w|^2 : T_p S \rightarrow \mathbb{R} \quad (1)$$

*is called the **first fundamental form** of the regular surface S at $p \in S$. In Linear Algebra terminology, $I_p(w)$ is called a **quadratic form** on the tangent space $T_p S$.*

Remark 1.3 *The first fundamental form is the expression of how the surface S inherits the natural inner product of \mathbb{R}^3 . One can use it to measure the length of curves on S or the area of a region on S .*

Definition 1.4 *If $\mathbf{x}(u, v) : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is a local parametrization at $p = \mathbf{x}(u_0, v_0) \in S$ with basis $\{\mathbf{x}_u, \mathbf{x}_v\}$, then the following three functions*

$$E(u_0, v_0) = \langle \mathbf{x}_u, \mathbf{x}_u \rangle_p, \quad F(u_0, v_0) = \langle \mathbf{x}_u, \mathbf{x}_v \rangle_p, \quad G(u_0, v_0) = \langle \mathbf{x}_v, \mathbf{x}_v \rangle_p, \quad (2)$$

*are called the **coefficients of the first fundamental form in the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$ of $T_p S$.***

Remark 1.5 *Since $\mathbf{x}_u(u, v)$ and $\mathbf{x}_v(u, v)$ are differentiable map of $(u, v) \in U$ into \mathbb{R}^3 , the three functions*

$$E(u, v), \quad F(u, v), \quad G(u, v) : (u, v) \in U \rightarrow \mathbb{R}$$

are all differentiable on U .

If $\alpha(t) = \mathbf{x}(u(t), v(t)) : (-\varepsilon, \varepsilon) \rightarrow S$ is a differentiable curve with $\alpha(0) = \mathbf{x}(u_0, v_0) = p$, then

$$\begin{aligned} I_p(\alpha'(0)) &= \langle \alpha'(0), \alpha'(0) \rangle_p \\ &= \langle \mathbf{x}_u(u_0, v_0) u'(0) + \mathbf{x}_v(u_0, v_0) v'(0), \mathbf{x}_u(u_0, v_0) u'(0) + \mathbf{x}_v(u_0, v_0) v'(0) \rangle_p \\ &= E(u_0, v_0) (u'(0))^2 + 2F(u_0, v_0) u'(0) v'(0) + G(u_0, v_0) (v'(0))^2 \\ &= \underbrace{(u'(0), v'(0)) \begin{pmatrix} E(u_0, v_0) & F(u_0, v_0) \\ F(u_0, v_0) & G(u_0, v_0) \end{pmatrix} \begin{pmatrix} u'(0) \\ v'(0) \end{pmatrix}} \end{aligned}$$

Lemma 1.6 *The matrix*

$$\begin{pmatrix} E(u_0, v_0) & F(u_0, v_0) \\ F(u_0, v_0) & G(u_0, v_0) \end{pmatrix} \quad (3)$$

is symmetric and positive definite.

Proof. It is clear that it is symmetric. To show positive definite, it is equivalent to saying that

$$E(u_0, v_0) > 0, \quad E(u_0, v_0)G(u_0, v_0) - F^2(u_0, v_0) > 0 \quad (4)$$

((4) will imply that $G(u_0, v_0) > 0$ also). This is easy by the Cauchy-Schwarz inequality. \square

Remark 1.7 One can also use definition to see that (3) is positive definite since it is the components of an inner product, which is positive definite, or one can see that

$$(a, b) \begin{pmatrix} E(u_0, v_0) & F(u_0, v_0) \\ F(u_0, v_0) & G(u_0, v_0) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} > 0, \quad \forall (a, b) \neq (0, 0) \in \mathbb{R}^2.$$

Example 1.8 Do Example 1, 2, 3 in p. 95.

We can use the first fundamental form I to answer metric questions on S without further references to the ambient space \mathbb{R}^3 . Let $\alpha(t) : 0 \in I \rightarrow S \subset \mathbb{R}^3$ be a curve on S . Its **arc length parameter** s is given by

$$\begin{aligned} s(t) &= \int_0^t |\alpha'(t)| dt = \int_0^t \sqrt{I(\alpha'(t))} dt, \quad \alpha'(t) = \mathbf{x}_u(u(t), v(t))u'(t) + \mathbf{x}_v(u(t), v(t))v'(t) \\ &= \int_0^t \sqrt{E(u(t), v(t))(u'(t))^2 + 2F(u(t), v(t))u'(t)v'(t) + G(u(t), v(t))(v'(t))^2} dt \\ &= \int_0^t \sqrt{E(u')^2 + 2Fu'v' + G(v')^2} dt, \quad t \in I. \end{aligned} \quad (5)$$

Remark 1.9 (Notation.) We usually write (5) as the convenient form

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2,$$

which means that if $\alpha(t) = \mathbf{x}(u(t), v(t))$ is a curve on S with arc length $s = s(t)$, then

$$\left(\frac{ds}{dt}\right)^2 = E(u(t), v(t))\left(\frac{du}{dt}\right)^2 + 2F(u(t), v(t))\frac{du}{dt}\frac{dv}{dt} + G(u(t), v(t))\left(\frac{dv}{dt}\right)^2.$$

Remark 1.10 If S is the **cylinder** in Example 2 in p. 95, we have $E \equiv 1$, $F \equiv 0$, $G \equiv 1$ and then

$$s(t) = \int_0^t |\alpha'(t)| dt = \int_0^t \sqrt{I(\alpha'(t))} dt = \int_0^t \sqrt{(u'(t))^2 + (v'(t))^2} dt, \quad t \in (-\varepsilon, \varepsilon),$$

i.e., the length between **any two points** of $\alpha(t)$ on S is equal to the length of its corresponding two points on $(u(t), v(t)) \in U$. In this case, we say the cylinder is **isometric** to the open set in \mathbb{R}^2 :

$$U = \{(u, v) \in \mathbb{R}^2 : 0 < u < 2\pi, -\infty < v < \infty\} \subset \mathbb{R}^2.$$

From the viewpoint of "metric geometry", the cylinder is the same as the plane. However, they have different curvature (more precisely, different mean curvature).

With the help of the first fundamental form I on S , we can also discuss the angle between two vectors on the same tangent space. In particular, if $\alpha(t) : I \rightarrow S$ and $\beta(t) : I \rightarrow S$ are two curves on S and they intersect at $t = t_0$, their intersection angle $\theta \in [0, \pi]$ between the two curves is defined as

$$\cos \theta = \frac{\langle \alpha'(t_0), \beta'(t_0) \rangle}{|\alpha'(t_0)| |\beta'(t_0)|} \in [-1, 1].$$

In particular, the angle φ between the two coordinate curves $\mathbf{x}(\cdot, v)$, $\mathbf{x}(u, \cdot)$ of a parametrization $\mathbf{x}(u, v)$ is given by

$$\cos \varphi = \frac{\langle \mathbf{x}_u, \mathbf{x}_v \rangle}{|\mathbf{x}_u| |\mathbf{x}_v|} = \frac{F}{\sqrt{EG}}.$$

Thus the coordinate curves of a parametrization $\mathbf{x}(u, v)$ are **orthogonal** everywhere if and only if $F(u, v) = 0$ for all (u, v) . Such a parametrization is called an **orthogonal parametrization**.

Example 1.11 Do Example 4 in p. 98.

Remark 1.12 The following integral formula is for your reference when you read p. 99:

$$\begin{aligned} \int \frac{1}{\sin \theta} d\theta &= \int \csc \theta d\theta = \int \frac{\csc^2 \theta - \csc \theta \cot \theta}{\csc \theta - \cot \theta} d\theta = \int \frac{\frac{d}{d\theta} (\csc \theta - \cot \theta)}{\csc \theta - \cot \theta} d\theta \\ &= \log |\csc \theta - \cot \theta| = \log \left| \frac{1 - \cos \theta}{\sin \theta} \right| = \log \left| \frac{2 \sin^2 \frac{\theta}{2}}{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}} \right| = \log \left| \tan \frac{\theta}{2} \right|. \end{aligned}$$

1.1.1 Area Formula on a Regular Surface (this is Section 2-5 of the textbook).

Definition 1.13 See p. 99 for the definition of a (regular) domain and region on S .

Remark 1.14 Explain the meaning of domain and region on S .

Recall that if a, b are two vectors in \mathbb{R}^3 , then

$$|a \wedge b| = \text{area of the parallelogram generated by } a, b.$$

Motivated by this, if $\mathbf{x} : U \rightarrow S$ is a local parametrization and $Q \subset U$ is a compact region with boundary a piecewise smooth simple closed curve, **we define the area of $R = \mathbf{x}(Q) \subset S$ as:**

$$A(R) = \iint_Q |\mathbf{x}_u \wedge \mathbf{x}_v| dudv. \quad (6)$$

We also use $\iint_R d\sigma$ to denote the area $A(R)$ of $R \subset S$.

Lemma 1.15 The above definition **does not** depend on the parametrization $\mathbf{x}(u, v)$.

Proof. Assume we have another parametrization $\bar{\mathbf{x}}(\bar{u}, \bar{v}) : \bar{U} \subset \mathbb{R}^2 \rightarrow S$ such that $\bar{\mathbf{x}}(\bar{Q}) = \mathbf{x}(Q) = R$, where $\bar{Q} \subset \bar{U}$ is a compact region with boundary a piecewise smooth simple closed curve. Then we have $Q = \mathbf{x}^{-1} \circ \bar{\mathbf{x}}(\bar{Q})$ and due to the change of parameter function $h = \mathbf{x}^{-1} \circ \bar{\mathbf{x}}$ one can express $(u, v) \in Q = h(\bar{Q})$ as a function of $(\bar{u}, \bar{v}) \in \bar{Q}$ (such a relation is a **diffeomorphism**). We now have the following identities

$$\begin{cases} (u, v) = h(\bar{u}, \bar{v}) = (u(\bar{u}, \bar{v}), v(\bar{u}, \bar{v})), \\ \bar{\mathbf{x}}(\bar{u}, \bar{v}) = \mathbf{x} \circ h(\bar{u}, \bar{v}) = \mathbf{x}(u(\bar{u}, \bar{v}), v(\bar{u}, \bar{v})), \end{cases}$$

and by the chain rule

$$\bar{\mathbf{x}}_{\bar{u}} = \frac{\partial u}{\partial \bar{u}} \mathbf{x}_u + \frac{\partial v}{\partial \bar{u}} \mathbf{x}_v, \quad \bar{\mathbf{x}}_{\bar{v}} = \frac{\partial u}{\partial \bar{v}} \mathbf{x}_u + \frac{\partial v}{\partial \bar{v}} \mathbf{x}_v,$$

we get

$$\begin{aligned} \bar{\mathbf{x}}_{\bar{u}} \wedge \bar{\mathbf{x}}_{\bar{v}} &= \left(\frac{\partial u}{\partial \bar{u}} \mathbf{x}_u + \frac{\partial v}{\partial \bar{u}} \mathbf{x}_v \right) \wedge \left(\frac{\partial u}{\partial \bar{v}} \mathbf{x}_u + \frac{\partial v}{\partial \bar{v}} \mathbf{x}_v \right) \\ &= \det \begin{pmatrix} \frac{\partial u}{\partial \bar{u}} & \frac{\partial u}{\partial \bar{v}} \\ \frac{\partial v}{\partial \bar{u}} & \frac{\partial v}{\partial \bar{v}} \end{pmatrix} \cdot (\mathbf{x}_u \wedge \mathbf{x}_v) \end{aligned} \quad (7)$$

More precisely, (7) is the same as

$$\begin{aligned} &\underbrace{\bar{\mathbf{x}}_{\bar{u}}(\bar{u}, \bar{v}) \wedge \bar{\mathbf{x}}_{\bar{v}}(\bar{u}, \bar{v})}_{=} \\ &= \det \begin{pmatrix} \frac{\partial u}{\partial \bar{u}} & \frac{\partial u}{\partial \bar{v}} \\ \frac{\partial v}{\partial \bar{u}} & \frac{\partial v}{\partial \bar{v}} \end{pmatrix} \cdot (\mathbf{x}_u(u(\bar{u}, \bar{v}), v(\bar{u}, \bar{v})) \wedge \mathbf{x}_v(u(\bar{u}, \bar{v}), v(\bar{u}, \bar{v}))) \\ &= \underbrace{(\mathbf{x}_u(h(\bar{u}, \bar{v})) \wedge \mathbf{x}_v(h(\bar{u}, \bar{v})))}_{=} \frac{\partial(u, v)}{\partial(\bar{u}, \bar{v})}, \quad \forall (\bar{u}, \bar{v}) \in \bar{Q}. \end{aligned} \quad (8)$$

Now by the change of variables formula for multiple integrals (see the book by Marsden "**Elementary Classical Analysis, 2nd edition**", p. 523), we have (note that $Q = h(\bar{Q})$)

$$\begin{aligned} & \iint_Q |\mathbf{x}_u(u, v) \wedge \mathbf{x}_v(u, v)| \, dudv \\ &= \iint_{\bar{Q}} |\mathbf{x}_u(h(\bar{u}, \bar{v})) \wedge \mathbf{x}_v(h(\bar{u}, \bar{v}))| \left| \frac{\partial(u, v)}{\partial(\bar{u}, \bar{v})} \right| d\bar{u}d\bar{v} = \iint_{\bar{Q}} |\bar{\mathbf{x}}_{\bar{u}}(\bar{u}, \bar{v}) \wedge \bar{\mathbf{x}}_{\bar{v}}(\bar{u}, \bar{v})| d\bar{u}d\bar{v} \end{aligned} \quad (9)$$

and we conclude

$$\iint_Q |\mathbf{x}_u \wedge \mathbf{x}_v| \, dudv = \iint_{\bar{Q}} |\bar{\mathbf{x}}_{\bar{u}} \wedge \bar{\mathbf{x}}_{\bar{v}}| d\bar{u}d\bar{v}, \quad (10)$$

i.e., the above definition is independent of the parametrizations we used. \square

Remark 1.16 *The assumption that the domain $R \subset S$ is contained in the image of a single parametrization is not very serious since in most examples there exists a parametrization \mathbf{x} which cover the entire surface except for some curves, which do not contribute to the area.*

Remark 1.17 *By the identity*

$$|\mathbf{x}_u \wedge \mathbf{x}_v|^2 + \langle \mathbf{x}_u, \mathbf{x}_v \rangle^2 = |\mathbf{x}_u|^2 |\mathbf{x}_v|^2,$$

we have

$$|\mathbf{x}_u \wedge \mathbf{x}_v|^2 = |\mathbf{x}_u|^2 |\mathbf{x}_v|^2 - \langle \mathbf{x}_u, \mathbf{x}_v \rangle^2,$$

i.e.

$$|\mathbf{x}_u \wedge \mathbf{x}_v| = \sqrt{EG - F^2}.$$

Hence we can express $A(R)$ as

$$A(R) = \iint_Q \sqrt{EG - F^2} \, dudv = A(R) = \iint_Q \sqrt{\begin{vmatrix} \langle \mathbf{x}_u, \mathbf{x}_u \rangle & \langle \mathbf{x}_u, \mathbf{x}_v \rangle \\ \langle \mathbf{x}_v, \mathbf{x}_u \rangle & \langle \mathbf{x}_v, \mathbf{x}_v \rangle \end{vmatrix}} \, dudv. \quad (11)$$

Example 1.18 *Do Example 5 in p. 101.*

1.2 Gradient on Surfaces (this is Exercise 14 in p. 104, Section 2-5, of the textbook).

Definition 1.19 *The **gradient** of a differentiable function $f : S \rightarrow \mathbb{R}$ is a differentiable map*

$$\text{grad } f : S \rightarrow \mathbb{R}^3, \quad (12)$$

*which assigns to each $p \in S$ a vector $\text{grad } f(p) \in T_p S \subset \mathbb{R}^3$ such that (the following identity is the **definition** of $\text{grad } f(p)$)*

$$\langle \text{grad } f(p), v \rangle_p = df_p(v) \quad \text{for all } v \in T_p S. \quad (13)$$

*By property in linear algebra, the vector $\text{grad } f(p)$ satisfying (13) **exists** and is **unique**. If there are two vectors $v_1, v_2 \in T_p S$ which satisfies*

$$\langle v_1, v \rangle = df_p(v) \quad \text{for all } v \in T_p S$$

and

$$\langle v_2, v \rangle = df_p(v) \quad \text{for all } v \in T_p S,$$

*then we must have $v_1 = v_2$. Geometrically, one can view $\text{grad } f : S \rightarrow \mathbb{R}^3$ as a **vector field** on S . It assigns each $p \in S$ a vector $\text{grad } f(p) \in T_p S$.*

Remark 1.20 For simplicity, we usually use the notation $\nabla_S f : S \rightarrow T_p S \subset \mathbb{R}^3$ to denote the gradient of $f : S \rightarrow \mathbb{R}$ on the surface S . Therefore, we have

$$\langle \nabla_S f(p), v \rangle = df_p(v) \quad \text{for all } v \in T_p S. \quad (14)$$

Since $\nabla_S f$ is a **vector field** on S , it can be expressed as a linear combination of the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$. Write

$$\nabla_S f = a\mathbf{x}_u + b\mathbf{x}_v.$$

Then

$$f_u = df_p(\mathbf{x}_u) = \langle \nabla_S f, \mathbf{x}_u \rangle = \langle a\mathbf{x}_u + b\mathbf{x}_v, \mathbf{x}_u \rangle = aE + bF, \quad f_u = \frac{\partial}{\partial u} f(\mathbf{x}(u, v)). \quad (15)$$

Similarly we have

$$f_v = df_p(\mathbf{x}_v) = \langle \nabla_S f, \mathbf{x}_v \rangle = \langle a\mathbf{x}_u + b\mathbf{x}_v, \mathbf{x}_v \rangle = aF + bG, \quad f_v = \frac{\partial}{\partial v} f(\mathbf{x}(u, v)). \quad (16)$$

Hence we get the matrix relation

$$\begin{pmatrix} f_u \\ f_v \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \iff \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} f_u \\ f_v \end{pmatrix}$$

and obtain, under a local parametrization $\mathbf{x}(u, v)$, the expression

$$\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{EG - F^2} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix} \begin{pmatrix} f_u \\ f_v \end{pmatrix},$$

which implies

$$\nabla_S f = a\mathbf{x}_u + b\mathbf{x}_v = \frac{f_u G - f_v F}{EG - F^2} \mathbf{x}_u + \frac{f_v E - f_u F}{EG - F^2} \mathbf{x}_v. \quad (17)$$

In particular, we see that $\nabla_S f$ is a differentiable function on $(u, v) \in U$. If $S = \mathbb{R}^2$ with Euclidean coordinates x, y , then we have $E = 1, F = 0, G = 1$, and the above becomes

$$\nabla_S f = \left(\frac{\partial f}{\partial x} \right) \mathbf{e}_1 + \left(\frac{\partial f}{\partial y} \right) \mathbf{e}_2$$

where $\{\mathbf{e}_1, \mathbf{e}_2\}$ is the standard basis of \mathbb{R}^2 . Moreover, for a given regular surface S **with orthogonal parametrization** $\mathbf{x}(u, v) : U \subset \mathbb{R}^2 \rightarrow S \subset \mathbb{R}^3$, (17) becomes (now we have $F \equiv 0$)

$$\nabla_S f = \frac{f_u}{E} \mathbf{x}_u + \frac{f_v}{G} \mathbf{x}_v. \quad (18)$$

Fix $p \in S$ and vary v in the **unit circle** $|v| = 1$ in $T_p S$ centered at $p \in S$ (denote this compact set as $S^1 \subset T_p S$), then $df_p(v) = \langle \nabla_S f(p), v \rangle \in \mathbb{R}$ attains its maximum value over $S^1 \subset T_p S$ at

$$v = \frac{\nabla_S f(p)}{|\nabla_S f(p)|}, \quad df_p(v) = |\nabla_S f(p)| \quad (19)$$

and attains its minimum value over $S^1 \subset T_p S$ at

$$v = -\frac{\nabla_S f(p)}{|\nabla_S f(p)|}, \quad df_p(v) = -|\nabla_S f(p)|. \quad (20)$$

This is clear from (13).

1.2.1 Comparing Euclidean Gradient and Surface Gradient.

Assume $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ is a differentiable function and $S \subset \mathbb{R}^3$ is a regular surface. We can restrict f onto S and $f|_S : S \rightarrow \mathbb{R}$ is also a differentiable function on S . For fixed $p \in S$ we have two gradient vectors at p , namely $\nabla f(p)$ (a vector in the space $T_p\mathbb{R}^3 \approx \mathbb{R}^3$ or one can say it is a vector in the ambient space \mathbb{R}^3) and $\nabla_S f(p)$ (a vector in the tangent space $T_pS \approx \mathbb{R}^2$). We have the following important result:

Theorem 1.21 *Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ be a differentiable function and $S \subset \mathbb{R}^3$ be a regular surface. The **projection** of the vector $\nabla f(p) \in \mathbb{R}^3$ onto T_pS is equal to $\nabla_S f(p) \in T_pS$.*

Proof. By the definition of gradient vector, we have the following:

$$\begin{cases} (1). \langle \nabla f(p), v \rangle = df_p(v), & \forall v \in \mathbb{R}^3, \\ (2). \langle \nabla_S f(p), v \rangle = df_p(v), & \forall v \in T_pS. \end{cases} \quad (21)$$

If we restrict v onto the subspace $T_pS \subset \mathbb{R}^3$, then (1) implies

$$\langle \nabla f(p), v \rangle = df_p(v), \quad \forall v \in T_pS. \quad (22)$$

Since $v \in T_pS$ in (22), we have

$$\begin{aligned} & \langle \nabla f(p), v \rangle \\ &= \langle (\text{normal part of } \nabla f(p) + \text{tangential part of } \nabla f(p)), v \rangle \\ &= \langle \text{tangential part of } \nabla f(p), v \rangle, \quad \forall v \in T_pS. \end{aligned} \quad (23)$$

Therefore, we conclude

$$df_p(v) = \langle \nabla f(p), v \rangle = \left\langle \underbrace{\text{tangential part of } \nabla f(p)}_{\text{tangential part of } \nabla f(p)}, v \right\rangle, \quad \forall v \in T_pS. \quad (24)$$

By (24) and (2) in (21) and uniqueness of gradient vector, we have

$$\nabla_S f(p) = \underbrace{\text{tangential part of } \nabla f(p)}_{\text{tangential part of } \nabla f(p)} = \text{the projection of } \nabla f(p) \text{ onto } T_pS. \quad (25)$$

The proof is done. □

1.3 Stereographic Projection of S^2 (this is Exercise 16 in p. 69, Section 2-2, of the textbook).

Consider the sphere S^2 given by

$$x^2 + y^2 + (z - 1)^2 = 1.$$

Its north pole N has coordinate $(0, 0, 2)$. For any $(x, y, z) \in S^2$, consider the line L joining N and (x, y, z) . This line L will intersect the xy -plane at a unique point $(u, v) \in \mathbb{R}^2$. The map $\pi : (x, y, z) \in S^2 \setminus (0, 0, 2) \rightarrow (u, v) \in \mathbb{R}^2$ is called **stereographic projection** of S^2 . To describe it, it is easier to look at its inverse π^{-1} . Using comparison between two right triangles, we get (here $(x, y, z) \in S^2$)

$$\frac{2 - z}{2} = \frac{\sqrt{x^2 + y^2}}{\sqrt{u^2 + v^2}} = \frac{\sqrt{1 - (z - 1)^2}}{\sqrt{u^2 + v^2}}$$

and let $2 - z = \lambda$ ($z = 2 - \lambda$) to get

$$\frac{\lambda}{2} = \frac{\sqrt{2\lambda - \lambda^2}}{\sqrt{u^2 + v^2}}.$$

This gives $(u^2 + v^2 + 4)\lambda^2 = 8\lambda$ and then

$$\lambda = \frac{8}{u^2 + v^2 + 4}, \quad z = 2 - \lambda = 2 - \frac{8}{u^2 + v^2 + 4} = \frac{2(u^2 + v^2)}{u^2 + v^2 + 4}$$

For y , we project these two right triangles onto yz -plane to get

$$\frac{2 - z}{2} = \frac{\sqrt{y^2}}{\sqrt{v^2}} = \frac{y}{v} \quad (\text{if } y > 0),$$

which gives

$$2 - \frac{2(u^2 + v^2)}{u^2 + v^2 + 4} = \frac{2y}{v}, \quad y = \frac{4v}{u^2 + v^2 + 4}.$$

Similarly, we project these two right triangles onto xz -plane to get

$$\frac{2 - z}{2} = \frac{\sqrt{x^2}}{\sqrt{u^2}} = \frac{x}{u} \quad (\text{if } x > 0),$$

which gives

$$x = \frac{4u}{u^2 + v^2 + 4} := \frac{4u}{H}, \quad \text{where } H = u^2 + v^2 + 4.$$

We conclude

$$\begin{aligned} \mathbf{x}(u, v) &:= \pi^{-1}(u, v) = (x(u, v), y(u, v), z(u, v)) \\ &= \left(\frac{4u}{u^2 + v^2 + 4}, \frac{4v}{u^2 + v^2 + 4}, \frac{2(u^2 + v^2)}{u^2 + v^2 + 4} \right) \\ &= \frac{1}{H} (4u, 4v, 2(u^2 + v^2)), \quad (u, v) \in \mathbb{R}^2. \end{aligned}$$

One can check that \mathbf{x} is a **homeomorphism** from \mathbb{R}^2 onto $S^2 - \{N\}$, **differentiable** from \mathbb{R}^2 into \mathbb{R}^3 . Moreover

$$\begin{aligned} \frac{\partial \mathbf{x}}{\partial u}(u, v) &= \frac{1}{H^2} (-4u^2 + 4v^2 + 16, -8uv, 16u), \quad H = u^2 + v^2 + 4 \\ \frac{\partial \mathbf{x}}{\partial v}(u, v) &= \frac{1}{H^2} (-8uv, 4u^2 - 4v^2 + 16, 16v) \end{aligned}$$

and so

$$\begin{aligned} \frac{\partial \mathbf{x}}{\partial u}(u, v) \wedge \frac{\partial \mathbf{x}}{\partial v}(u, v) \\ = \frac{1}{H^4} \left(-64u(u^2 + v^2 + 4), -64v(u^2 + v^2 + 4), 16[16 - (u^2 + v^2)^2] \right). \end{aligned}$$

In particular, we also note that

$$\begin{aligned} \left\langle \frac{\partial \mathbf{x}}{\partial u}(u, v), \frac{\partial \mathbf{x}}{\partial v}(u, v) \right\rangle \\ = \frac{1}{H^4} [(-4u^2 + 4v^2 + 16)(-8uv) + (-8uv)(4u^2 - 4v^2 + 16) + (16u)(16v)] \\ = 0, \quad \forall (u, v) \in \mathbb{R}^2. \end{aligned}$$

We easily see that $\frac{\partial \mathbf{x}}{\partial u}(u, v) \wedge \frac{\partial \mathbf{x}}{\partial v}(u, v) \neq (0, 0, 0)$ for all $(u, v) \in \mathbb{R}^2$ (because $\frac{\partial \mathbf{x}}{\partial u}(u, v) \perp \frac{\partial \mathbf{x}}{\partial v}(u, v)$ and both are nonzero vectors). Finally we see that one can choose two stereographic projections to cover the whole S^2 . One misses the north pole and the other misses the south pole.

Finally, we compute the coefficients E , F , G of the first fundamental form. We have

$$\begin{aligned} E(u, v) &= \langle \mathbf{x}_u, \mathbf{x}_u \rangle \\ &= \frac{1}{H^4} \left[(-4u^2 + 4v^2 + 16)^2 + (-8uv)^2 + (16u)^2 \right] = \frac{16}{H^2} \end{aligned} \quad (26)$$

and

$$\begin{aligned} G(u, v) &= \langle \mathbf{x}_v, \mathbf{x}_v \rangle \\ &= \frac{1}{H^4} \left[(-8uv)^2 + (4u^2 - 4v^2 + 16)^2 + (16v)^2 \right] = \frac{16}{H^2} \end{aligned} \quad (27)$$

and

$$F(u, v) = \langle \mathbf{x}_u, \mathbf{x}_v \rangle = 0. \quad (28)$$

We shall see later on that this parametrization $\mathbf{x}(u, v) : \mathbb{R}^2 \rightarrow S^2 \setminus \{(0, 0, 2)\}$ is a **conformal diffeomorphism** due to

$$E(u, v) = G(u, v) = \frac{16}{H^2}, \quad F(u, v) = 0, \quad \forall (u, v) \in \mathbb{R}^2 \text{ (domain of } \mathbf{x}\text{)}. \quad (29)$$

2 Chapter 3: The Geometry of the Gauss Map.

2.1 The Definition of the Gauss Map and its Fundamental Properties (this is Section 3-2 of the book).

2.1.1 Orientation of a Regular Surface $S \subset \mathbb{R}^3$.

We first recall the following fact: Let $\mathbf{x} : U$ (open set) $\subset \mathbb{R}^2 \rightarrow S \subset \mathbb{R}^3$ be a parametrization of a regular surface. The vector

$$N(q) = \frac{\mathbf{x}_u \wedge \mathbf{x}_v}{|\mathbf{x}_u \wedge \mathbf{x}_v|}(q) \in \mathbb{R}^3, \quad q \in U \quad (30)$$

is called **the unit normal vector field** on $\mathbf{x}(U) \subset S$ **induced by the parametrization \mathbf{x}** . Note: $N(q)$ is normal to S at the point $p = \mathbf{x}(q)$ and $N : \mathbf{x}(U) \subset S \rightarrow \mathbb{R}^3$ is a **differentiable map** on $\mathbf{x}(U)$.

Remark 2.1 *Since I did not teach Section 2.6 of the book, I will adopt the following definition for a regular surface $S \subset \mathbb{R}^3$ to be **orientable**.*

Definition 2.2 *A regular surface $S \subset \mathbb{R}^3$ is said to be **orientable** if there exists a **differentiable field of unit normal vectors** $N : S \rightarrow \mathbb{R}^3$ on the **whole** surface S . For simplicity, we just call it a **unit normal vector field on S** .*

Remark 2.3 *There exist regular surfaces in \mathbb{R}^3 which are not orientable. One famous example is the **Möbius strip**. See the discussion in p. 108 for it (we will not discuss it).*

Remark 2.4 *If there exists a parametrization $\mathbf{x} : U \subset \mathbb{R}^2 \rightarrow S \subset \mathbb{R}^3$ such that $\mathbf{x}(U) = S$, then **the unit normal vector field** N given by (30) is defined on the **whole** surface S . In such a case, S is orientable. In particular, if S is the graph of a differentiable function $f(x, y)$ defined on some open set U of \mathbb{R}^2 , then it is orientable.*

Remark 2.5 *We know that every regular surface S is **locally orientable**. Hence whether S is orientable or not is a **global** property.*

In case S is orientable, a choice of N on S is called an **orientation** on S (this is equivalent to a choice of **compatible** coordinate neighborhoods covering S). If S is connected, there are only two choices of N on it. Therefore, if S is connected, then it has exactly **two** orientations. If S has k connected components, then it has 2^k orientations.

If S is a regular surface with an orientation N , a basis $\{v, w\}$ on $T_p S$ is called **positive** if $v \wedge w$ is pointing in the direction of $N(p)$, i.e. $\det(v, w, N(p)) > 0$. Otherwise, we say it is **negative**. If $\{v, w\}$ and $\{\tilde{v}, \tilde{w}\}$ are two positive bases on $T_p S$, then its change of coordinates has **positive determinant**.

The following says that the inverse image of a regular value of a differentiable function is also orientable. Thus, in general, it is difficult to find nonorientable surfaces in \mathbb{R}^3 .

Lemma 2.6 *Let $f : U \subset \mathbb{R}^3 \rightarrow \mathbb{R}$ be a differentiable function and $a \in f(U)$ is a **regular value** of f . Then the surface*

$$S = \{(x, y, z) \in U : f(x, y, z) = a\}$$

is orientable.

Proof. We know that the gradient vector (it is a **nonzero** vector since a is a regular value)

$$\nabla f(x, y, z) = \left(\frac{\partial f}{\partial x}(x, y, z), \frac{\partial f}{\partial y}(x, y, z), \frac{\partial f}{\partial z}(x, y, z) \right), \quad (x, y, z) \in S$$

is everywhere perpendicular to S . Thus

$$N(x, y, z) = \frac{\nabla f(x, y, z)}{|\nabla f(x, y, z)|}, \quad (x, y, z) \in S$$

is a unit normal field on S . By Definition 2.2, S is orientable. □

Another important result is the following:

Lemma 2.7 *Any **compact** regular surface $S \subset \mathbb{R}^3$ is orientable. Therefore, spheres and ellipsoids in \mathbb{R}^3 are both orientable.*

Proof. Omit it. □

2.1.2 Gauss Map of a Regular Surface $S \subset \mathbb{R}^3$.

Throughout this chapter (Chapter 3 of the textbook), unless otherwise stated, we always assume that S is orientable with a chosen orientation N . For simplicity, we call S a **regular surface with an orientation** N .

Definition 2.8 *Let $S \subset \mathbb{R}^3$ be a regular surface with an orientation N . The map $N : S \rightarrow \mathbb{R}^3$ has its values in the unit sphere*

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}.$$

Thus we can write it as $N : S \rightarrow S^2$ and call it the **Gauss map** of S .

Remark 2.9 *Since $N : S \rightarrow \mathbb{R}^3$ is differentiable (this is due to definition), by p. 77, Example 3, we know that $N : S \rightarrow S^2$ is also differentiable. One can also use local parametrization to verify this. Its differential $dN_p : T_p S \rightarrow T_{N(p)} S^2$ can be viewed as a linear map from $T_p S \rightarrow T_p S$ because one can identify $T_{N(p)} S^2$ and $T_p S$ (they are parallel planes in \mathbb{R}^3 ; both planes are normal to $N(p)$). For any $v \in T_p S$, choose $\alpha(t) \in S$, $t \in (-\varepsilon, \varepsilon)$, so that $\alpha(0) = p$, $\alpha'(0) = v$. Then we have*

$$dN_p(v) = \left. \frac{d}{dt} \right|_{t=0} N(\alpha(t)), \quad (31)$$

or more generally,

$$dN_{\alpha(t)}(\alpha'(t)) = \frac{d}{dt}N(\alpha(t)), \quad \forall t \in (-\varepsilon, \varepsilon).$$

If we write $\alpha(t)$ as $\mathbf{x}(u(t), v(t))$ with $\alpha(0) = \mathbf{x}(u(0), v(0)) = p$ then

$$N(\alpha(t)) = N(\mathbf{x}(u(t), v(t)))$$

and for simplicity we will just write $N(\mathbf{x}(u(t), v(t)))$ as $N(u(t), v(t))$ with the understanding that $N(u, v)$ is actually $N(\mathbf{x}(u, v))$. By this, we have

$$\frac{d}{dt}N(\alpha(t)) = \frac{d}{dt}N(u(t), v(t)) = N_u(u(t), v(t))u'(t) + N_v(u(t), v(t))v'(t), \quad \forall t \in (-\varepsilon, \varepsilon)$$

and

$$\left. \frac{d}{dt} \right|_{t=0} N(\alpha(t)) = \underbrace{N_u(p)u'(0) + N_v(p)v'(0)}.$$

On the other hand, we can also write the above $\left. \frac{d}{dt} \right|_{t=0} N(\alpha(t))$ as

$$\begin{aligned} \left. \frac{d}{dt} \right|_{t=0} N(\alpha(t)) \\ = dN_p(\alpha'(0)) = dN_p[u'(0)\mathbf{x}_u + v'(0)\mathbf{x}_v] = \underbrace{u'(0)dN_p(\mathbf{x}_u) + v'(0)dN_p(\mathbf{x}_v)}. \end{aligned}$$

In particular, we note that

$$N_u(p) = dN_p(\mathbf{x}_u), \quad N_v(p) = dN_p(\mathbf{x}_v), \quad (32)$$

where \mathbf{x}_u and \mathbf{x}_v in (32) are evaluated at $(u(0), v(0)) \in U$.

Example 2.10 Do Example 2 in p. 139.

Example 2.11 Do Example 3 in p. 141. In this example, we have

$$dN(x'(t), y'(t), z'(t)) = (-x'(t), -y'(t), 0),$$

which can be written as

$$dN[(x'(t), y'(t), 0) + (0, 0, z'(t))] = (-x'(t), -y'(t), 0).$$

Thus the differential $dN_p : T_p S \rightarrow T_p S$ has two eigenvalues 0 and -1 .

Example 2.12 Do Example 4 in p. 141. In this example, we have $\mathbf{x}(u, v) = (u, v, v^2 - u^2)$, $(u, v) \in \mathbb{R}^2$, and we choose

$$\begin{aligned} N(u, v) = N(\mathbf{x}(u, v)) &= \frac{\mathbf{x}_u \wedge \mathbf{x}_v}{|\mathbf{x}_u \wedge \mathbf{x}_v|}(u, v) \\ &= \left(\frac{u}{\sqrt{u^2 + v^2 + \frac{1}{4}}}, \frac{-v}{\sqrt{u^2 + v^2 + \frac{1}{4}}}, \frac{1}{2\sqrt{u^2 + v^2 + \frac{1}{4}}} \right). \end{aligned}$$

Hence at $p = (0, 0, 0) = \mathbf{x}(0, 0)$ we have $\mathbf{x}_u(0, 0) = (1, 0, 0)$ and $\mathbf{x}_v(0, 0) = (0, 1, 0)$ and

$$\begin{aligned} dN_p((1, 0, 0)) = dN_p(\mathbf{x}_u(0, 0)) &= \left. \frac{\partial}{\partial u} \right|_{u=0} N(\mathbf{x}(u, 0)) = \left. \frac{\partial}{\partial u} \right|_{u=0} N(u, 0) \\ &= \left. \frac{\partial}{\partial u} \right|_{u=0} \left(\frac{u}{\sqrt{u^2 + \frac{1}{4}}}, 0, \frac{1}{2\sqrt{u^2 + \frac{1}{4}}} \right) = (2, 0, 0), \end{aligned}$$

and similarly

$$\begin{aligned} dN_p((0, 1, 0)) &= dN_p(\mathbf{x}_v(0, 0)) = \left. \frac{\partial}{\partial v} \right|_{v=0} N(\mathbf{x}(0, v)) = \left. \frac{\partial}{\partial v} \right|_{v=0} N(0, v) \\ &= \left. \frac{\partial}{\partial v} \right|_{v=0} \left(0, \frac{-v}{\sqrt{v^2 + \frac{1}{4}}}, \frac{1}{2\sqrt{v^2 + \frac{1}{4}}} \right) = (0, -2, 0). \end{aligned}$$

In general, we have

$$\begin{aligned} \left. \frac{d}{dt} \right|_{t=0} N(u(t), v(t)) &= \left. \frac{d}{dt} \right|_{t=0} N(\mathbf{x}(u(t), v(t))) \\ &= dN_p(u'(0)\mathbf{x}_u(0, 0) + v'(0)\mathbf{x}_v(0, 0)) = (2u'(0), -2v'(0), 0), \end{aligned}$$

i.e.,

$$dN_p(u'(0), v'(0), 0) = (2u'(0), -2v'(0), 0).$$

Thus $dN_p : T_p S \rightarrow T_p S$ has two eigenvalues 2 and -2 with corresponding eigenvectors $\mathbf{x}_u(0, 0) = (1, 0, 0)$ and $\mathbf{x}_v(0, 0) = (0, 1, 0)$.

Example 2.13 Do Example 5 in p. 140. In this example, we have $\mathbf{x}(u, v) = (u, v, u^2 + kv^2)$, $(u, v) \in \mathbb{R}^2$, where

$$\begin{cases} \mathbf{x}_u(u, v) = (1, 0, 2u), & \mathbf{x}_v(u, v) = (0, 1, 2kv), \\ \mathbf{x}_u(u, v) \wedge \mathbf{x}_v(u, v) = (-2u, -2kv, 1), \end{cases}$$

and we choose

$$\begin{aligned} N(u, v) &= N(\mathbf{x}(u, v)) = -\frac{\mathbf{x}_u \wedge \mathbf{x}_v}{|\mathbf{x}_u \wedge \mathbf{x}_v|}(u, v) \\ &= \left(\frac{u}{\sqrt{u^2 + k^2v^2 + \frac{1}{4}}}, \frac{kv}{\sqrt{u^2 + k^2v^2 + \frac{1}{4}}}, \frac{-1}{2\sqrt{u^2 + k^2v^2 + \frac{1}{4}}} \right). \end{aligned}$$

Hence at $p = (0, 0, 0)$ we have $\mathbf{x}_u(0, 0) = (1, 0, 0)$ and $\mathbf{x}_v(0, 0) = (0, 1, 0)$ and

$$dN_p((1, 0, 0)) = dN_p(\mathbf{x}_u(0, 0)) = \left. \frac{\partial}{\partial u} \right|_{u=0} N(u, 0) = (2, 0, 0),$$

and similarly

$$dN_p((0, 1, 0)) = dN_p(\mathbf{x}_v(0, 0)) = \left. \frac{\partial}{\partial v} \right|_{v=0} N(0, v) = (0, 2k, 0).$$

In general, we have

$$\begin{aligned} \left. \frac{d}{dt} \right|_{t=0} N(u(t), v(t)) &= \left. \frac{d}{dt} \right|_{t=0} N(\mathbf{x}(u(t), v(t))) \\ &= dN_p[u'(0)\mathbf{x}_u(0, 0) + v'(0)\mathbf{x}_v(0, 0)] = (2u'(0), 2kv'(0), 0), \end{aligned}$$

i.e.,

$$dN_p(u'(0), v'(0), 0) = (2u'(0), 2kv'(0), 0).$$

Thus $dN_p : T_p S \rightarrow T_p S$ has two eigenvalues 2 and $2k$ with corresponding eigenvectors $\mathbf{x}_u(0, 0) = (1, 0, 0)$ and $\mathbf{x}_v(0, 0) = (0, 1, 0)$.

Proposition 2.14 (This is Proposition 1 in p. 142.) For each $p \in S$. the differential $dN_p : T_p S \rightarrow T_p S$ of the Gauss map is a **self-adjoint** linear map.

Remark 2.15 Read the appendix "self-adjoint linear map and quadratic forms" in p.217-219 by yourself.

Remark 2.16 Note that with respect to an **orthonormal basis** $\{v_1, v_2\}$ on $T_p S$, the matrix representation M for $dN_p : T_p S \rightarrow T_p S$ is **symmetric**. If the basis $\{v_1, v_2\}$ is **not orthonormal**, M may not be symmetric in general. In particular, the matrix for the linear map $dN_p : T_p S \rightarrow T_p S$ with respect to a parametrization basis $\{\mathbf{x}_u, \mathbf{x}_v\}$ may not be symmetric in general.

Proof. Fix $p \in S$ and assume $\mathbf{x}(u, v) : U \subset \mathbb{R}^2 \rightarrow S \subset \mathbb{R}^3$ is a parametrization of S around p with $\mathbf{x}(q) = p$. We already know that $dN_p : T_p S \rightarrow T_p S$ is linear and $\{\mathbf{x}_u, \mathbf{x}_v\}$ (evaluated at $q \in U$) is a basis on $T_p S$. To show that it is self-adjoint, we need to check that

$$\langle dN_p(v), w \rangle = \langle v, dN_p(w) \rangle \quad \text{for all } v, w \in T_p S. \quad (33)$$

By linearity, it suffices to check that

$$\langle dN_p(\mathbf{x}_u), \mathbf{x}_v \rangle = \langle \mathbf{x}_u, dN_p(\mathbf{x}_v) \rangle, \quad (34)$$

which is the same as

$$\langle N_u(u, v), \mathbf{x}_v(u, v) \rangle = \langle N_v(u, v), \mathbf{x}_u(u, v) \rangle \quad \text{at } q \in U, \quad (35)$$

where $N(u, v)$ means $N(\mathbf{x}(u, v))$, where $(u, v) \in U$.

Note that for a parametrization $\mathbf{x}(u, v)$ we have

$$\langle N(u, v), \mathbf{x}_u(u, v) \rangle = \langle N(u, v), \mathbf{x}_v(u, v) \rangle = 0 \quad \text{for all } (u, v) \in U.$$

By differentiation with respect to u and v respectively, the above will imply

$$\begin{cases} \langle N_u(u, v), \mathbf{x}_u(u, v) \rangle = -\langle N(u, v), \mathbf{x}_{uu}(u, v) \rangle, & \text{where } N_u(u, v) = dN(\mathbf{x}_u(u, v)) \\ \langle N_v(u, v), \mathbf{x}_v(u, v) \rangle = -\langle N(u, v), \mathbf{x}_{vv}(u, v) \rangle, & \text{where } N_v(u, v) = dN(\mathbf{x}_v(u, v)) \\ \langle N_u(u, v), \mathbf{x}_v(u, v) \rangle = \langle N_v(u, v), \mathbf{x}_u(u, v) \rangle = -\langle N(u, v), \mathbf{x}_{uv}(u, v) \rangle, & \mathbf{x}_{uv}(u, v) = \mathbf{x}_{vu}(u, v) \end{cases}$$

for all $(u, v) \in U$. In particular, at the point $p \in S$, we have the identity (34).

The proof is done. □

Another useful result is the following:

Lemma 2.17 If the differential $dN_p : T_p S \rightarrow T_p S$ of the Gauss map satisfies

$$\langle dN_p(v), v \rangle = 0, \quad \forall v \in T_p S, \quad (36)$$

then $dN_p(v) = 0$ for all $v \in T_p S$.

Proof. By the assumption, we have

$$\langle dN_p(v+w), v+w \rangle = 0, \quad \forall v, w \in T_p S,$$

which gives

$$\langle dN_p(v), w \rangle + \langle dN_p(w), v \rangle = 2\langle dN_p(v), w \rangle = 0, \quad \forall v, w \in T_p S.$$

Hence for fixed $v \in T_p S$ we have $\langle dN_p(v), w \rangle = 0$ for all $w \in T_p S$. This implies $dN_p(v) = 0$. But since $v \in T_p S$ can be arbitrary, we conclude $dN_p(v) = 0$ for all $v \in T_p S$. □

The most important property of a self-adjoint linear map from an **inner product vector space** V with $\dim V = 2$ is the following:

Theorem 2.18 Let V be an **inner product vector space** with $\dim V = 2$ and assume $A : V \rightarrow V$ is a **self-adjoint** linear map. Then there exists an orthonormal basis $\{e_1, e_2\}$ of V such that

$$Ae_1 = \lambda_1 e_1, \quad Ae_2 = \lambda_2 e_2 \quad (37)$$

for some $\lambda_1, \lambda_2 \in \mathbb{R}$ (without loss of generality we may assume $\lambda_1 \geq \lambda_2$), i.e. e_1, e_2 are eigenvectors and λ_1, λ_2 are eigenvalues of A . Moreover, we have

$$\lambda_1 = \max_{v \in V, |v|=1} \langle Av, v \rangle, \quad \lambda_2 = \min_{v \in V, |v|=1} \langle Av, v \rangle. \quad (38)$$

Proof. See p. 219 of the textbook. We omit it. \square

Remark 2.19 (Interesting observation.) Assume $A : V \rightarrow V$ is a **self-adjoint** linear map with $\dim V = 2$. If $\lambda_1 > \lambda_2$ are two eigenvalues of A with corresponding unit eigenvectors v_1, v_2 , then we must have $v_1 \perp v_2$ and

$$\lambda_1 = \max_{v \in V, |v|=1} \langle Av, v \rangle, \quad \lambda_2 = \min_{v \in V, |v|=1} \langle Av, v \rangle.$$

To see this, by the identities

$$(\lambda_1 - \lambda_2) \langle v_1, v_2 \rangle = \langle Av_1, v_2 \rangle - \langle v_1, Av_2 \rangle = 0, \quad \lambda_1 - \lambda_2 \neq 0,$$

we have $v_1 \perp v_2$. Finally, for any $v \in V$ with $|v| = 1$ we can express it as

$$v = \alpha v_1 + \beta v_2 \quad \text{for some } \alpha, \beta \in \mathbb{R}, \quad \alpha^2 + \beta^2 = 1$$

and by

$$\langle Av, v \rangle = \langle \alpha \lambda_1 v_1 + \beta \lambda_2 v_2, \alpha v_1 + \beta v_2 \rangle = \lambda_1 \alpha^2 + \lambda_2 \beta^2,$$

where

$$\lambda_2 = \lambda_2 \alpha^2 + \lambda_2 \beta^2 \leq \lambda_1 \alpha^2 + \lambda_2 \beta^2 \leq \lambda_1 \alpha^2 + \lambda_1 \beta^2 = \lambda_1,$$

we obtain

$$\lambda_1 = \max_{v \in V, |v|=1} \langle Av, v \rangle, \quad \lambda_2 = \min_{v \in V, |v|=1} \langle Av, v \rangle. \quad (39)$$

2.1.3 Second Fundamental Form, Normal Curvature, and Geodesic Curvature.

Definition 2.20 The quadratic form $II_p(v) := -\langle dN_p(v), v \rangle : T_p S \rightarrow \mathbb{R}$ is called the **second fundamental form** of S at p . Note that we have

$$II_p(-v) = II_p(v) \quad \text{for all } v \in T_p S \quad (40)$$

and the linear map $dN_p : T_p S \rightarrow T_p S$ is self-adjoint.

Definition 2.21 Let C be a **regular curve** in S (with unit normal N) parametrized by $\alpha(s)$ (with Frenet frame $\{t(s), n(s), b(s)\}$), where $s \in I$ is arc length parameter. The **unit vector**

$$\mathbf{n}_{int}(s) = N(s) \wedge \alpha'(s) = N(s) \wedge t(s), \quad s \in I \quad (N(s) \text{ means } N(\alpha(s))) \quad (41)$$

is called the **intrinsic normal** of α at s . It is an unit vector **lying on** $T_{\alpha(s)} S$ and **normal to the curve** α at s , i.e.

$$\mathbf{n}_{int}(s) \in T_{\alpha(s)} S, \quad \langle \mathbf{n}_{int}(s), t(s) \rangle = \langle \mathbf{n}_{int}(s), \alpha'(s) \rangle = 0, \quad s \in I. \quad (42)$$

That is why we call it the **intrinsic normal** of α . Now the three vectors

$$\{t(s), \mathbf{n}_{int}(s), N(s)\} \quad (43)$$

form an **orthonormal frame** at $\alpha(s)$ (it also has the the previous **Frenet frame** $\{t(s), n(s), b(s)\}$ at $\alpha(s)$). The tangent plane $T_{\alpha(s)} S$ is spanned by the orthonormal basis $\{t(s), \mathbf{n}_{int}(s)\}$ and

$$t(s) \wedge \mathbf{n}_{int}(s) = N(s). \quad (44)$$

Remark 2.22 (Important.) Note that $\mathbf{n}_{int}(s) = N(s) \wedge t(s)$ is always defined even if $n(s)$ of α is undefined (i.e., when $\alpha''(s) = 0$).

Remark 2.23 (Important.) The **projection** of the normal vector $n(s) \in \mathbb{R}^3$ of α onto the **tangent plane** $T_{\alpha(s)}S$ is given by:

$$\begin{aligned} & \text{projection of } n(s) \text{ onto } T_{\alpha(s)}S \\ &= \underbrace{\langle n(s), t(s) \rangle}_{\text{projection of } n(s) \text{ onto } T_{\alpha(s)}S} t(s) + \underbrace{\langle n(s), \mathbf{n}_{int}(s) \rangle}_{\text{projection of } n(s) \text{ onto } \mathbf{n}_{int}(s)} \mathbf{n}_{int}(s) = \underbrace{\langle n(s), \mathbf{n}_{int}(s) \rangle}_{\text{projection of } n(s) \text{ onto } \mathbf{n}_{int}(s)} \mathbf{n}_{int}(s). \end{aligned} \quad (45)$$

This explains why we call $\mathbf{n}_{int}(s)$ the **intrinsic normal** of α because it is the projection of $n(s)$ onto $T_{\alpha(s)}S$.

Remark 2.24 (Important.) To study the geometry of a curve $\alpha(s)$ on S , it is better to use the frame $\{t(s), \mathbf{n}_{int}(s), N(s)\}$ (it respects the surface) than the Frenet frame $\{t(s), n(s), b(s)\}$.

For a parametrized curve $\alpha(s) \in S$ (with normal N), $s \in I$, since we have $\langle \alpha''(s), \alpha'(s) \rangle = 0$ everywhere, the vector $\alpha''(s)$ must lie on the plane spanned by $\mathbf{n}_{int}(s)$ and $N(s)$. Therefore, we have

$$\alpha''(s) = k(s) n(s) = \underbrace{\langle \alpha''(s), \mathbf{n}_{int}(s) \rangle}_{\text{projection of } \alpha''(s) \text{ onto } \mathbf{n}_{int}(s)} \mathbf{n}_{int}(s) + \underbrace{\langle \alpha''(s), N(s) \rangle}_{\text{projection of } \alpha''(s) \text{ onto } N(s)} N(s), \quad s \in I. \quad (46)$$

Definition 2.25 In (245), the quantity

$$\langle \alpha''(s), \mathbf{n}_{int}(s) \rangle \quad (\text{denoted as } k_g(s))$$

is called the **geodesic curvature** of α ; and the quantity

$$\langle \alpha''(s), N(s) \rangle \quad (\text{denoted as } k_n(s))$$

is called the **normal curvature** of α . Therefore, we have the identity

$$\alpha''(s) = k(s) n(s) = k_g(s) \mathbf{n}_{int}(s) + k_n(s) N(s), \quad s \in I.$$

Since we also have $|\alpha''(s)| = k(s) \geq 0$ (the curvature of $\alpha(s)$ as a curve in \mathbb{R}^3), we conclude the important identity

$$k^2(s) = |\alpha''(s)|^2 = k_g^2(s) + k_n^2(s), \quad s \in I, \quad (47)$$

which is the same as

$$k(s) = \sqrt{k_g^2(s) + k_n^2(s)}, \quad s \in I.$$

If $\alpha''(s) \neq 0$ (then $n(s)$ is defined), we also have

$$\begin{cases} k_g(s) = \underbrace{\langle \alpha''(s), \mathbf{n}_{int}(s) \rangle}_{\text{projection of } \alpha''(s) \text{ onto } \mathbf{n}_{int}(s)} = k(s) \langle n(s), \mathbf{n}_{int}(s) \rangle, & \text{where } \mathbf{n}_{int}(s) = N(s) \wedge \alpha'(s) \\ k_n(s) = \underbrace{\langle \alpha''(s), N(s) \rangle}_{\text{projection of } \alpha''(s) \text{ onto } N(s)} = k(s) \langle n(s), N(s) \rangle, & s \in I. \end{cases} \quad (48)$$

Finally, if $\alpha''(s) = 0$ (then $n(s)$ is undefined), we have $k_g(s) = k_n(s) = k(s) = 0$.

Remark 2.26 If we change the orientation of S , then both $\mathbf{n}_{int}(s)$ and $N(s)$ change sign and so do $k_n(s)$ and $k_g(s)$.

Assume $\alpha(0) = p \in S$. We have the following important identity:

$$\begin{aligned} II_p(\alpha'(0)) &= -\langle dN_p(\alpha'(0)), \alpha'(0) \rangle = -\langle N'(0), \alpha'(0) \rangle = \langle N(0), \alpha''(0) \rangle \\ &= \langle N(0), k(0) \mathbf{n}(0) \rangle = k_n(0) \quad (\text{or denote it as } k_n(p)) = \mathbf{normal curvature at } p. \end{aligned} \quad (49)$$

We conclude that the value of **the second fundamental form** $II_p(v)$ at a unit vector $v \in T_pS$ is equal to the **normal curvature** of a regular curve α passing through p and tangent to v , i.e.

$$II_{\alpha(s)}(\alpha'(s)) = -\langle dN_p(\alpha'(s)), \alpha'(s) \rangle = k_n(s), \quad \forall s \in I. \quad (50)$$

Thus the normal curvature of α at p is actually to measure the "geometry" of S , NOT the curvature of α . It is the component of the tangent vector $-dN_p(\alpha'(s))$ in the direction $\alpha'(s)$, $s \in I$.

By the identity (224), we have the following important observation:

Lemma 2.27 *The normal curvature $k_n(s)$ measures the "geometry" of S in \mathbb{R}^3 (along some direction $v \in T_pS$ given by $\alpha'(0)$) and the geodesic curvature $k_g(s)$ measures how α is curving in S . The geodesic curvature of α is the **intrinsic curvature** of α in S , i.e., the curvature viewed by the surface S . However, the **curvature** $k(s)$ of $\alpha \in S$ in \mathbb{R}^3 depends on the "geometry" of S in \mathbb{R}^3 and how α is curving in S , i.e. $k(s)$ depends on $k_n(s)$ and $k_g(s)$.*

Another interesting observation is the following:

Proposition 2.28 *(This is Proposition 2 in p. 144.) (Meusnier.) All curves (parametrized by arc length parameter) lying on S passing through $p \in S$ and having the same tangent line at p have the same normal curvature at p .*

Remark 2.29 *(Important.) By the above Proposition, we can talk about the normal curvature along a given direction $v \in T_pS$ at p (here both v and $-v$ are regarded as having the same direction). Moreover, if we change the direction v into $-v$, we get the same normal curvature.*

Proof. Let $\alpha(s)$ and $\beta(s)$ be two regular curves lying on S with $\alpha(0) = \beta(0) = p \in S$ and $\alpha'(0) = v \in T_pS$. By the assumption we have $\beta'(0) = \pm v$ and by (224), the normal curvature of α at p is

$$k_n^{(\alpha)}(0) = II_p(\alpha'(0)) = II_p(v).$$

On the other hand, the normal curvature of β at p is

$$k_n^{(\beta)}(0) = II_p(\beta'(0)) = II_p(\pm v) = II_p(v).$$

The proof is done. □

Example 2.30 *(Curve lying on S^2 .) Let $S^2 \subset \mathbb{R}^3$ be the unit sphere in \mathbb{R}^3 centered at the origin $O = (0,0,0)$ with chosen orientation $N(p) = -p$ (inward) for all $p \in S^2$. Let $\alpha(s) : I \rightarrow S^2$ be a regular curve lying on S^2 . We have $N(s) = -\alpha(s)$ and the following three useful identities*

$$\langle \alpha(s), \alpha(s) \rangle \equiv 1, \quad \langle \alpha(s), \alpha'(s) \rangle \equiv 0, \quad \langle \alpha'(s), \alpha'(s) \rangle \equiv 1, \quad \forall s \in I,$$

which give

$$\langle \alpha''(s), \alpha(s) \rangle = -\langle \alpha'(s), \alpha'(s) \rangle \equiv -1, \quad \langle \alpha''(s), \alpha'(s) \rangle \equiv 0, \quad \forall s \in I,$$

By definition

$$\mathbf{n}_{int}(s) = N(s) \wedge \alpha'(s) = -\alpha(s) \wedge \alpha'(s) \quad (51)$$

and the frame $\{t(s), \mathbf{n}_{int}(s), N(s)\}$ is given by

$$\{t(s), \mathbf{n}_{int}(s), N(s)\} = \{\alpha'(s), -\alpha(s) \wedge \alpha'(s), -\alpha(s)\}.$$

The geodesic curvature and normal curvature are given by

$$\begin{cases} k_g(s) = \langle \alpha''(s), \mathbf{n}_{int}(s) \rangle = -\langle \alpha''(s), \alpha(s) \wedge \alpha'(s) \rangle = -\det(\alpha(s), \alpha'(s), \alpha''(s)) \\ k_n(s) = \langle \alpha''(s), N(s) \rangle = \langle \alpha''(s), -\alpha(s) \rangle = \langle \alpha'(s), \alpha'(s) \rangle \equiv 1, \quad s \in I \end{cases} \quad (52)$$

and the vector $\alpha''(s)$ can be decomposed as

$$\begin{aligned} \alpha''(s) &= k(s) n(s) = k_g(s) \mathbf{n}_{int}(s) + k_n(s) N(s) \\ &= -\det(\alpha(s), \alpha'(s), \alpha''(s)) \mathbf{n}_{int}(s) + 1 \cdot N(s), \quad s \in I, \end{aligned} \quad (53)$$

which gives the curvature identity

$$k(s) = \sqrt{1 + k_g^2(s)} = \sqrt{1 + [\det(\alpha(s), \alpha'(s), \alpha''(s))]^2}, \quad s \in I. \quad (54)$$

One can also compute $k_n(s)$ by the identity (224) and get

$$\begin{aligned} k_n(s) &= -\langle dN_{\alpha(s)}(\alpha'(s)), \alpha'(s) \rangle \\ &= -\langle N'(s), \alpha'(s) \rangle = \langle \alpha'(s), \alpha'(s) \rangle \equiv 1, \quad \forall s \in I. \end{aligned}$$

Finally, we note that if we change the orientation of S^2 by choosing $N(p) = p$ (outward) for all $p \in S^2$, both $\mathbf{n}_{int}(s)$ and $N(s)$ change sign and so do $k_n(s)$ and $k_g(s)$. In such a case, we have the nice identity

$$k_g(s) = \det(\alpha(s), \alpha'(s), \alpha''(s)), \quad s \in I. \quad (55)$$

This example confirms our observation in Lemma 2.27. The normal curvature $k_n(s) \equiv 1$ describes the geometry of S^2 , not the geometry of $\alpha(s)$.

Example 2.31 (Curve lying on \mathbb{R}^2 .) If $\alpha(s)$ is a curve lying on the plane $\mathbb{R}^2 \subset \mathbb{R}^3$, then $k_n(s) \equiv 0$ (since the plane has **no** "curvature") and $k_g(s) = k(s)$, where $k(s)$ is the **signed curvature** of α in the plane. More precisely, we choose $N(p) = (0, 0, 1)$ (upward) for all $p \in \mathbb{R}^2$ and so $dN_p(v) = 0$ for all $p \in \mathbb{R}^2$ and all $v \in T_p S$. If we write the unit tangent vector $\alpha'(s)$ as $\alpha'(s) = (\cos \theta(s), \sin \theta(s), 0)$, we get

$$\mathbf{n}_{int}(s) = N(s) \wedge \alpha'(s) = (0, 0, 1) \wedge (\cos \theta(s), \sin \theta(s), 0) = (-\sin \theta(s), \cos \theta(s), 0)$$

and then $\mathbf{n}_{int}(s)$ is the same as the normal vector $n(s)$ for plane curve $\alpha(s)$. Hence

$$k_g(s) = \langle \alpha''(s), \mathbf{n}_{int}(s) \rangle = \langle \alpha''(s), n(s) \rangle = k(s) = \text{signed curvature of } \alpha(s). \quad (56)$$

There is other way to express the geodesic curvature $k_g(s)$. Assume the Frenet frame $\{t(s), n(s), b(s)\}$ of $\alpha(s)$ exists. We have

$$\begin{aligned} k_g(s) &= \langle \alpha''(s), \mathbf{n}_{int}(s) \rangle = k(s) \langle n(s), N(s) \wedge t(s) \rangle \\ &= k \langle N(s), t(s) \wedge n(s) \rangle = k(s) \langle b(s), N(s) \rangle \end{aligned} \quad (57)$$

and conclude

$$\begin{cases} k_n(s) = \langle \alpha''(s), N(s) \rangle = \underbrace{k(s) \langle n(s), N(s) \rangle}_{= \text{normal curvature}}, \\ k_g(s) = \langle \alpha''(s), \mathbf{n}_{int}(s) \rangle = \underbrace{k(s) \langle b(s), N(s) \rangle}_{= \text{geodesic curvature}}. \end{cases} \quad (58)$$

Definition 2.32 Let $S \subset \mathbb{R}^3$ be a regular surface with a chosen orientation N . Given a unit vector $v \in T_p S$, the set

$$S \cap P$$

is called the **normal section** of S at p **in the direction** v . Here P is the plane passing through p and spanned by $N(p)$ and $v \in T_p S$.

Since S is a regular surface and near p it is like the graph of a function $z = f(x, y)$, $(x, y) \in U$ (some open set containing $(0, 0)$) with $f(0, 0) = 0$, $f_x(0, 0) = 0$, $f_y(0, 0) = 0$; see Exercise 26 in p. 93), therefore near p the **normal section** of S at p (along any direction $v \in T_p S$) is a **regular plane curve** C lying on S (here "regular curve" is in the sense of p. 77-78 of the textbook). If we parametrize it by $\alpha(s)$, $s \in (-\varepsilon, \varepsilon)$, with $\alpha(0) = p$, then $\alpha(s)$ is a curve on S (and also on the plane P) with $\alpha'(0) = v \in T_p S$ and either (in the following, the curvature $k(s)$ of $\alpha(s)$ is defined as $k(s) = |\alpha''(s)| \geq 0$)

$$\alpha''(0) \neq 0, \quad \alpha''(0) = k(0) \mathbf{n}(0) \quad \text{with} \quad k(0) = |\alpha''(0)| > 0 \quad (59)$$

or

$$\alpha''(0) = 0, \quad \text{with} \quad k(0) = |\alpha''(0)| = 0 \quad \text{and} \quad \mathbf{n}(0) \text{ is undefined.}$$

where in (59) the curvature $k(0)$ is defined as $|\alpha''(0)|$. In the first case, $\mathbf{n}(0)$ is defined with

$$\mathbf{n}(0) = N(0) \quad \text{or} \quad \mathbf{n}(0) = -N(0).$$

In the second case, $\mathbf{n}(0)$ is undefined.

By the relation $k_n(0) = \langle \alpha''(0), N(0) \rangle$ we have the following possibilities:

$$\begin{cases} k_n(0) = k(0) > 0 & \text{if } \mathbf{n}(0) = N(0), \\ k_n(0) = -k(0) < 0 & \text{if } \mathbf{n}(0) = -N(0), \\ k_n(0) = k(0) = 0 & \text{if } \alpha''(0) = 0 \text{ (}\mathbf{n}(0) \text{ is undefined)} \end{cases} \quad (60)$$

In any case, we conclude (note that here, similar to space curves, $k(0)$ is defined as $|\alpha''(0)| \geq 0$)

$$|k_n(0)| = k(0) = |\alpha''(0)|. \quad (61)$$

In particular, we see that the **normal section** of S at p has **no geodesic curvature** due to the identity $k^2(0) = k_g^2(0) + k_n^2(0)$.

Remark 2.33 The purpose of the normal section is to find a curve $\alpha(s)$, $\alpha(0) = p$, on S such that its curvature $k(0)$ is equal to the absolute value of the **normal curvature** $k_n(0)$.

Example 2.34 Let $S = S^2$ centered at $(0, 0, 0)$ with inward normal N and $p \in S^2$. Each of a normal section of S at p in the direction $v \in T_p S$ is a great circle with 0 geodesic curvature. We have $k_n(0) = 1$ for all $v \in T_p S$.

Example 2.35 (*Skip this in class.*) Do Example 6 in p. 145. Here we use the fact from Lemma 2.17 to conclude that $dN_p = 0$, $p = (0, 0, 0)$. To be more precise, for each possible **normal section** $\alpha(s)$ of S at p in the direction v we have curvature $k(0) = 0$ (due to the equation $z = y^4$) and so $k_n(0) = 0$, which gives

$$-\langle dN_p(\alpha'(0)), \alpha'(0) \rangle = k_n(0) = 0$$

for all possible $\alpha'(0) \in T_p S$.

Example 2.36 (Curve lying on cylinder.) Let S be the cylinder in \mathbb{R}^3 given by

$$\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\} \quad (62)$$

and we choose $N(x, y, z) = (-x, -y, 0)$. For $p \in S$, we can look at all possible **normal sections** of S at p in all possible directions $\pm v \in T_p S$ and conclude that, at $p \in S$, the maximal normal curvature is $+1$ and the minimal normal curvature is 0 .

Definition 2.37 For $p \in S$, since the map $-dN_p : T_p S \rightarrow T_p S$ is self-adjoint, by Theorem 2.18 there exists orthonormal basis $\{e_1, e_2\}$ such that

$$-dN_p(e_1) = k_1 e_1, \quad -dN_p(e_2) = k_2 e_2, \quad (63)$$

for some $k_1, k_2 \in \mathbb{R}$ (without loss of generality we may assume $k_1 \geq k_2$), where k_1, k_2 satisfy

$$\begin{cases} k_1 = \max_{v \in T_p S, |v|=1} \langle -dN_p(v), v \rangle = \mathbf{maximal\ normal\ curvature\ at\ } p, \\ k_2 = \min_{v \in T_p S, |v|=1} \langle -dN_p(v), v \rangle = \mathbf{minimal\ normal\ curvature\ at\ } p. \end{cases}$$

We call them the **principal curvatures** of S at p . The directions given by $\pm e_1, \pm e_2$ are called the **principal directions** of S at p . Therefore, along any possible direction $\pm v \in T_p S$ at p , $|v| = 1$, the normal curvature $\langle -dN_p(\pm v), \pm v \rangle$ satisfies (now we denote k_1, k_2 at p as $k_1(p)$ and $k_2(p)$)

$$k_2(p) \leq \langle -dN_p(\pm v), \pm v \rangle \leq k_1(p), \quad \forall \text{ unit vector } v \in T_p S,$$

i.e. $\langle -dN_p(\pm v), \pm v \rangle$ lies on the interval $[k_2(p), k_1(p)]$. We call it the **normal curvature interval** at $p \in S$.

Remark 2.38 (Be careful.) In the above definition, we assume $\{e_1, e_2\}$ to be orthonormal even if $k_1 = k_2$. So by default, **principal directions** are **perpendicular** to each other. However, note that when $k_1 = k_2$ (denote it as k), we have $-dN_p = kI$ and any vector $v \neq 0 \in T_p S$ is an **eigenvector** and we usually say that the direction given by $v \neq 0 \in T_p S$ is a principal direction. This may be a little bit confusing sometimes !!!

Remark 2.39 Summary: maximal and minimal normal curvatures are eigenvalues and principal directions are eigenvector directions, for the linear map $-dN$.

Lemma 2.40 For each $p \in S$ and each number $\lambda \in [k_2(p), k_1(p)]$, there is some direction $\pm v_0 \in T_p S$ such that

$$\langle -dN_p(\pm v_0), \pm v_0 \rangle = \lambda.$$

Remark 2.41 We will see in Section 3.3 that if we choose $k_1(p) \geq k_2(p)$ for all $p \in S$, then $k_1(p)$ and $k_2(p)$ are continuous functions on S .

Proof. This is obvious since the function $\langle -dN_p(\pm v), \pm v \rangle : v (\text{unit vectors}) \in T_p S \rightarrow \mathbb{R}$ is continuous. Therefore, its image is a connected closed interval equal to $[k_2(p), k_1(p)]$. The result follows due to the intermediate value theorem. \square

Example 2.42 Let $S = xy$ -plane in \mathbb{R}^3 . Then at any $p \in S$, all directions are principal directions and all normal curvatures at p are 0 .

Example 2.43 Let $S = S^2$ in \mathbb{R}^3 with $N(p) = -p$ (inward) on S^2 . Then at any $p \in S$ all directions are principal directions and all normal curvatures at p are 1 .

Example 2.44 Let S be the cylinder (62) in \mathbb{R}^3 with inward normal. At any $p \in S$, the directions parallel to the z -axis and the directions perpendicular to the z -axis are principal directions. The **principal curvatures** of S at p are 0 and 1.

Example 2.45 Let S be the hyperbolic paraboloid in Example 4 in p. 141. It has global parametrization $\mathbf{x}(u, v) = (u, v, v^2 - u^2)$, $(u, v) \in \mathbb{R}^2$, and we choose $N(u, v) = N(\mathbf{x}(u, v)) = \frac{\mathbf{x}_u \wedge \mathbf{x}_v}{|\mathbf{x}_u \wedge \mathbf{x}_v|}(u, v)$. At $p = (0, 0, 0) = \mathbf{x}(0, 0)$ we have $\mathbf{x}_u(0, 0) = (1, 0, 0)$ and $\mathbf{x}_v(0, 0) = (0, 1, 0)$, hence $T_p S = xy$ -plane. We note that (see the computation in Example 2.12)

$$-dN_p((1, 0, 0)) = (-2, 0, 0) = -2(1, 0, 0), \quad -dN_p((0, 1, 0)) = (0, 2, 0) = 2(0, 1, 0).$$

Therefore, $-dN_p$ has two eigenvalues -2 and 2 and corresponding eigenvectors $(1, 0, 0)$ and $(0, 1, 0)$. The **principal curvatures** of S at p are -2 and 2 . The two directions along the x -axis and y -axis are **principal directions**.

2.1.4 Line of Curvature.

From now on, we always assume that $S \subset \mathbb{R}^3$ is a regular surface with a chosen orientation N .

Definition 2.46 Let $C \subset S$ be a connected regular curve with the property that the **tangent line** L at any $p \in C$ is a **principal direction** of S at p . Then we say C is a **line of curvature** on S .

Remark 2.47 Note that C is in general a curve on S , **not** a straight line on S (i.e. do not be misled by the name). On S^2 , any curve $\alpha(s) \in S^2$ is a line of curvature. Similarly, on \mathbb{R}^2 , any curve $\alpha(s) \in \mathbb{R}^2$ is a line of curvature.

Proposition 2.48 (**This is Proposition 3 in p. 147.**) (**Olinde Rodrigues.**) A necessary and sufficiently for a connected regular C on S to be a **line of curvature** is that

$$N'(t) = \lambda(t) \alpha'(t) \quad (\text{same as } -N'(t) = -\lambda(t) \alpha'(t)) \quad (64)$$

for any parametrization $\alpha(t)$ of C and any t in the domain of α , where $N(t) = N(\alpha(t))$ and $\lambda(t)$ is a differentiable function of t . In this case, $-\lambda(t)$ is the **principal curvature** of S at $\alpha(t)$ along $\alpha'(t)$.

Remark 2.49 (**Important.**) If a curve $\alpha(s) \in S$, $s \in I$, is a **line of curvature**, then each $\alpha'(s)$ is an eigenvector of $dN_{\alpha(s)}$ and by

$$-\langle dN_p(\alpha'(s)), \alpha'(s) \rangle = -\langle N'(s), \alpha'(s) \rangle = \langle N(s), \alpha''(s) \rangle = k_n(s) = \text{normal curvature at } \alpha(s), \quad (65)$$

we see that

$$k_n(s) = k_1(s) \quad \text{or} \quad k_2(s), \quad \forall s \in I. \quad (66)$$

Proof. For any parametrization $\alpha(t)$ of C , if it is a line of curvature, then $\alpha'(t)$ is a principal direction and we have

$$N'(t) = dN(\alpha'(t)) = \lambda(t) \alpha'(t)$$

for some function $\lambda(t)$. The function $\lambda(t)$ is **differentiable** due to the identity

$$\lambda(t) = \frac{\langle N'(t), \alpha'(t) \rangle}{\langle \alpha'(t), \alpha'(t) \rangle}, \quad \langle \alpha'(t), \alpha'(t) \rangle > 0 \text{ for all } t \in \text{domain of } \alpha.$$

Conversely, if we have the identity (64) for all t , it means that the vector $\alpha'(t) \neq 0$ is an eigenvector with eigenvalue $-\lambda(t)$ for the map $-dN_{\alpha(t)}$. Hence the curve C is a line of curvature. \square

2.1.5 Euler Formula, Gauss Curvature, Mean Curvature, and Umbilical Point.

Let $S \subset \mathbb{R}^3$ be a regular surface with an orientation N . Let $\{e_1, e_2\}$ be an orthonormal basis on $T_p S$ (eigenvectors) corresponding to the two principal curvatures $k_1(p) \geq k_2(p)$. We also assume that they have positive orientation, i.e. they satisfy $e_1 \wedge e_2 = N(p)$ on $T_p S$. If $k_1(p) > k_2(p)$, such a basis on $T_p S$ is **unique (modulo the choice $\{-e_1, -e_2\}$)**. For any unit vector $v \in T_p S$, one can express it as

$$v = (\cos \theta) e_1 + (\sin \theta) e_2, \quad |v| = 1,$$

where θ is the angle from e_1 to v in the orientation of $T_p S$ (which means $e_1 \wedge v = N(p)$). The **normal curvature along v** is now given by

$$\begin{aligned} k_n(p) &= II_p(v) = -\langle dN_p(v), v \rangle \\ &= -\langle dN_p((\cos \theta) e_1 + (\sin \theta) e_2), (\cos \theta) e_1 + (\sin \theta) e_2 \rangle \\ &= \langle (k_1 \cos \theta) e_1 + (k_2 \sin \theta) e_2, (\cos \theta) e_1 + (\sin \theta) e_2 \rangle = k_1(p) \cos^2 \theta + k_2(p) \sin^2 \theta. \end{aligned} \quad (67)$$

The formula in (67) is known as the **Euler formula**, which is the expression of the second fundamental form $II_p(\cdot)$ on $T_p S$ with respect to the basis $\{e_1, e_2\}$.

Remark 2.50 *In Euler formula, it suffices to focus on $\theta \in [0, \pi)$ due to $II_p(-v) = II_p(v)$.*

Remark 2.51 (Interesting observation.) *The formula (67) automatically implies the inequality*

$$k_2(p) \leq k_n(p) \leq k_1(p), \quad \forall v \in T_p S, \quad |v| = 1$$

due to the linear combination

$$k_n(p) = \rho k_1 + (1 - \rho) k_2, \quad 0 \leq \rho = \cos^2 \theta \leq 1. \quad (68)$$

Definition 2.52 *Let $k_1 \geq k_2$ be the two **principal curvatures** of S at p . The two numbers*

$$K := k_1 k_2, \quad H := \frac{k_1 + k_2}{2} \quad (69)$$

*are called the **Gauss curvature** of S at p and the **mean curvature** of S at p respectively. We note that*

$$K = \det(-dN_p), \quad H = \frac{1}{2} \text{Tr}(-dN_p) \left(= \frac{1}{2} \text{Trace}(-dN_p) \right). \quad (70)$$

Remark 2.53 *If we change the orientation of S (i.e. replace N by $-N$), each principal curvature changes sign and so the mean curvature H changes its sign. However, the Gauss curvature K is unchanged (since $\dim T_p S = 2$).*

Definition 2.54 *A point $p \in S$ is called*

1. **Elliptic** if $\det(dN_p) > 0$ (i.e. $k_1(p)$ and $k_2(p)$ have the same sign).
2. **Hyperbolic** if $\det(dN_p) < 0$ (i.e. $k_1(p)$ and $k_2(p)$ have opposite sign).
3. **Parabolic** if $\det(dN_p) = 0$, but $dN_p \neq 0$. (i.e. either $k_1(p)$ or $k_2(p)$ is zero, but not both).
4. **Planar** if $dN_p = 0$ (i.e. both $k_1(p)$ and $k_2(p)$ are zero).

Remark 2.55 *Note that the above definition does not depend on the choice of the orientation N on S .*

Example 2.56 (See **Example 5** in p. 142.) For the graphic surface $S : z = x^2 + ky^2$ ($k > 0$ is a constant) with **upward normal**, the point $p = (0, 0, 0) \in S$ is an **elliptic point** with $k_1 = 2$, $k_2 = 2k$ and all normal curvatures have the same sign. All curves $\alpha(s) \in S$, $\alpha(0) = p$, passing through p are bending towards the same side of $T_p S$ due to the constant sign of $\langle \alpha''(0), N(p) \rangle$.

Example 2.57 For the graphic surface $S : z = y^2 - x^2$ with **upward normal**, the point $p = (0, 0, 0) \in S$ is a **hyperbolic point** with $k_1 = 2$, $k_2 = -2$. There are curves passing through p bending towards one side of $T_p S$ and there are curves passing through p bending towards the other side of $T_p S$.

Example 2.58 Let S be the cylinder in Example 3 in p. 141 with **inward normal**. At each $p \in S$, we have $k_1 = 1$, $k_2 = 0$. Therefore, all points on S are **parabolic points**.

Definition 2.59 Let $k_1 \geq k_2$ be the two **principal curvatures** of S at p . If we have $k_1 = k_2$, then the point $p \in S$ is called an **umbilical point**. In particular, any **planar point** is an umbilical point.

Example 2.60 On S^2 , any point $p \in S^2$ is an umbilical point. Similarly, on \mathbb{R}^2 , any point $p \in \mathbb{R}^2$ is an umbilical point.

We have the following interesting result related to the above example:

Proposition 2.61 (**This is Proposition 4** in p. 149.) Let $S \subset \mathbb{R}^3$ be a **connected** surface and all points on S are **umbilical**. Then S is contained either in a sphere (not necessarily unit sphere) or in a plane.

Proof. Let $p \in S$ and $\mathbf{x}(u, v) : U \subset \mathbb{R}^2 \rightarrow S$ a parametrization near p and we take U to be connected open set in \mathbb{R}^2 . For each $q \in V = \mathbf{x}(U)$ and any vector $w = a\mathbf{x}_u + b\mathbf{x}_v \in T_q S$, we have

$$dN_q(w) = \lambda(q)w, \quad \forall w \in T_q S, \quad q \in V, \quad (71)$$

where $\lambda(q) : V \rightarrow \mathbb{R}$ is a differentiable function on V (you can see this from (72) below). Since $w = a\mathbf{x}_u + b\mathbf{x}_v$, we have

$$aN_u + bN_v = \lambda(q)(a\mathbf{x}_u + b\mathbf{x}_v).$$

Since $w \in T_q S$ is arbitrary, we have (pick $a = 1$, $b = 0$ and $a = 0$, $b = 1$ respectively)

$$N_u = \lambda(q)\mathbf{x}_u, \quad N_v = \lambda(q)\mathbf{x}_v, \quad \lambda(q) = \frac{\langle N_u, \mathbf{x}_u \rangle}{E(u, v)} = \frac{\langle N_u, \mathbf{x}_u \rangle}{G(u, v)}, \quad (72)$$

which gives (look at $N_{uv} - N_{vu} = 0$)

$$\lambda_v(q)\mathbf{x}_u - \lambda_u(q)\mathbf{x}_v = 0, \quad \forall q \in V$$

and so

$$\lambda_v(q) = \lambda_u(q) = 0, \quad \forall q \in V. \quad (73)$$

The above implies that $\lambda(q)$ is a **constant function** on V since $V \subset S$ is connected.

Case 1: $\lambda(q) \equiv 0$ on V .

In this case, we have $N_u = N_v \equiv 0$ on V , which implies $N(u, v) = \text{const. } N_0$ on V . In particular we get

$$\frac{\partial}{\partial u} \langle \mathbf{x}(u, v), N_0 \rangle = \frac{\partial}{\partial v} \langle \mathbf{x}(u, v), N_0 \rangle = 0 \quad \text{on } (u, v) \in U.$$

It means that all points $\mathbf{x}(u, v)$, $(u, v) \in U$, lie on a plane P perpendicular to N_0 .

Case 2: $\lambda(q) \equiv \lambda \neq 0$ on V .

In this case, we have

$$N_u = \lambda \mathbf{x}_u, \quad N_v = \lambda \mathbf{x}_v, \quad \forall (u, v) \in U,$$

which gives

$$\left(\frac{N}{\lambda} - \mathbf{x} \right)_u = \left(\frac{N}{\lambda} - \mathbf{x} \right)_v = 0, \quad \forall (u, v) \in U,$$

i.e.

$$\mathbf{x}(u, v) - \frac{N(u, v)}{\lambda} = \text{const. vector } v_0 \quad \text{on } U.$$

We conclude

$$|\mathbf{x}(u, v) - v_0|^2 = \frac{1}{\lambda^2} \quad \text{on } U$$

and know that all points of V are contained in a sphere with radius $1/|\lambda|$ centered at v_0 .

Finally, for any $p, q \in S$, since S is **connected**, there exists a continuous path $\alpha(t) \in S$, $t \in [0, 1]$, such that $\alpha(0) = p$, $\alpha(1) = q$. By compactness of the set $\alpha([0, 1])$, there exists finitely many connected coordinate neighborhoods V_1, \dots, V_k satisfying

$$\alpha([0, 1]) \subset \bigcup_{i=1}^k V_i$$

and without loss of generality we may assume that

$$V_1 \cap V_2 \text{ (open set)} \neq \emptyset, \quad V_2 \cap V_3 \text{ (open set)} \neq \emptyset, \quad \dots, \quad V_{k-1} \cap V_k \text{ (open set)} \neq \emptyset.$$

and moreover, each coordinate neighborhoods V_i is lying either on a plane or on a sphere with radius $r_i > 0$.

If V_1 lies on a plane, then V_2 also **lies on the same plane**. This is because if V_2 lies on a sphere or a different plane, then it is impossible for $V_1 \cap V_2$ to be an **open set** on the surface S . By induction, all V_i , $i = 3, 4, \dots, k$, will all lie on the same plane. Hence p and q lie on the same plane. By fixing $p \in S$ and letting $q \in S$ be arbitrary, we see that the whole surface S lies on some plane.

Similarly, if V_1 lies on a sphere with radius $r > 0$, the same argument implies that all V_i , $1 \leq i \leq k$, lies on the same sphere with radius $r > 0$. Hence p, q lie on the same sphere with radius $r > 0$. By fixing $p \in S$ and letting $q \in S$ be arbitrary, we see that the whole surface S lies on a sphere with radius $r > 0$. The proof is done. \square

2.1.6 Asymptotic Direction, Asymptotic Curve, Dupin Indicatrix, and Conjugate Directions.

Definition 2.62 Let $p \in S$. A direction $v \in T_p S$ is called an **asymptotic direction** at p if the normal curvature of S at p along v is 0. A connected regular curve $C \subset S$ is called an **asymptotic curve** if for each $p \in C$ the **tangent line** L at $p \in C$ is along an asymptotic **direction**.

Remark 2.63 (Comparison.) If $\alpha(s) \in S$, $s \in I$, is a regular curve on S , we have:

1. If normal curvature $k_n(s)$ at $\alpha(s)$ along $\alpha'(s)$ is either $-k_1(s)$ or $-k_2(s)$ for all $s \in I$, then $\alpha(s)$ is a **line of curvature**.
2. If normal curvature $k_n(s)$ at $\alpha(s)$ along $\alpha'(s)$ is 0 everywhere, then $\alpha(s)$ is an **asymptotic curve**.

Lemma 2.64 We have the following:

1. If $p \in S$ is an **elliptic** point ($k_1(p) k_2(p) > 0$), there is **no asymptotic direction** at p .
2. If $p \in S$ is a **hyperbolic** point ($k_1(p) k_2(p) < 0$), we have **exactly two asymptotic directions** at p . Moreover, if $k_1(p) + k_2(p) = 0$ (i.e. mean curvature is 0), these two directions are **orthogonal**.
3. If $p \in S$ is a **parabolic** point ($k_1(p) k_2(p) = 0$, but one of them is nonzero), there is **exactly one asymptotic direction** at p .

Remark 2.65 (Important.) For 2 (hyperbolic point), we also have: if we have two **orthogonal** asymptotic directions at p , then we must have $H(p) = 0$. One can use Euler formula to check it.

Proof. (1) is obvious.

For (2), let $\{e_1, e_2\}$ an orthonormal basis on $T_p S$ (eigenvectors of $-dN_p$) corresponding to the two principal curvatures $k_1(p) \geq k_2(p)$. For unit vector $v \in T_p S$ with $v = (\cos \theta) e_1 + (\sin \theta) e_2$, the **normal curvature** $k_n(p)$ **along** v is given by

$$k_n(p) = k_1(p) \cos^2 \theta + k_2(p) \sin^2 \theta = [k_1(p) - k_2(p)] \cos^2 \theta + k_2(p), \quad \theta \in [0, \pi), \quad (74)$$

where now $k_1(p) > 0$ and $k_2(p) < 0$. We want to find $\theta \in [0, \pi)$ such that $k_n(p) = 0$, i.e. want to solve the equation for $\theta \in [0, \pi)$:

$$\cos^2 \theta = -\frac{k_2(p)}{k_1(p) - k_2(p)} \in (0, 1),$$

which gives

$$\cos \theta = \pm \sqrt{-\frac{k_2(p)}{k_1(p) - k_2(p)}}. \quad (75)$$

The equation has exactly two solutions $\theta_0 \in (0, \pi/2)$ and $\pi - \theta_0 \in (\pi/2, \pi)$. Therefore, we have **exactly two** asymptotic directions at p (draw a picture for this).

If $k_1(p) + k_2(p) = 0$, then equation (75) becomes $\cos \theta = \pm 1/\sqrt{2}$ and the two solutions are $\theta_0 = \pi/4, 3\pi/4$. These two directions are orthogonal.

For (3), if $k_1(p) > 0, k_2(p) = 0$, then equation $k_n(p) = 0$ becomes

$$k_1(p) \cos^2 \theta = 0, \quad \theta \in [0, \pi).$$

The only solution is $\theta = \pi/2$ and there is exactly one asymptotic direction at p . On the other hand, if $k_1(p) = 0, k_2(p) < 0$, then equation $k_n(p) = 0$ becomes

$$k_2(p) \sin^2 \theta = 0, \quad \theta \in [0, \pi).$$

The only solution is $\theta = 0$ and there is exactly one asymptotic direction at p . □

Definition 2.66 Let $p \in S$. The **Dupin indicatrix** at p is the set of vectors on $T_p S$ given by

$$\{w \in T_p S : II_p(w) = -\langle dN_p(w), w \rangle = \pm 1\}. \quad (76)$$

Let $\{e_1, e_2\}$ an orthonormal basis on $T_p S$ (eigenvectors of $-dN_p$) corresponding to the two principal curvatures $k_1(p) \geq k_2(p)$. For $w \in T_p S$, we can express it as

$$w = xe_1 + ye_2 = (r \cos \theta) e_1 + (r \sin \theta) e_2, \quad r = |w|$$

and obtain

$$II_p(w) = -\langle dN_p(xe_1 + ye_2), xe_1 + ye_2 \rangle = k_1 x^2 + k_2 y^2$$

or

$$\begin{aligned} II_p(w) &= -\langle dN_p((r \cos \theta) e_1 + (r \sin \theta) e_2), (r \cos \theta) e_1 + (r \sin \theta) e_2 \rangle \\ &= r^2 (k_1 \cos^2 \theta + k_2 \sin^2 \theta) = r^2 II_p(v), \quad \text{where } v = \frac{w}{r}. \end{aligned}$$

Therefore, **the Dupin indicatrix is described by**

$$k_1 x^2 + k_2 y^2 = \pm 1, \tag{77}$$

which is a **symmetric quadratic curve** on the plane $T_p S$.

Similar to Lemma 2.64, we have:

Lemma 2.67 *We have the following:*

1. If $p \in S$ is an **elliptic** point ($k_1(p) k_2(p) > 0$), the Dupin indicatrix is an **ellipse** on $T_p S$. Moreover, if $k_1(p) = k_2(p)$, it is a **circle**.
2. If $p \in S$ is a **hyperbolic** point ($k_1(p) k_2(p) < 0$), the Dupin indicatrix is a **hyperbola** and **its two asymptotes are pointing to the two asymptotic directions** at p .
3. If $p \in S$ is a **parabolic** point ($k_1(p) k_2(p) = 0$, but one of them is nonzero), the Dupin indicatrix is a pair of **parallel lines** pointing to the **asymptotic direction** at p .

Proof. (1) is obvious.

For (2), it suffices to verify the last statement. The hyperbola satisfies the equation

$$k_1(p) x^2 + k_2(p) y^2 = \pm 1, \quad k_1(p) > 0, \quad k_2(p) < 0.$$

We can decompose it as

$$\left(\sqrt{k_1(p)} x - \sqrt{-k_2(p)} y \right) \left(\sqrt{k_1(p)} x + \sqrt{-k_2(p)} y \right) = \pm 1$$

and its two asymptotes are the two lines L_1, L_2 given by

$$L_1 : \sqrt{k_1(p)} x - \sqrt{-k_2(p)} y = 0, \quad L_2 : \sqrt{k_1(p)} x + \sqrt{-k_2(p)} y = 0.$$

The line L_1 intersects the unit circle on $T_p S$ at $(\cos \theta_0, \sin \theta_0)$ on the first quadrant $(+, +)$ at

$$\cos \theta_0 = \sqrt{-\frac{k_2(p)}{k_1(p) - k_2(p)}}, \quad \sin \theta_0 = \sqrt{\frac{k_1(p)}{k_1(p) - k_2(p)}} \tag{78}$$

and the line L_2 intersects the unit circle on $T_p S$ at $(\cos \theta_0, \sin \theta_0)$ on the second quadrant $(-, +)$ at

$$\cos \theta_0 = -\sqrt{-\frac{k_2(p)}{k_1(p) - k_2(p)}}, \quad \sin \theta_0 = \sqrt{\frac{k_1(p)}{k_1(p) - k_2(p)}}. \tag{79}$$

The above two directions are asymptotic directions due to (75).

For (3), we may assume $k_1(p) > 0, k_2(p) = 0$. Now the Dupin indicatrix becomes

$$k_1(p) x^2 = \pm 1,$$

which is a pair of two parallel lines $x = \pm 1/\sqrt{k_1(p)}$ pointing in the y -axis direction (the direction with $\theta = \pi/2$, which is the only asymptotic direction at p). \square

Definition 2.68 Let $p \in S$ and $w_1, w_2 \in T_p S$ are two **nonzero** vectors (may or may not be unit vectors). If we have

$$\langle dN_p(w_1), w_2 \rangle = \langle w_1, dN_p(w_2) \rangle = 0, \quad (80)$$

we say these two vectors are **conjugate**. The two directions r_1, r_2 given by w_1 (or $-w_1$) and w_2 (or $-w_2$) are called **conjugate directions**.

Remark 2.69 (Be careful.) Be careful that the definition (80) implies $dN_p(w_1) \perp w_2$ and $dN_p(w_2) \perp w_1$. However, in general, it **does not** imply that $dN_p(w_1) = \lambda_1 w_1$ and $dN_p(w_2) = \lambda_2 w_2$ for some constants λ_1, λ_2 (unless $w_1 \perp w_2$). The vector $dN_p(w_1)$ may have component in w_2 direction and the vector $dN_p(w_2)$ may have component in w_1 direction.

Example 2.70 Assume at $p \in S$ we have $k_1 \neq k_2$ and **both are not 0**. Then two principal directions are conjugate.

Example 2.71 If $k_1 \neq k_2$ but one them is 0, say $k_1 > 0$ (with unit eigenvector e_1), $k_2 = 0$ (with unit eigenvector e_2 , $e_1 \perp e_2$), then for **any nonzero vector** $w \in T_p S$, the two vectors $w, e_2 \in T_p S$ are conjugate due to

$$\langle dN_p(w), e_2 \rangle = \langle w, dN_p(e_2) \rangle = \langle w, 0 \rangle = 0. \quad (81)$$

Example 2.72 An asymptotic direction (normal curvature zero direction) is conjugate to itself.

Example 2.73 If $k_1 = k_2 \neq 0$ (denote the common value as λ , $\lambda \neq 0$), any pair of **orthogonal directions** are conjugate. This is because $-dN_p = \lambda I$ and

$$\langle dN_p(w_1), w_2 \rangle = \langle -\lambda w_1, w_2 \rangle = -\lambda \langle w_1, w_2 \rangle = 0$$

if and only if $\langle w_1, w_2 \rangle = 0$. If $k_1 = k_2 = 0$, any pair of directions are conjugate.

Lemma 2.74 (Conjugate directions in terms of polar coordinates.) Let $p \in S$ and $\{e_1, e_2\}$ is the orthonormal basis at $T_p S$ satisfying

$$e_1 \wedge e_2 = N(p), \quad -dN_p(e_1) = k_1 e_1, \quad -dN_p(e_2) = k_2 e_2, \quad (82)$$

where $k_1 \neq k_2$ are principal curvatures. Let r_1, r_2 be two directions and θ, φ are the angles from e_1 to r_1, r_2 respectively in the orientation of $T_p S$. Then r_1, r_2 are **conjugate** if and only if

$$k_1 \cos \theta \cos \varphi + k_2 \sin \theta \sin \varphi = 0. \quad (83)$$

Remark 2.75 If we replace θ by $\theta + \pi$ (or φ by $\varphi + \pi$), the identity (83) still holds.

Proof. By the assumption, r_1, r_2 are conjugate if and only if the two vectors

$$w_1 = (\cos \theta) e_1 + (\sin \theta) e_2, \quad w_2 = (\cos \varphi) e_1 + (\sin \varphi) e_2 \quad (84)$$

are conjugate. We compute

$$\begin{aligned} & \langle dN_p(w_1), w_2 \rangle \\ &= \langle dN_p((\cos \theta) e_1 + (\sin \theta) e_2), (\cos \varphi) e_1 + (\sin \varphi) e_2 \rangle \\ &= -k_1 \cos \theta \cos \varphi - k_2 \sin \theta \sin \varphi \end{aligned}$$

and so $\langle dN_p(w_1), w_2 \rangle = 0$ if and only if (83) holds. \square

Lemma 2.76 (*Conjugate directions in terms of Euclidean coordinates.*) Same assumption as in Lemma 2.74 with $k_1 \neq k_2$. Assume the direction r_1 is given by some **nonzero** vector $w_1 = xe_1 + ye_2 \in T_p S$ and the direction r_2 is given by another **nonzero** vector $w_2 = \tilde{x}e_1 + \tilde{y}e_2 \in T_p S$. Then r_1, r_2 are **conjugate** if and only if

$$k_1 x \tilde{x} + k_2 y \tilde{y} = 0. \quad (85)$$

In particular, for any **nonzero** vector $w_1 = xe_1 + ye_2 \in T_p S$, if

$$w_2 = (-k_2 y) e_1 + (k_1 x) e_2 \in T_p S \quad (86)$$

is a **nonzero vector**, then w_1 and w_2 are conjugate.

Remark 2.77 The above lemma is still correct if $k_1 = k_2 \neq 0$. In case $k_1 > 0, k_2 = 0$, (85) becomes $k_1 x \tilde{x} = 0$. In such a case, any nonzero $(0, y)$ and any nonzero (\tilde{x}, \tilde{y}) are conjugate. See Example 2.70.

Proof. We can write $w_1 = \sqrt{x^2 + y^2} (\cos \theta, \sin \theta)$ and $w_2 = \sqrt{\tilde{x}^2 + \tilde{y}^2} (\cos \varphi, \sin \varphi)$. By (83), they are conjugate if and only if

$$k_1 \cos \theta \cos \varphi + k_2 \sin \theta \sin \varphi = 0,$$

which is the same as

$$k_1 \left(\sqrt{x^2 + y^2} \cos \theta \right) \left(\sqrt{\tilde{x}^2 + \tilde{y}^2} \cos \varphi \right) + k_2 \left(\sqrt{x^2 + y^2} \sin \theta \right) \left(\sqrt{\tilde{x}^2 + \tilde{y}^2} \sin \varphi \right) = 0,$$

i.e. if and only if (85) holds. In particular, the two nonzero vectors $(x, y), (-k_2 y, k_1 x)$ are conjugate. \square

Example 2.78 (*Using Dupin indicatrix to find conjugate directions.*) Using the property that any two **nonzero** vectors $w_1 = (x, y), w_2 = (-k_2 y, k_1 x)$ are conjugate to each other, one can explain the picture in the textbook, p. 152 for the elliptic case. The construction is also valid for the hyperbolic case. **Draw two pictures on blackboard** (one for elliptic case and one for hyperbolic case).

2.1.7 Conclusion for Conjugate Directions.

Let $p \in S$ and $\{e_1, e_2\}$ is the orthonormal basis at $T_p S$ satisfying

$$e_1 \wedge e_2 = N(p), \quad -dN_p(e_1) = k_1 e_1, \quad -dN_p(e_2) = k_2 e_2, \quad (87)$$

where k_1, k_2 are principal curvatures. Let r_1 be a direction on $T_p S$ given by $w_1 = (\cos \theta) e_1 + (\sin \theta) e_2$. By Lemma 2.74, we can conclude the following:

(1). Assume $k_1 \neq k_2$ and $k_1 k_2 > 0$ (i.e. $p \in S$ is an **elliptic** point) (without loss of generality, we may assume $k_1 > 0$ and $k_2 > 0$). In such a case the vector $(k_1 \cos \theta, k_2 \sin \theta) \neq (0, 0)$ is nonzero and there is an **unique conjugate direction** r_2 given by (note that $-w_2$ also gives rise to the **same** direction)

$$w_2 = (\cos \varphi) e_1 + (\sin \varphi) e_2,$$

where the direction $(\cos \varphi, \sin \varphi)$ is perpendicular to the direction $(k_1 \cos \theta, k_2 \sin \theta)$, i.e.

$$k_1 \cos \theta \cos \varphi + k_2 \sin \theta \sin \varphi = 0 \text{ (same as } \begin{pmatrix} k_1 \cos \theta \\ k_2 \sin \theta \end{pmatrix} \cdot \begin{pmatrix} \cos \varphi \\ \sin \varphi \end{pmatrix} = 0). \quad (88)$$

Moreover, the direction r_2 is **different from** r_1 (due to $k_1 \cos^2 \theta + k_2 \sin^2 \theta > 0$). In particular, if r_1 is the **principal direction** e_1 (same as $\theta = 0$), then the identity (88) is the same as $\cos \varphi = 0$ (i.e. $\varphi = \frac{\pi}{2}$) and we have r_2 is the **principal direction** e_2 . Similarly, if r_1 is the **principal direction** e_2 , then r_2 is the **principal direction** e_1 .

(2). If $k_1 \neq k_2$ and $k_1 k_2 < 0$ (i.e. $p \in S$ is a **hyperbolic** point), then we have the same conclusion as in (1), but the direction r_2 may be the same as r_1 if we have

$$k_1 \cos^2 \theta + k_2 \sin^2 \theta = 0 \text{ (note that } k_1 > 0, k_2 < 0\text{),}$$

which means that r_1 is an **asymptotic direction**. Therefore, as long as r_1 is **not an asymptotic direction**, the conjugate direction r_2 is **unique and different from r_1** . **But if r_1 is an asymptotic direction, the only direction conjugate to r_1 is r_1 itself**. Finally, if r_1 is the **principal direction e_1 (e_2)**, then r_2 is the **principal direction e_2 (e_1)**.

(3). If $k_1 \neq k_2$ and $k_1 k_2 = 0$ (i.e. $p \in S$ is a **parabolic** point) (without loss of generality, we may assume $k_1 > 0, k_2 = 0$), then the identity (88) is the same as $\cos \theta \cos \varphi = 0$. Therefore, if $\theta = \frac{\pi}{2}$ (same as $w_1 = e_2$, eigenvector for eigenvalue $k_2 = 0$), then **any nonzero vector $w_2 \in T_p S$** is conjugate to r_1 . But if $\theta \neq \frac{\pi}{2}$ (same as $w_1 \neq e_2$), then we have $\cos \varphi = 0$ and **the only direction conjugate to r_1 is the eigenvector direction e_2** .

Therefore, for a parabolic point, we have the property: **if two nonzero vectors $w_1, w_2 \in T_p S$ are conjugate to each other, then one of them must be pointing to the eigenvector direction e_2 , where $k_2 = 0$** .

(4). If $k_1 = k_2 = \lambda \neq 0$, then we have $-dN_p = \lambda I$, $\lambda \neq 0$, and for any nonzero vectors $w_1, w_2 \in T_p S$, we have

$$\langle dN_p(w_1), w_2 \rangle = \langle -\lambda w_1, w_2 \rangle = -\lambda \langle w_1, w_2 \rangle.$$

Therefore, $w_1, w_2 \in T_p S$ are conjugate if and only if $\langle w_1, w_2 \rangle = 0$.

(5). If $k_1 = k_2 = \lambda = 0$, then $-dN_p = 0$ and any pair of directions are conjugate.

2.1.8 Three Interesting Identities Related to Gauss Curvature and Mean Curvature.

Lemma 2.79 *Let $p \in S$ and $\{e_1, e_2\}$ be the orthonormal basis at $T_p S$ satisfying*

$$e_1 \wedge e_2 = N(p), \quad -dN_p(e_1) = k_1 e_1, \quad -dN_p(e_2) = k_2 e_2, \quad (89)$$

where k_1, k_2 are principal curvatures. We have:

1. If $H(p) = 0$ with $k_1(p) > 0, k_2(p) < 0$, we have the identity

$$\langle dN_p(w_1), dN_p(w_2) \rangle = -K(p) \langle w_1, w_2 \rangle, \quad \forall w_1, w_2 \in T_p(S), \quad (90)$$

where $K(p) < 0$ is the Gauss curvature at p . In particular, if we assume $H \equiv 0$ on S with $k_1 > 0, k_2 < 0$ everywhere, then the angle of two intersecting curves on S and the angle of their spherical images are equal **up to a sign** (in such a case, for any $p \in S$, dN_p is a **conformal** map because it preserves angles).

2. We have

$$dN_p(w_1) \wedge dN_p(w_2) = K(p) (w_1 \wedge w_2), \quad \forall w_1, w_2 \in T_p(S). \quad (91)$$

3. We have

$$dN_p(w_1) \wedge w_2 + w_1 \wedge dN_p(w_2) = -2H(p) (w_1 \wedge w_2), \quad \forall w_1, w_2 \in T_p(S). \quad (92)$$

Remark 2.80 In the case when we have $w_1 \wedge w_2 = N$, then (91) and (92) give the following

$$K(p) = \langle dN_p(w_1) \wedge dN_p(w_2), N(p) \rangle \quad (93)$$

and

$$-2H(p) = \langle dN_p(w_1) \wedge w_2 + w_1 \wedge dN_p(w_2), N(p) \rangle. \quad (94)$$

Proof. We prove (2) and (3) first. For any $w_1, w_2 \in T_p(S)$, we can express them as

$$w_1 = a_1 e_1 + a_2 e_2, \quad w_2 = b_1 e_1 + b_2 e_2 \in T_p(S) \quad (95)$$

for some constants a_1, a_2, b_1, b_2 . We have

$$w_1 \wedge w_2 = (a_1 b_2 - a_2 b_1) e_1 \wedge e_2$$

and

$$\begin{aligned} & dN_p(w_1) \wedge dN_p(w_2) \\ &= (-a_1 k_1 e_1 - a_2 k_2 e_2) \wedge (-b_1 k_1 e_1 - b_2 k_2 e_2) \\ &= (a_1 k_1 e_1 + a_2 k_2 e_2) \wedge (b_1 k_1 e_1 + b_2 k_2 e_2) = (a_1 k_1 b_2 k_2 - a_2 k_2 b_1 k_1) e_1 \wedge e_2 \\ &= k_1 k_2 (a_1 b_2 - a_2 b_1) e_1 \wedge e_2 = K(p) (w_1 \wedge w_2), \quad \forall w_1, w_2 \in T_p(S). \end{aligned}$$

We also have

$$\begin{aligned} & dN_p(w_1) \wedge w_2 + w_1 \wedge dN_p(w_2) \\ &= (-a_1 k_1 e_1 - a_2 k_2 e_2) \wedge (b_1 e_1 + b_2 e_2) + (a_1 e_1 + a_2 e_2) \wedge (-b_1 k_1 e_1 - b_2 k_2 e_2) \\ &= [-a_1 k_1 b_2 + a_2 k_2 b_1 + a_1 (-b_2 k_2) + a_2 b_1 k_1] e_1 \wedge e_2 \\ &= [a_1 b_2 (-k_1 - k_2) + a_2 b_1 (k_1 + k_2)] e_1 \wedge e_2 = -2H(p) (a_1 b_2 - a_2 b_1) e_1 \wedge e_2 \\ &= -2H(p) (w_1 \wedge w_2), \quad \forall w_1, w_2 \in T_p(S). \end{aligned}$$

For (1), we first have

$$K(p) = k_1(p) k_2(p) = -k_1^2(p) = -k_2^2(p)$$

and

$$\begin{aligned} \langle dN_p(w_1), dN_p(w_2) \rangle &= \langle -a_1 k_1 e_1 - a_2 k_2 e_2, -b_1 k_1 e_1 - b_2 k_2 e_2 \rangle \\ &= a_1 b_1 k_1^2 + a_2 b_2 k_2^2 = -(a_1 b_1 + a_2 b_2) K(p) = -K(p) \langle w_1, w_2 \rangle, \quad \forall w_1, w_2 \in T_p(S), \end{aligned}$$

where $K(p) < 0$. In particular, we have

$$|dN_p(w)|^2 = -K(p) |w|^2 \quad (\text{same as } |dN_p(w)| = \sqrt{-K(p)} |w|), \quad \forall w \in T_p S. \quad (96)$$

If we assume $H \equiv 0$ on S with $k_1 > 0, k_2 < 0$ everywhere, then for any two regular curves $\alpha(s), \beta(s)$ on S with intersection point at $\alpha(0) = \beta(0) = q \in S$ with angle θ , we have

$$\langle dN_q(\alpha'(0)), dN_q(\beta'(0)) \rangle = -K(q) \langle \alpha'(0), \beta'(0) \rangle,$$

where we know that

$$dN_q(\alpha'(0)) = \left. \frac{d}{ds} \right|_{s=0} N(\alpha(s)), \quad dN_q(\beta'(0)) = \left. \frac{d}{ds} \right|_{s=0} N(\beta(s)).$$

The intersection angle θ_N of the two spherical image curves $N(\alpha(s)), N(\beta(s))$ at $N(q)$ satisfies

$$\begin{aligned} \cos \theta_N &= \frac{\langle dN_q(\alpha'(0)), dN_q(\beta'(0)) \rangle}{|dN_q(\alpha'(0))| |dN_q(\beta'(0))|} \\ &= \frac{-K(q) \langle \alpha'(0), \beta'(0) \rangle}{\sqrt{-K(q)} |\alpha'(0)| \sqrt{-K(q)} |\beta'(0)|} = \frac{\langle \alpha'(0), \beta'(0) \rangle}{|\alpha'(0)| |\beta'(0)|} = \cos \theta, \end{aligned}$$

which implies $\theta_N = \pm \theta$. The proof is done. \square

Remark 2.81 (A different proof of Lemma 2.79.) For (2), we may assume $\{w_1, w_2\} \subset T_p(S)$ is a **basis** and so

$$dN_p(w_1) = aw_1 + bw_2, \quad dN_p(w_2) = cw_1 + dw_2$$

for some coefficients a, b, c, d . With respect to $\{w_1, w_2\}$, the map $dN_p : T_pS \rightarrow T_pS$ has matrix representation

$$M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}, \quad K(p) = ad - bc, \quad H(p) = -\frac{1}{2}(a + d)$$

and

$$\begin{aligned} dN_p(w_1) \wedge dN_p(w_2) \\ = (aw_1 + bw_2) \wedge (cw_1 + dw_2) = (ad - bc)w_1 \wedge w_2 = K(p)w_1 \wedge w_2. \end{aligned}$$

For (3), if $\{w_1, w_2\} \subset T_p(S)$ is a **basis**, then

$$\begin{aligned} dN_p(w_1) \wedge w_2 + w_1 \wedge dN_p(w_2) \\ = (aw_1 + bw_2) \wedge w_2 + w_1 \wedge (cw_1 + dw_2) \\ = (aw_1) \wedge w_2 + w_1 \wedge (dw_2) = (a + d)w_1 \wedge w_2 = -2H(p)w_1 \wedge w_2. \end{aligned} \quad (97)$$

If $\{w_1, w_2\}$ is not a basis, then $w_1 = \lambda w_2$ for some $\lambda \in \mathbb{R}$ and then

$$dN_p(w_1) \wedge w_2 + w_1 \wedge dN_p(w_2) = \lambda dN_p(w_2) \wedge w_2 + \lambda w_2 \wedge dN_p(w_2) = 0$$

and similarly $w_1 \wedge w_2 = 0$. In such a case we also have the identity (97). Finally, for (1), we assume $H(p) = 0$ (i.e. $a + d = 0$) with $k_1(p) > 0$, $k_2(p) < 0$. If $\{w_1, w_2\} \subset T_p(S)$ is a **basis**, then

$$\begin{aligned} \langle dN_p(w_1), dN_p(w_2) \rangle &= \langle aw_1 + bw_2, cw_1 + dw_2 \rangle \\ &= ac \langle w_1, w_1 \rangle + (ad + bc) \langle w_1, w_2 \rangle + bd \langle w_2, w_2 \rangle \end{aligned} \quad (98)$$

and, on the other hand, we have

$$-K(p) \langle w_1, w_2 \rangle = -(ad - bc) \langle w_1, w_2 \rangle. \quad (99)$$

To go further, since $dN_p : T_pS \rightarrow T_pS$ is self-adjoint, we have $\langle dN_p(w_1), w_2 \rangle = \langle w_1, dN_p(w_2) \rangle$, which gives

$$\langle aw_1 + bw_2, w_2 \rangle = \langle w_1, cw_1 + dw_2 \rangle,$$

i.e.

$$\underbrace{c \langle w_1, w_1 \rangle - b \langle w_2, w_2 \rangle}_{\text{self-adjoint cond.}} = (a - d) \langle w_1, w_2 \rangle, \quad (100)$$

and by $a + d = 0$ ($d = -a$), the above becomes

$$\boxed{c \langle w_1, w_1 \rangle - b \langle w_2, w_2 \rangle = 2a \langle w_1, w_2 \rangle}. \quad (101)$$

Therefore,

$$\begin{aligned} &\underbrace{\langle dN_p(w_1), dN_p(w_2) \rangle + K(p) \langle w_1, w_2 \rangle}_{\text{self-adjoint cond.}} \\ &= ac \langle w_1, w_1 \rangle + (ad + bc) \langle w_1, w_2 \rangle + bd \langle w_2, w_2 \rangle + (ad - bc) \langle w_1, w_2 \rangle \\ &= ac \langle w_1, w_1 \rangle + 2ad \langle w_1, w_2 \rangle + bd \langle w_2, w_2 \rangle, \quad d = -a \\ &= a [c \langle w_1, w_1 \rangle - 2a \langle w_1, w_2 \rangle - b \langle w_2, w_2 \rangle] = 0 \end{aligned} \quad (102)$$

and the identity is verified. If $\{w_1, w_2\}$ is not a basis, then without loss of generality we may assume $w_1 = w_2$ (denote it as w) and express it as $w = a_1e_1 + a_2e_2$ (see (95)). One can easily verify the identity (102).

2.1.9 The Geodesic Torsion for a Regular Curve on a Regular Surface $S \subset \mathbb{R}^3$ (this is Exercise 19 in p. 155).

Let $\alpha(s)$, $s \in I$, be a regular curve on S . In the following, we shall compare the **Frenet frame** equations for the moving frame $\{t(s), n(s), b(s)\}$ along $\alpha(s) \in \mathbb{R}^3$ and the differential equations for the **surface moving frame** $\{t(s), \mathbf{n}_{int}(s), N(s)\}$ along $\alpha(s) \in S$. Recall that $\mathbf{n}_{int}(s)$ is the **intrinsic normal** of α at s , defined as

$$\mathbf{n}_{int}(s) = N(s) \wedge t(s) = N(s) \wedge \alpha'(s), \quad s \in I \quad (N(s) \text{ means } N(\alpha(s))). \quad (103)$$

By (245), we have

$$\begin{aligned} t'(s) &= \underbrace{\alpha''(s)} = k(s)n(s) = \underbrace{\langle \alpha''(s), \mathbf{n}_{int}(s) \rangle}_{k_g(s)} \mathbf{n}_{int}(s) + \underbrace{\langle \alpha''(s), N(s) \rangle}_{k_n(s)} N(s) \\ &= \underbrace{k_g(s) \mathbf{n}_{int}(s) + k_n(s) N(s)}, \quad s \in I, \end{aligned} \quad (104)$$

where $k_n(s)$ is the **geodesic curvature** of α and $k_n(s)$ is the **normal curvature** of α .

To continue, we would like to express $\{t(s), n(s), b(s)\}$ in terms of $\{t(s), \mathbf{n}_{int}(s), N(s)\}$. The following lemma is straightforward.

Lemma 2.82 *We have the following expression:*

$$\begin{cases} n(s) = \underbrace{\langle n(s), \mathbf{n}_{int}(s) \rangle}_{k_g(s)} \mathbf{n}_{int}(s) + \underbrace{\langle n(s), N(s) \rangle}_{k_n(s)} N(s) \\ b(s) = -\underbrace{\langle n(s), N(s) \rangle}_{k_n(s)} \mathbf{n}_{int}(s) + \underbrace{\langle n(s), \mathbf{n}_{int}(s) \rangle}_{k_g(s)} N(s). \end{cases} \quad (105)$$

In terms of formal matrix notation we have

$$\begin{pmatrix} t(s) \\ n(s) \\ b(s) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \langle n(s), \mathbf{n}_{int}(s) \rangle & \langle n(s), N(s) \rangle \\ 0 & -\langle n(s), N(s) \rangle & \langle n(s), \mathbf{n}_{int}(s) \rangle \end{pmatrix} \begin{pmatrix} t(s) \\ \mathbf{n}_{int}(s) \\ N(s) \end{pmatrix}, \quad s \in I. \quad (106)$$

*where the coefficient matrix is **orthogonal**.*

Proof. The first identity in (105) is clear. For the second identity in (105), we have

$$\begin{aligned} b(s) &= t(s) \wedge n(s) = t(s) \wedge \left[\underbrace{\langle n(s), \mathbf{n}_{int}(s) \rangle}_{k_g(s)} \mathbf{n}_{int}(s) + \underbrace{\langle n(s), N(s) \rangle}_{k_n(s)} N(s) \right] \\ &= \underbrace{\langle n(s), \mathbf{n}_{int}(s) \rangle}_{k_g(s)} N(s) - \underbrace{\langle n(s), N(s) \rangle}_{k_n(s)} \mathbf{n}_{int}(s). \end{aligned}$$

The proof is done. □

Next, we compute $N'(s)$ (note that $N(s)$ means $N(\alpha(s))$) and can express it as

$$N'(s) = A(s)t(s) + Q(s)\mathbf{n}_{int}(s) + C(s)N(s), \quad s \in I$$

for some coefficients $A(s)$, $Q(s)$, $C(s)$ and we see that $C(s) \equiv 0$ (since $\langle N'(s), N(s) \rangle = 0$ for all s) and

$$A(s) = \langle N'(s), t(s) \rangle = \langle N'(s), \alpha'(s) \rangle = -\langle N(s), \alpha''(s) \rangle = -k_n(s),$$

which is the normal curvature. Hence we conclude (we will give $Q(s)$ a name later on)

$$N'(s) = -k_n(s)t(s) + Q(s)\mathbf{n}_{int}(s), \quad s \in I. \quad (107)$$

Finally, we compute

$$\begin{aligned}
\mathbf{n}'_{int}(s) &= \frac{d}{ds} [N(s) \wedge \alpha'(s)] = N'(s) \wedge \alpha'(s) + N(s) \wedge \alpha''(s) \\
&= [-k_n(s) t(s) + Q(s) \mathbf{n}_{int}(s)] \wedge t(s) + N(s) \wedge [k_g(s) \mathbf{n}_{int}(s) + k_n(s) N(s)] \\
&= -k_g(s) t(s) - Q(s) N(s), \quad s \in I.
\end{aligned}$$

We define the following:

Definition 2.83 (See textbook p. 155.) The quantity $Q(s)$ is called the **geodesic torsion** of α at $\alpha(s)$ and denote it as $\tau_g(s)$. Note that we can also express it as

$$\tau_g(s) = \langle N'(s), \mathbf{n}_{int}(s) \rangle = -\langle N(s), \mathbf{n}'_{int}(s) \rangle, \quad s \in I. \quad (108)$$

Remark 2.84 (Important.) The geodesic torsion $\tau_g(s)$ is a **new quantity**. It cannot be expressed in terms of $k_n(s)$ and $k_g(s)$. Note that if we do the following:

$$\begin{aligned}
\tau_g(s) &= \langle N'(s), \mathbf{n}_{int}(s) \rangle = -\langle N(s), \mathbf{n}'_{int}(s) \rangle = -\left\langle N(s), \frac{d}{ds} (N(s) \wedge t(s)) \right\rangle \\
&= -\langle N(s), N'(s) \wedge t(s) + N(s) \wedge t'(s) \rangle = -\langle N(s), N'(s) \wedge t(s) \rangle \\
&= -\langle N(s), [-k_n(s) t(s) + \tau_g(s) \mathbf{n}_{int}(s)] \wedge t(s) \rangle = -\langle N(s), \tau_g(s) \mathbf{n}_{int}(s) \wedge t(s) \rangle = \tau_g(s),
\end{aligned}$$

we get nothing useful at all.

We can summarize the following:

Lemma 2.85 (Moving frame equations on S .) The **surface moving frame** $\{t(s), \mathbf{n}_{int}(s), N(s)\}$ along $\alpha(s) \in S$ (with normal $N(s)$) satisfies the equation

$$\begin{pmatrix} t'(s) \\ \mathbf{n}'_{int}(s) \\ N'(s) \end{pmatrix} = \begin{pmatrix} 0 & k_g(s) & k_n(s) \\ -k_g(s) & 0 & -\tau_g(s) \\ -k_n(s) & \tau_g(s) & 0 \end{pmatrix} \begin{pmatrix} t(s) \\ \mathbf{n}_{int}(s) \\ N(s) \end{pmatrix}, \quad s \in I. \quad (109)$$

Remark 2.86 Compare with the Frenet frame equations for $\alpha(s) \in \mathbb{R}^3$, given by

$$\begin{pmatrix} t'(s) \\ n'(s) \\ b'(s) \end{pmatrix} = \begin{pmatrix} 0 & k(s) & 0 \\ -k(s) & 0 & -\tau(s) \\ 0 & \tau(s) & 0 \end{pmatrix} \begin{pmatrix} t(s) \\ n(s) \\ b(s) \end{pmatrix}, \quad \begin{pmatrix} t(s) \\ n(s) \\ b(s) \end{pmatrix} \in \mathbb{R}^9. \quad (110)$$

The following lemma is about the relation between **geodesic torsion** and **principal curvatures** of S :

Lemma 2.87 Let $p \in S$ and $\alpha(s) \in S$, $s \in I$, is a regular curve with $\alpha(0) = p$ and let $\{e_1, e_2\}$ be the orthonormal basis at $T_p S$ satisfying

$$e_1 \wedge e_2 = N(p), \quad -dN_p(e_1) = k_1 e_1, \quad -dN_p(e_2) = k_2 e_2. \quad (111)$$

Denote the angle from e_1 to $t(0) = \alpha'(0)$ by φ . Then we have

$$\tau_g(0) = (k_1 - k_2) \cos \varphi \sin \varphi. \quad (112)$$

Proof. By (109), we have (note that $N(0) \wedge e_1 = e_2$, $N(0) \wedge e_2 = -e_1$)

$$\begin{aligned}\tau_g(0) &= \langle N'(0), \mathbf{n}_{int}(0) \rangle = \langle N'(0), N(0) \wedge t(0) \rangle \\ &= \left\langle N'(0), N(0) \wedge \left[\underbrace{(\cos \varphi) e_1 + (\sin \varphi) e_2} \right] \right\rangle = \langle N'(0), (\cos \varphi) e_2 - (\sin \varphi) e_1 \rangle \\ &= (\cos \varphi) \langle N'(0), e_2 \rangle - (\sin \varphi) \langle N'(0), e_1 \rangle.\end{aligned}$$

Now we note that

$$\begin{aligned}N'(0) &= dN_p(\alpha'(0)) = dN_p\left(\underbrace{(\cos \varphi) e_1 + (\sin \varphi) e_2}\right) \\ &= (\cos \varphi) (-k_1 e_1) + (\sin \varphi) (-k_2 e_2),\end{aligned}$$

which gives

$$\begin{aligned}\tau_g(0) &= (\cos \varphi) \langle N'(0), e_2 \rangle - (\sin \varphi) \langle N'(0), e_1 \rangle \\ &= (\cos \varphi) (\sin \varphi) (-k_2) - (\sin \varphi) (\cos \varphi) (-k_1) = (k_1 - k_2) (\cos \varphi) (\sin \varphi).\end{aligned}$$

□

Lemma 2.88 (*Relation between torsion and geodesic torsion.*) Let $p \in S$ and $\alpha(s) \in S$, $s \in I$, is a regular curve with $\alpha(0) = p$. By (106), we have

$$n(s) = \langle n(s), \mathbf{n}_{int}(s) \rangle \mathbf{n}_{int}(s) + \langle n(s), N(s) \rangle N(s), \quad s \in I$$

and if we write it as

$$n(s) = \underbrace{\langle n(s), N(s) \rangle}_{\cos \theta(s)} N(s) + \underbrace{\langle n(s), \mathbf{n}_{int}(s) \rangle}_{\sin \theta(s)} \mathbf{n}_{int}(s), \quad s \in I$$

and let (i.e. the angle between $n(s)$ and $N(s)$ is denoted as $\theta(s)$)

$$\langle n(s), N(s) \rangle = \cos \theta(s), \quad \langle n(s), \mathbf{n}_{int}(s) \rangle = \sin \theta(s), \quad s \in I, \quad (113)$$

then we have

$$\theta'(s) = \tau(s) - \tau_g(s), \quad s \in I, \quad (114)$$

where $\tau(s)$ is the torsion of $\alpha(s)$ as a curve in \mathbb{R}^3 .

Remark 2.89 If we let $\langle n(s), N(s) \rangle = \sin \theta(s)$, then $\theta'(s) = \tau_g(s) - \tau(s)$, $s \in I$.

Proof. By (109) and (110), we have

$$\begin{aligned}(-\sin \theta(s)) \theta'(s) &= \frac{d}{ds} \langle n(s), N(s) \rangle = \langle n'(s), N(s) \rangle + \langle n(s), N'(s) \rangle \\ &= \langle -k(s) t(s) - \tau(s) b(s), N(s) \rangle + \langle n(s), -k_n(s) t(s) + \tau_g(s) \mathbf{n}_{int}(s) \rangle \\ &= -\tau(s) \langle b(s), N(s) \rangle + \tau_g(s) \langle n(s), \mathbf{n}_{int}(s) \rangle \\ &= -\tau(s) \left\langle -\underbrace{\langle n(s), N(s) \rangle}_{\cos \theta(s)} \mathbf{n}_{int}(s) + \underbrace{\langle n(s), \mathbf{n}_{int}(s) \rangle}_{\sin \theta(s)} N(s), N(s) \right\rangle + \tau_g(s) \langle n(s), \mathbf{n}_{int}(s) \rangle \\ &= \underbrace{\langle n(s), \mathbf{n}_{int}(s) \rangle}_{\sin \theta(s)} (\tau_g(s) - \tau(s)), \quad \text{where } \langle n(s), \mathbf{n}_{int}(s) \rangle = \sin \theta(s).\end{aligned}$$

The proof is done. □

Lemma 2.90 (*Lines of curvature have zero geodesic torsion.*) Let $p \in S$ and $\alpha(s) \in S$, $s \in I$, is a regular curve with $\alpha(0) = p$. Then $\alpha(s)$ is a **line of curvature** if and only if $\tau_g(s) \equiv 0$ along the whole curve $\alpha(s)$, $s \in I$.

Proof. (\implies) Assume $\alpha(s)$ is a line of curvature. We have

$$N'(s) = dN_{\alpha(s)}(\alpha'(s)) = \lambda(s)\alpha'(s), \quad \forall s \in I$$

for some function $\lambda(s)$ and by (109) we also have

$$N'(s) = -k_n(s)t(s) + \tau_g(s)\mathbf{n}_{int}(s), \quad s \in I. \quad (115)$$

Comparing the above two identities, we have $\tau_g(s) \equiv 0$ along the whole curve $\alpha(s)$, $s \in I$.

(\impliedby) Assume $\tau_g(s) \equiv 0$ for all $s \in I$. By the identity (115), we must have $N'(s) = -k_n(s)t(s)$ for all $s \in I$, which means that $\alpha(s)$ is a line of curvature. \square

2.2 The Gauss Map in Local Coordinates (this is Section 3-3 of the book).

The purpose of this section is to use local coordinates to study the second fundamental form and the differential of Gauss map. Let $S \subset \mathbb{R}^3$ be a regular surface with an orientation N . All local parametrizations $\mathbf{x}(u, v)$ of S in this section are assumed to be compatible with the orientation N of S . That is

$$N(u, v) = N(\mathbf{x}(u, v)) = \frac{\mathbf{x}_u \wedge \mathbf{x}_v}{|\mathbf{x}_u \wedge \mathbf{x}_v|}$$

for all (u, v) in the domain of \mathbf{x} .

Let $\mathbf{x}(u, v)$ be a parametrization near $p \in S$. Let $\alpha(t) = \mathbf{x}(u(t), v(t))$ be a curve in S with $\alpha(0) = p$. In the following all computations are evaluated at the point p and at $t = 0$ unless otherwise stated. We have

$$dN(\alpha') = dN(\mathbf{x}_u u' + \mathbf{x}_v v') = \frac{d}{dt}N(u, v) = N_u u' + N_v v',$$

where $N(u, v)$ means $N(\mathbf{x}(u, v))$ and we know that $dN(\mathbf{x}_u) = N_u$, $dN(\mathbf{x}_v) = N_v$.

Since $N_u, N_v \in T_p S$, we can write

$$\begin{cases} N_u = dN(\mathbf{x}_u) = a_{11}\mathbf{x}_u + a_{21}\mathbf{x}_v, \\ N_v = dN(\mathbf{x}_v) = a_{12}\mathbf{x}_u + a_{22}\mathbf{x}_v, \end{cases} \quad (116)$$

for some constants a_{ij} , $1 \leq i, j \leq 2$. The above means that the matrix representation M for the linear map $dN_p : T_p S \rightarrow T_p S$, with respect to the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$, is given by

$$M = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}. \quad (117)$$

In particular, we have

$$\begin{aligned} dN(\alpha') &= dN(\mathbf{x}_u u' + \mathbf{x}_v v') = (a_{11}\mathbf{x}_u + a_{21}\mathbf{x}_v)u' + (a_{12}\mathbf{x}_u + a_{22}\mathbf{x}_v)v' \\ &= (a_{11}u' + a_{12}v')\mathbf{x}_u + (a_{21}u' + a_{22}v')\mathbf{x}_v, \end{aligned}$$

which, in terms of matrix formulation, is

$$M \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix}. \quad (118)$$

Remark 2.91 (Also see Remark 2.95.) Note that the matrix representation M for dN_p with respect to the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$ is **not necessarily symmetric** even if dN_p is **self-adjoint**. The matrix M is symmetric if $\{\mathbf{x}_u, \mathbf{x}_v\}$ is **orthonormal** due to

$$\begin{cases} \langle dN(\mathbf{x}_u), \mathbf{x}_v \rangle = \langle a_{11}\mathbf{x}_u + a_{21}\mathbf{x}_v, \mathbf{x}_v \rangle = a_{21}, \\ \langle dN(\mathbf{x}_v), \mathbf{x}_u \rangle = \langle a_{12}\mathbf{x}_u + a_{22}\mathbf{x}_v, \mathbf{x}_u \rangle = a_{12}, \\ \langle dN(\mathbf{x}_u), \mathbf{x}_u \rangle = \langle dN(\mathbf{x}_v), \mathbf{x}_v \rangle. \end{cases}$$

Now we look at an example: let $T(x, y) = (2x, -3y) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a **self-adjoint** linear map and choose the **non-orthonormal** basis $\{v_1, v_2\} = \{(1, 0), (1, 1)\}$. We have

$$Tv_1 = 2v_1, \quad Tv_2 = 5v_1 - 3v_2.$$

Therefore, the matrix representation for $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ with respect to $\{v_1, v_2\}$ is

$$M = \begin{pmatrix} 2 & 5 \\ 0 & -3 \end{pmatrix},$$

which is not symmetric.

Now we compare two different ways to express $II_p(\alpha')$, where $\alpha' = \mathbf{x}_u u' + \mathbf{x}_v v'$.

First way:

We have

$$\begin{aligned} II_p(\alpha') &= -\langle dN(\alpha'), \alpha' \rangle = -\left\langle \underbrace{N_u u' + N_v v'}_{\text{tangent vector}}, \mathbf{x}_u u' + \mathbf{x}_v v' \right\rangle \\ &= \boxed{(u', v') \begin{pmatrix} e & f \\ f & g \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix}} = e(u')^2 + 2fu'v' + g(v')^2, \end{aligned} \quad (119)$$

where the quantities e, f, g are given by

$$\begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} -\langle N_u, \mathbf{x}_u \rangle & -\langle N_u, \mathbf{x}_v \rangle \\ -\langle N_v, \mathbf{x}_u \rangle & -\langle N_v, \mathbf{x}_v \rangle \end{pmatrix} = \begin{pmatrix} \langle N, \mathbf{x}_{uu} \rangle & \langle N, \mathbf{x}_{vu} \rangle \\ \langle N, \mathbf{x}_{uv} \rangle & \langle N, \mathbf{x}_{vv} \rangle \end{pmatrix}. \quad (120)$$

By (119), we conclude the formula:

$$II_p(A\mathbf{x}_u + B\mathbf{x}_v) = eA^2 + 2fAB + gB^2, \quad \forall \text{ constants } A, B. \quad (121)$$

Definition 2.92 If $\mathbf{x}(u, v) : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is a local parametrization at $p \in S$ with basis $\{\mathbf{x}_u, \mathbf{x}_v\}$, then the above three quantities e, f, g are called **the coefficients of the second fundamental form** with respect to the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$ on $T_p S$.

Remark 2.93 Similar to E, F, G , the three quantities e, f, g are **differentiable functions** on their domain. For computational purpose, it is easier to use the formula

$$e = \langle N, \mathbf{x}_{uu} \rangle, \quad f = \langle N, \mathbf{x}_{uv} \rangle, \quad g = \langle N, \mathbf{x}_{vv} \rangle$$

to find e, f, g .

Remark 2.94 The first-derivative vectors $\mathbf{x}_u, \mathbf{x}_v, N_u, N_v$ are tangential, but their second derivatives can point to any directions in \mathbb{R}^3 .

Second way:

We have

$$\begin{aligned}
II_p(\alpha') &= -\langle dN(\alpha'), \alpha' \rangle \\
&= -\left\langle \underbrace{(a_{11}\mathbf{x}_u + a_{21}\mathbf{x}_v)u' + (a_{12}\mathbf{x}_u + a_{22}\mathbf{x}_v)v'}_{\mathbf{x}_u u' + \mathbf{x}_v v'}, \mathbf{x}_u u' + \mathbf{x}_v v' \right\rangle \\
&= -(u', v') \begin{pmatrix} \langle a_{11}\mathbf{x}_u + a_{21}\mathbf{x}_v, \mathbf{x}_u \rangle & \langle a_{11}\mathbf{x}_u + a_{21}\mathbf{x}_v, \mathbf{x}_v \rangle \\ \langle a_{12}\mathbf{x}_u + a_{22}\mathbf{x}_v, \mathbf{x}_u \rangle & \langle a_{12}\mathbf{x}_u + a_{22}\mathbf{x}_v, \mathbf{x}_v \rangle \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix} \\
&= \boxed{-(u', v') \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix}},
\end{aligned}$$

where $\{E, F, G\}$ are the coefficients of the first fundamental form. Thus, by comparison, the coefficients e, f, g of the second fundamental form and the coefficients E, F, G of the first fundamental form are related by

$$\begin{pmatrix} e & f \\ f & g \end{pmatrix} = - \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}, \quad (122)$$

or equivalently,

$$\boxed{\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = - \begin{pmatrix} e & f \\ f & g \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}}. \quad (123)$$

We call (123) **the equations of Weingarten**.

By

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} = \frac{1}{\Delta} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix}, \quad \Delta = EG - F^2 > 0.$$

we can express a_{ij} , $1 \leq i, j \leq 2$, in (123) explicitly as

$$a_{11} = \frac{fF - eG}{\Delta}, \quad a_{12} = \frac{gF - fG}{\Delta}, \quad a_{21} = \frac{eF - fE}{\Delta}, \quad a_{22} = \frac{fF - gE}{\Delta}. \quad (124)$$

Remark 2.95 From (124), we see that for the matrix $M = (a_{ij})$ to be symmetric we must have $a_{12} = a_{21}$, which is the same as

$$gF - fG = eF - fE \quad (\text{equivalent to } (E - G)f = (e - g)F). \quad (125)$$

The identity will not hold in general. But if we have $E = G$ and $F = 0$, then it will hold. In such a case, the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$ is **orthonormal**.

Remark 2.96 By (124), we see that a_{ij} , $1 \leq i, j \leq 2$, are **differentiable functions** on their domain.

Remark 2.97 Note that in (123) both matrices on the right hand side are symmetric. But the product of two symmetric matrices are not symmetric in general.

Remark 2.98 We always have $EG - F^2 > 0$, but we **do not** have $eg - f^2 > 0$ in general.

Since the determinant of $-dN_p$ (same as the determinant of dN_p) is the **Gauss curvature** K of S at p , we immediately have

$$K = k_1 k_2 = \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \frac{eg - f^2}{EG - F^2}. \quad (126)$$

Note that, in general, $eg - f^2$ may not be positive. We also have the **mean curvature** H of S at p , given by

$$H = \frac{1}{2} \text{Tr}(-dN_p) = -\frac{1}{2}(a_{11} + a_{22}) = \frac{1}{2} \frac{eG - 2fF + gE}{EG - F^2}. \quad (127)$$

Lemma 2.99 One can also use e, f, g to classify the following: Let $p \in S$. We have:

1. p is an **elliptic** point if $eg - f^2 > 0$.
2. p is a **hyperbolic** point if $eg - f^2 < 0$.
3. p is a **parabolic** point if $eg - f^2 = 0$, but at least one of e, f, g is not zero.
4. p is a **planar** point if $e = f = g = 0$ is zero.

Proof. 1 and 2 are clear. For 3 and 4, if $dN_p = 0$, we have $N_u = N_v = 0$, which implies $e = f = g = 0$ due to (120). Conversely, if $e = f = g = 0$, then by (123) we have $a_{ij} = 0$ for all $1 \leq i, j \leq 2$, which implies $dN_p = 0$. From this observation, 3 and 4 are clear. \square

Lemma 2.100 The two **principal curvatures** k_1, k_2 at $p \in S$ are given by

$$k_1 = H + \sqrt{H^2 - K}, \quad k_2 = H - \sqrt{H^2 - K}.$$

where K is the **Gauss curvature** of S at p and H is the **mean curvature** of S at p . In particular, we see that k_1, k_2 are continuous functions on their domain and differentiable except at umbilical points (where $H^2 = K$).

Proof. Since k_1, k_2 are eigenvalues of $-dN_p$, we have

$$\det(-dN_p - k_i I) = \det(dN_p + k_i I) = 0, \quad i = 1, 2,$$

i.e. k_1, k_2 are the two roots of the characteristic polynomial

$$\det(-dN_p - \lambda I) = \det \begin{pmatrix} a_{11} + \lambda & a_{21} \\ a_{12} & a_{22} + \lambda \end{pmatrix} = \lambda^2 - 2H\lambda + K = 0.$$

The result follows. \square

Example 2.101 Consider the parametrization of the **torus** given by ($a > r > 0$)

$$\mathbf{x}(u, v) = ((a + r \cos u) \cos v, (a + r \cos u) \sin v, r \sin u), \quad 0 < u < 2\pi, \quad 0 < v < 2\pi.$$

We compute

$$\begin{cases} \mathbf{x}_u = (-r \sin u \cos v, -r \sin u \sin v, r \cos u) \\ \mathbf{x}_v = (-(a + r \cos u) \sin v, (a + r \cos u) \cos v, 0) \\ \mathbf{x}_{uu} = (-r \cos u \cos v, -r \cos u \sin v, -r \sin u) \\ \mathbf{x}_{uv} = (r \sin u \sin v, -r \sin u \cos v, 0) \\ \mathbf{x}_{vv} = (-(a + r \cos u) \cos v, -(a + r \cos u) \sin v, 0) \end{cases}$$

and get

$$E = \langle \mathbf{x}_u, \mathbf{x}_u \rangle = r^2, \quad F = \langle \mathbf{x}_u, \mathbf{x}_v \rangle = 0, \quad G = \langle \mathbf{x}_v, \mathbf{x}_v \rangle = (a + r \cos u)^2.$$

Also we have

$$\begin{aligned} e &= \langle N, \mathbf{x}_{uu} \rangle = \frac{1}{|\mathbf{x}_u \wedge \mathbf{x}_v|} \langle \mathbf{x}_u \wedge \mathbf{x}_v, \mathbf{x}_{uu} \rangle \\ &= \frac{1}{\sqrt{EG - F^2}} \det(\mathbf{x}_u, \mathbf{x}_v, \mathbf{x}_{uu}) = \frac{r^2(a + r \cos u)}{r(a + r \cos u)} = r. \end{aligned}$$

Similarly, we obtain

$$f = \frac{\det(\mathbf{x}_u, \mathbf{x}_v, \mathbf{x}_{uv})}{r(a + r \cos u)} = 0, \quad g = \frac{\det(\mathbf{x}_u, \mathbf{x}_v, \mathbf{x}_{vv})}{r(a + r \cos u)} = (a + r \cos u) \cos u.$$

Hence the Gauss curvature $K(u, v)$ at the point $\mathbf{x}(u, v)$ is given by

$$K(u, v) = \frac{eg - f^2}{EG - F^2} = \frac{\cos u}{r(a + r \cos u)}, \quad 0 < u < 2\pi. \quad (128)$$

By (128), we can easily locate elliptic points, hyperbolic points and parabolic points on the torus. See the picture in p. 160 of the book.

Proposition 2.102 (This is Proposition 1 in p. 160.) Let $p \in S$ be an **elliptic** point. Then there exists a neighborhood V of p in S such that all points in V belong to the same side of $T_p S$. If $p \in S$ is a **hyperbolic** point, then in each neighborhood of p there exist points of S in both sides of $T_p S$.

Remark 2.103 (Important.) There is no result similar to the above proposition for a parabolic or planar point.

Proof. We use Taylor series expansion. Let $\mathbf{x}(u, v)$ be a parametrization near p with $\mathbf{x}(0, 0) = p$. The distance from the point $\mathbf{x}(u, v)$ to the tangent plane $T_p S$ is given by

$$d(u, v) = \langle \mathbf{x}(u, v) - \mathbf{x}(0, 0), N(p) \rangle.$$

By Taylor series expansion (for vector-valued functions), we have

$$\mathbf{x}(u, v) = \mathbf{x}(0, 0) + \mathbf{x}_u u + \mathbf{x}_v v + \frac{1}{2} (\mathbf{x}_{uu} u^2 + 2\mathbf{x}_{uv} uv + \mathbf{x}_{vv} v^2) + \bar{R},$$

where the derivatives are evaluated at $(0, 0)$ and $\bar{R} = \bar{R}(u, v)$ satisfies (note that \bar{R} is a vector)

$$\lim_{(u,v) \rightarrow (0,0)} \frac{\bar{R}}{u^2 + v^2} = 0.$$

Now we can plug the above into $d(u, v)$ to get

$$\begin{aligned} d(u, v) &= \left\langle \mathbf{x}_u u + \mathbf{x}_v v + \underbrace{\frac{1}{2} (\mathbf{x}_{uu} u^2 + 2\mathbf{x}_{uv} uv + \mathbf{x}_{vv} v^2)} + \bar{R}, N(p) \right\rangle \\ &= \underbrace{\frac{1}{2} (eu^2 + 2fuv + gv^2)} + R = \underbrace{\frac{1}{2} II_p(w)} + R \end{aligned} \quad (129)$$

where $w = \mathbf{x}_u u + \mathbf{x}_v v \in T_p S$ and

$$II_p(w) = -\langle dN_p(\mathbf{x}_u u + \mathbf{x}_v v), \mathbf{x}_u u + \mathbf{x}_v v \rangle = \frac{1}{2} (eu^2 + 2fuv + gv^2)$$

and $R = \langle \bar{R}, N(p) \rangle$, with

$$\lim_{(u,v) \rightarrow (0,0)} \frac{R}{u^2 + v^2} = 0. \quad (130)$$

We can rewrite (129) as (we may assume $w \neq 0$)

$$\boxed{d(u, v) = |w|^2 \left(\frac{1}{2} II_p \left(\frac{w}{|w|} \right) + \frac{R}{|w|^2} \right)}, \quad \frac{w}{|w|} \text{ is unit vector} \quad (131)$$

and note that

$$II_p \left(\frac{w}{|w|} \right) = \mathbf{normal\ curvature\ along\ direction} \frac{w}{|w|} \in T_p S.$$

Moreover, we also have (see (130) and Remark 2.104 below)

$$\begin{aligned} \lim_{(u,v) \rightarrow (0,0)} \frac{R}{|w|^2} &= \lim_{(u,v) \rightarrow (0,0)} \frac{R}{|\mathbf{x}_u u + \mathbf{x}_v v|^2} \\ &= \lim_{(u,v) \rightarrow (0,0)} \left(\frac{u^2 + v^2}{|\mathbf{x}_u u + \mathbf{x}_v v|^2} \frac{R}{u^2 + v^2} \right) = 0. \end{aligned} \quad (132)$$

If p is an elliptic point (assume that $k_1(p) \geq k_2(p) > 0$), then

$$II_p \left(\frac{w}{|w|} \right) \geq k_2(p) > 0 \quad \text{for any nonzero } w \in T_p S.$$

Hence for $|w| \neq 0$ small enough, we have

$$\frac{1}{2} II_p \left(\frac{w}{|w|} \right) + \frac{R}{|w|^2} \geq k_2(p) + \frac{R}{|w|^2} > 0, \quad (133)$$

i.e., for all (u, v) close to $(0, 0)$, we have $d(u, v) > 0$. Thus all such $\mathbf{x}(u, v)$ lies on the same side of $T_p S$.

If p is a hyperbolic point with $k_1(p) > 0$ and $k_2(p) < 0$, then there exist (u, v) and (\bar{u}, \bar{v}) such that $d(u, v) > 0$ and $d(\bar{u}, \bar{v}) < 0$. Hence (u, v) and (\bar{u}, \bar{v}) lie on different sides of $T_p S$. The proof is done. \square

Remark 2.104 *This is to explain that the quantity $(u^2 + v^2) / |\mathbf{x}_u u + \mathbf{x}_v v|^2$ stays **bounded** as $(u, v) \rightarrow (0, 0)$. We have*

$$\frac{u^2 + v^2}{|\mathbf{x}_u u + \mathbf{x}_v v|^2} = \frac{u^2 + v^2}{u^2 |\mathbf{x}_u(u, v)|^2 + 2uv \langle \mathbf{x}_u(u, v), \mathbf{x}_v(u, v) \rangle + v^2 |\mathbf{x}_v(u, v)|^2} \quad (134)$$

and as $(u, v) \rightarrow (0, 0)$, we have

$$\begin{cases} |\mathbf{x}_u(u, v)|^2 \rightarrow |\mathbf{x}_u(0, 0)|^2, & |\mathbf{x}_v(u, v)|^2 \rightarrow |\mathbf{x}_v(0, 0)|^2 \\ \langle \mathbf{x}_u(u, v), \mathbf{x}_v(u, v) \rangle \rightarrow \langle \mathbf{x}_u(0, 0), \mathbf{x}_v(0, 0) \rangle. \end{cases}$$

If we write (u, v) as $(r \cos \theta, r \sin \theta)$, we get

$$\begin{aligned} & \frac{u^2 + v^2}{u^2 |\mathbf{x}_u(u, v)|^2 + 2uv \langle \mathbf{x}_u(u, v), \mathbf{x}_v(u, v) \rangle + v^2 |\mathbf{x}_v(u, v)|^2} \\ &= \frac{1}{(r^2 \cos^2 \theta) |\mathbf{x}_u(*)|^2 + (2r^2 \cos \theta \sin \theta) \langle \mathbf{x}_u(*), \mathbf{x}_v(*) \rangle + (r^2 \sin^2 \theta) |\mathbf{x}_v(*)|^2} \\ &= \frac{1}{(\cos^2 \theta) |\mathbf{x}_u(*)|^2 + (2 \cos \theta \sin \theta) \langle \mathbf{x}_u(*), \mathbf{x}_v(*) \rangle + (\sin^2 \theta) |\mathbf{x}_v(*)|^2}, \end{aligned} \quad (135)$$

where $(*) = (r \cos \theta, r \sin \theta)$. As $(u, v) \rightarrow (0, 0)$, the denominator in (135) is like (the angle θ may not have a limit as $(u, v) \rightarrow (0, 0)$, it will move around in the interval $[0, 2\pi]$)

$$(\cos \theta, \sin \theta) \begin{pmatrix} |\mathbf{x}_u(0, 0)|^2 & \langle \mathbf{x}_u(0, 0), \mathbf{x}_v(0, 0) \rangle \\ \langle \mathbf{x}_u(0, 0), \mathbf{x}_v(0, 0) \rangle & |\mathbf{x}_v(0, 0)|^2 \end{pmatrix} \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}. \quad (136)$$

Note that the symmetric matrix

$$A := \begin{pmatrix} |\mathbf{x}_u(0,0)|^2 & \langle \mathbf{x}_u(0,0), \mathbf{x}_v(0,0) \rangle \\ \langle \mathbf{x}_u(0,0), \mathbf{x}_v(0,0) \rangle & |\mathbf{x}_v(0,0)|^2 \end{pmatrix}$$

is **positive-definite** and so as θ runs over $[0, 2\pi]$, the minimal quantity in (136) is equal to

$$\min_{v \in \mathbb{R}^2, |v|=1} \langle Av, v \rangle = \lambda_2 > 0, \quad (137)$$

where $\lambda_2 > 0$ is the **minimum eigenvalue** of A . Hence we have

$$(\cos \theta, \sin \theta) \begin{pmatrix} |\mathbf{x}_u(0,0)|^2 & \langle \mathbf{x}_u(0,0), \mathbf{x}_v(0,0) \rangle \\ \langle \mathbf{x}_u(0,0), \mathbf{x}_v(0,0) \rangle & |\mathbf{x}_v(0,0)|^2 \end{pmatrix} \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \geq \lambda_2 > 0 \quad (138)$$

for all $\theta \in [0, 2\pi]$. This will imply the assertion.

At a **parabolic** or **planar** point, there is no result similar to the above proposition. For simplicity, we only look at the case of a planar point. We can compare the following two examples.

Example 2.105 (*This is Example 6 in p. 145.*) (**Read this example by yourself.**) Consider the surface of revolution S obtained by rotating the curve $z = y^4$ about the z -axis. At the point $p = (0, 0, 0)$ we have $dN_p = 0$. This is because each **normal section** of S at p has **zero** curvature (since the curve $z = y^4$ has zero curvature at $(0, 0)$). Therefore, along any unit vector direction $v \in T_p S$, the normal curvature is $\langle -dN_p(v), v \rangle = 0$. By Lemma 2.17, we have $dN_p = 0$ and so p is a planar point. For this example, the whole surface S lies on one side of $T_p S$.

Example 2.106 (*This is Example 2 in p. 161.*) (**Read this example by yourself.**) Consider the **Monkey Saddle** parametrized by $\mathbf{x}(u, v) = (x(u, v), y(u, v), z(u, v))$, where

$$x(u, v) = u, \quad y(u, v) = v, \quad z(u, v) = u^3 - 3v^2u.$$

One can check that at the point $(0, 0, 0)$, the coefficients of the second fundamental form are $e = f = g = 0$ (due to $\mathbf{x}_{uu} = \mathbf{x}_{uv} = \mathbf{x}_{vv} = 0$ at p). Hence it is a planar point. However, in any neighborhood of this point, there are points on both sides of the tangent plane at $(0, 0, 0)$ (look at $\mathbf{x}(u, u) = (u, u, -2u^3)$, $u \in (-\infty, \infty)$). See picture in p. 162.

2.2.1 The Differential Equations for the Asymptotic Curves and the Lines of Curvature.

The equation for asymptotic curves. Let $\mathbf{x}(u, v)$ be a parametrization near $p = \mathbf{x}(0, 0) \in S$ and let $e = e(u, v)$, $f = f(u, v)$, $g = g(u, v)$ be the coefficients of the second fundamental form. Let $\alpha(t) = \mathbf{x}(u(t), v(t))$, $t \in I$, be a regular curve on S which is an asymptotic curve. Then we have

$$II_{\alpha(t)}(\alpha'(t)) = 0 \quad \text{for all } t \in I. \quad (139)$$

We recall that

$$\begin{aligned} II_{\alpha(t)}(\alpha'(t)) &= -\left\langle dN_{\alpha(t)}(\alpha'(t)), \alpha'(t) \right\rangle \\ &= -\langle N_u u' + N_v v', \mathbf{x}_u u' + \mathbf{x}_v v' \rangle = e(u')^2 + 2f u' v' + g(v')^2. \end{aligned} \quad (140)$$

Therefore, in local coordinates, the **differential equation for an asymptotic curve** $\mathbf{x}(u(t), v(t))$ is given by

$$e(u')^2 + 2f u' v' + g(v')^2 = 0, \quad t \in I. \quad (141)$$

Note that (141) is equivalent to (139).

Remark 2.107 *Be careful that the identity*

$$\left\langle dN_{\alpha(t)}(\alpha'(t)), \alpha'(t) \right\rangle = 0$$

does not imply that $dN_{\alpha(t)}(\alpha'(t)) = 0$. For example, we have the following

$$M = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}, \quad \left\langle \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \end{pmatrix} \right\rangle = 0.$$

Remark 2.108 (Important.) *The equation (141) is **independent of reparametrization** of $\alpha(t)$. Therefore, (144) is valid regardless of whether $\alpha(t)$ is parametrized by arc length s or not.*

In particular, if $p \in S$ is a **hyperbolic point** (i.e. $eg - f^2 < 0 < 0$ at p), then Lemma 2.64 says that there are **two** asymptotic directions at p . By continuity we know $eg - f^2 < 0$ in some neighborhood of p and all points on this neighborhood are hyperbolic and have two asymptotic directions. In such a case, we have the following interesting fact :

Lemma 2.109 *Assume all points on some neighborhood V around $p \in S$ are **hyperbolic point**. A necessary and sufficient condition for the **coordinate curves** ($u = u_0$, $v = v(t)$ or $u = u(t)$, $v = v_0$) near p to be **asymptotic curves** is $e = g = 0$ in that neighborhood.*

Proof. (\implies) If in some neighborhood V of p the two family of coordinate curves $u = u_0$, $v = v(t)$ and $u = u(t)$, $v = v_0$. are all asymptotic curves, then for the first case we have $u' \equiv 0$ and the differential equation (141) is satisfied. Therefore, we obtain

$$e(u')^2 + 2fu'v' + g(v')^2 = g(v')^2 = 0, \quad t \in I,$$

which implies $g = 0$ on V . Similarly for the second case we have $e = 0$ on V .

(\impliedby) If we have $e = g = 0$ on V , then equation (141) becomes $2fu'v' = 0$, where we know $f \neq 0$ everywhere on V (since all points of V are hyperbolic). We see that any coordinate curve in V can satisfy the equation $2fu'v' = 0$. The proof is done. \square

Remark 2.110 *The above lemma says that if we can find $\mathbf{x}(u, v)$ so that $e(u, v) = g(u, v) = 0$ in a neighborhood near p , then **all coordinate curves** of $\mathbf{x}(u, v)$ in that neighborhood are asymptotic curves.*

Remark 2.111 *Note that if $eg - f^2 > 0$ at p (elliptic point), there is **no asymptotic curve** near p .*

What happens for a **parabolic point** $p \in S$, i.e. $eg - f^2 = 0$ at p (but not $e = f = g = 0$ at p). Unlike elliptic or hyperbolic point, a parabolic point can be **isolated**. In such a case, there is not much to discuss at all. However, if we have $eg - f^2 \equiv 0$ on some neighborhood around p , we have:

Lemma 2.112 *Assume all points on some neighborhood V around $p \in S$ are **parabolic points** (for example, a cylinder). A necessary and sufficient condition for the **coordinate curves** $u = u_0$, $v = v(t)$, $t \in I$, lying inside V to be **asymptotic curves** is $e \neq 0$ (everywhere on V) and $g = f = 0$ (everywhere on V).*

Remark 2.113 *If the coordinate curves have the form $u = u(t)$, $v = v_0$, $t \in I$, then the condition becomes $g \neq 0$ (everywhere on V) and $e = f = 0$ (everywhere on V).*

Proof. By Lemma 2.64, we know there is **exactly one asymptotic direction** at each point of V . Assume the coordinate curve $u = u_0$, $v = v(t)$, $t \in I$, is an **asymptotic curve** lying inside V , then by (141) we have (note that $v'(t) \neq 0$ everywhere, for simplicity, we may assume $v(t) = t$, $t \in I$)

$$g(u_0, v(t))(v'(t))^2 = 0, \quad t \in I,$$

which gives $g(u_0, v(t)) \equiv 0$ for all $t \in I$ and so $g = 0$ everywhere on V . Since we also have $eg - f^2 = 0$ everywhere on V , we must have $f = 0$ everywhere on V . Moreover, since all points on V are parabolic points, we must have $e \neq 0$ everywhere on V .

Conversely, if we have $e \neq 0$, $g = f = 0$ everywhere on V , then the equation (141) becomes

$$e(u')^2 + 2fu'v' + g(v')^2 = e(u')^2 = 0. \quad (142)$$

Therefore, any coordinate curve of the form $u = u_0$, $v = v(t)$, $t \in I$, is an **asymptotic curve** as long as it lies inside V . \square

The equation for lines of curvature. We now turn to the differential equation for a **line of curvature**. For a line of curvature we have the condition

$$dN_{\alpha(t)}(\alpha'(t)) = \lambda(t)\alpha'(t), \quad \text{for all } t \in I, \quad \text{for some function } \lambda(t),$$

where $\alpha' = \mathbf{x}_u u' + \mathbf{x}_v v'$ and by

$$dN \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix}$$

we have

$$dN_{\alpha(t)}(\alpha'(t)) = (a_{11}u' + a_{12}v')\mathbf{x}_u + (a_{21}u' + a_{22}v')\mathbf{x}_v.$$

we conclude that $u'(t)$ and $v'(t)$ satisfy the 2×2 system of equations

$$\begin{cases} \frac{fF - eG}{EG - F^2}u' + \frac{gF - fG}{EG - F^2}v' = \lambda u' \dots (1) \\ \frac{eF - fE}{EG - F^2}u' + \frac{fF - gE}{EG - F^2}v' = \lambda v' \dots (2) \end{cases} \quad (143)$$

due to the identity

$$\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = \frac{1}{EG - F^2} \begin{pmatrix} e & f \\ f & g \end{pmatrix} \begin{pmatrix} -G & F \\ F & -E \end{pmatrix}.$$

The system (143) is **not self-contained (which means we cannot solve it directly)**, so we need to **eliminate** the function λ . We consider (1) $\cdot v' - (2) \cdot u'$ and get

$$(fE - eF)(u')^2 + (gE - eG)u'v' + (gF - fG)(v')^2 = 0, \quad (144)$$

which may also be written as as

$$\begin{vmatrix} (v')^2 & -u'v' & (u')^2 \\ E & F & G \\ e & f & g \end{vmatrix} = 0. \quad (145)$$

We call (144) the **differential equation for a line of curvature**. Note that in (144), the coefficients E , F , G , e , f , g are all functions of (u, v) .

Remark 2.114 (Important.) The equation (144) is **independent of reparametrization** of $\alpha(t)$. Therefore, (144) is valid regardless of whether $\alpha(t)$ is parametrized by arc length s or not.

We note the following:

Lemma 2.115 (*Read this lemma by yourself.*) *The 2×2 system of equations (143) and the single equation (144) are equivalent.*

Proof. If we have equation (144), it clearly imply the system (143).

Conversely, if (144) is satisfied, then we can infer

$$\left(\frac{fF - eG}{EG - F^2} u' + \frac{gF - fG}{EG - F^2} v' \right) v' = \left(\frac{eF - fE}{EG - F^2} u' + \frac{fF - gE}{EG - F^2} v' \right) u'$$

and if $u'(t) \neq 0$ and $v'(t) \neq 0$, we can write the above as

$$\frac{1}{u'} \left(\frac{fF - eG}{EG - F^2} u' + \frac{gF - fG}{EG - F^2} v' \right) = \frac{1}{v'} \left(\frac{eF - fE}{EG - F^2} u' + \frac{fF - gE}{EG - F^2} v' \right) := \lambda(t),$$

which gives the system (143) if we use the above function as $\lambda(t)$. If $u'(t) \neq 0$ and $v'(t) = 0$, then (144) implies $(fE - eF)(u(t), v(t)) = 0$ and we can deduce the system (143) again using

$$\lambda(t) = \frac{fF - eG}{EG - F^2} (u(t), v(t)).$$

Similarly, for $u'(t) = 0$ and $v'(t) \neq 0$, then (144) implies $(gF - fG)(u(t), v(t)) = 0$ and we can deduce the system (143) again using

$$\lambda(t) = \frac{fF - gE}{EG - F^2} (u(t), v(t)).$$

Thus we conclude that (143) is **equivalent to** (144). □

We conclude:

Lemma 2.116 *Assume all points on some neighborhood V around p are **nonumbilical** point. A necessary and sufficient condition for the **coordinate curves** ($u = u_0, v = v(t)$ or $u = u(t), v = v_0$) lying inside V to be **lines of curvature** is $F = f = 0$ on V .*

Remark 2.117 *In the above lemma, for the statement in the direction (\Leftarrow) we do not have to assume that $p \in S$ is a **nonumbilical** point.*

Proof. For (\Leftarrow), assume $\mathbf{x}(u, v)$ is a parametrization with the property $F = f \equiv 0$ on V . Then (144) becomes

$$(gE - eG) u' v' = 0. \tag{146}$$

Hence coordinate curves of $\mathbf{x}(u, v)$ are lines of curvature. Note that for this part we **do not** need p to be a **nonumbilical** point.

For (\Rightarrow), assume **coordinate curves of a parametrization $\mathbf{x}(u, v)$ are lines of curvature**, then take $u' = 0, v' = 1$ and $u' = 1, v' = 0$ respectively in (144) to get

$$gF - fG = 0 \quad \text{and} \quad fE - eF = 0, \quad \text{respectively on } V. \tag{147}$$

Since all points on V are **nonumbilical** and **coordinate curves are lines of curvature (the tangent vectors \mathbf{x}_u and \mathbf{x}_v of coordinate curves are pointing to principal directions and the principal directions are perpendicular)**, we must have $\mathbf{x}_u \perp \mathbf{x}_v$ everywhere and so $F \equiv 0$ on V . The above identity (147) becomes

$$fG = 0 \quad \text{and} \quad fE = 0. \tag{148}$$

Since $E > 0$ and $G > 0$ everywhere, we must have $f \equiv 0$ on V . The proof is done. □

2.2.2 Gauss and Mean Curvature for Surfaces of Revolution (Example 4 in p. 163).

Let C be a **regular connected curve** lying on xz -plane parametrized by **arc length parameter** $v \in (a, b)$ (here we use notation v instead of s):

$$(x, 0, z) = (\varphi(v), 0, \psi(v)), \quad v \in (a, b),$$

where $(\varphi'(v))^2 + (\psi'(v))^2 \equiv 1$ and $\varphi(v) > 0$ for all $v \in (a, b)$. Consider the surface of revolution S generated by C parametrized by

$$\mathbf{x}(u, v) = (\varphi(v) \cos u, \varphi(v) \sin u, \psi(v)), \quad 0 < u < 2\pi, \quad a < v < b,$$

where $\varphi(v) > 0$.

The coefficients E, F, G of the first fundamental form are

$$E = \varphi^2(v), \quad F = 0, \quad G = (\varphi'(v))^2 + (\psi'(v))^2 = 1, \quad \sqrt{EG - F^2} = \varphi(v).$$

We compute the coefficients e, f, g of the second fundamental form:

$$\begin{aligned} e &= \frac{1}{\sqrt{EG - F^2}} \det(\mathbf{x}_u, \mathbf{x}_v, \mathbf{x}_{uu}) \\ &= \frac{1}{\varphi(v)} \begin{vmatrix} -\varphi(v) \sin u & \varphi(v) \cos u & 0 \\ \varphi'(v) \cos u & \varphi'(v) \sin u & \psi'(v) \\ -\varphi(v) \cos u & -\varphi(v) \sin u & 0 \end{vmatrix} = -\varphi(v) \psi'(v). \end{aligned}$$

Similarly, we obtain

$$f = \frac{1}{\sqrt{EG - F^2}} \det(\mathbf{x}_u, \mathbf{x}_v, \mathbf{x}_{uv}) = 0$$

and

$$\begin{aligned} g &= \frac{1}{\sqrt{EG - F^2}} \det(\mathbf{x}_u, \mathbf{x}_v, \mathbf{x}_{vv}) \\ &= \frac{1}{\varphi(v)} \begin{vmatrix} -\varphi(v) \sin u & \varphi(v) \cos u & 0 \\ \varphi'(v) \cos u & \varphi'(v) \sin u & \psi'(v) \\ \varphi''(v) \cos u & \varphi''(v) \sin u & \psi''(v) \end{vmatrix} = \psi'(v) \varphi''(v) - \psi''(v) \varphi'(v). \end{aligned}$$

Since $F = f = 0$ (this fact remains true even if we do not use arc length parameter), we conclude that the **parallels** ($v = \text{const.}$) and the **meridians** ($u = \text{const.}$) are **lines of curvature** on S (because they are coordinate curves). See Remark 2.117 also.

The Gauss curvature is

$$\begin{aligned} K &= \frac{eg - f^2}{EG - F^2} = \frac{eg}{EG} \\ &= -\frac{\varphi(v) \psi'(v) \cdot [\psi'(v) \varphi''(v) - \psi''(v) \varphi'(v)]}{\varphi^2(v)} = -\frac{\varphi''(v)}{\varphi(v)}, \end{aligned} \tag{149}$$

where, for the last identity in (149), we have used the identity

$$\frac{d}{dv} (\varphi'(v))^2 + (\psi'(v))^2 = 2\varphi'(v) \varphi''(v) + 2\psi'(v) \psi''(v) \equiv 0$$

in the numerator. In particular, we conclude: $p \in S$ is a **parabolic** point if either one of the following occurs (but not both)

$$\begin{cases} \psi'(v) = 0 \text{ (same as } e = 0), \\ \psi'(v) \varphi''(v) - \psi''(v) \varphi'(v) = 0 \text{ (same as } g = 0 \text{ or curvature of } C \text{ is 0)}. \end{cases} \tag{150}$$

However, if both of the above identities holds, then $p \in S$ is a **planar** point.

To find the principal curvatures, by the **equations of Weingarten** (note that $F = f = 0$), we have

$$\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = - \begin{pmatrix} e & f \\ f & g \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} = - \begin{pmatrix} \frac{e}{E} & 0 \\ 0 & \frac{g}{G} \end{pmatrix}$$

and see that the two eigenvalues k_1, k_2 of $-dN$ are

$$k_1 = \frac{e}{E} = \frac{-\varphi(v)\psi'(v)}{\varphi^2(v)} = \frac{-\psi'(v)}{\varphi(v)} \quad (151)$$

and

$$k_2 = \frac{g}{G} = \psi'(v)\varphi''(v) - \psi''(v)\varphi'(v). \quad (152)$$

Thus

$$K = \frac{eg}{EG} = -\frac{\varphi''(v)}{\varphi(v)}$$

and

$$H = \frac{1}{2} \frac{eG + gE}{EG} = -\frac{1}{2} \frac{\psi'(v)}{\varphi(v)} + \frac{1}{2} [\psi'(v)\varphi''(v) - \psi''(v)\varphi'(v)]. \quad (153)$$

We note that k_1, k_2, K, H are all **independent** of u . **They depend only on v** . This is intuitively obvious.

To end this example, we state one more interesting result for general surfaces:

Lemma 2.118 *If $\mathbf{x}(u, v), (u, v) \in U \subset \mathbb{R}^2$, is a local parametrization of a regular surface $S \subset \mathbb{R}^3$ (not necessarily a surface of revolution) with the property*

$$F(u, v) = f(u, v) = 0, \quad \forall (u, v) \in U, \quad (154)$$

then we must have

$$k_1 = \frac{e}{E}, \quad k_2 = \frac{g}{G} \quad \text{on } U. \quad (155)$$

Proof. If $\mathbf{x}(u, v)$ satisfies (154), the **equations of Weingarten** becomes

$$\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = - \begin{pmatrix} e & f \\ f & g \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} = - \begin{pmatrix} \frac{e}{E} & 0 \\ 0 & \frac{g}{G} \end{pmatrix},$$

which means the $-dN$ has the two eigenvalues $\frac{e}{E}$ and $\frac{g}{G}$ and they are k_1 and k_2 respectively. The proof is done. \square

2.2.3 Gauss and Mean Curvature for Graphs (Example 5 in p. 165).

Assume that $S : z = h(x, y)$ is the graph of a differentiable function defined on some open set $U \subset \mathbb{R}^2$. Clearly we can parametrize S by

$$\mathbf{x}(x, y) = (x, y, h(x, y)), \quad (x, y) \in U$$

and get

$$\begin{cases} \mathbf{x}_x = (1, 0, h_x), & \mathbf{x}_y = (0, 1, h_y), \\ \mathbf{x}_{xx} = (0, 0, h_{xx}), & \mathbf{x}_{xy} = (0, 0, h_{xy}), & \mathbf{x}_{yy} = (0, 0, h_{yy}), \\ E(x, y) = 1 + h_x^2, & F(x, y) = h_x h_y, & G(x, y) = 1 + h_y^2, \\ EG - F^2 = 1 + h_x^2 + h_y^2, \end{cases}$$

and then

$$N(x, y) = \frac{(-h_x, -h_y, 1)}{(1 + h_x^2 + h_y^2)^{1/2}}$$

and

$$\begin{cases} e = \langle N, \mathbf{x}_{xx} \rangle = \frac{h_{xx}}{(1 + h_x^2 + h_y^2)^{1/2}}, \\ f = \langle N, \mathbf{x}_{xy} \rangle = \frac{h_{xy}}{(1 + h_x^2 + h_y^2)^{1/2}}, \\ g = \langle N, \mathbf{x}_{yy} \rangle = \frac{h_{yy}}{(1 + h_x^2 + h_y^2)^{1/2}}. \end{cases} \quad (156)$$

We conclude:

Lemma 2.119 *The Gauss curvature and mean curvature of a graphic surface $z = h(x, y)$, $(x, y) \in U$, are given by*

$$K(x, y) = \frac{eg - f^2}{EG - F^2} = \frac{h_{xx}h_{yy} - h_{xy}^2}{(1 + h_x^2 + h_y^2)^2}, \quad (x, y) \in U \quad (157)$$

and

$$\begin{aligned} H(x, y) &= \frac{1}{2} \frac{eG - 2fF + gE}{EG - F^2} \\ &= \frac{1}{2} \frac{(1 + h_y^2)h_{xx} - 2h_xh_y + (1 + h_x^2)h_{yy}}{(1 + h_x^2 + h_y^2)^{3/2}}, \quad (x, y) \in U. \end{aligned} \quad (158)$$

2.2.4 Gauss and Mean Curvature for Graphs with Special Coordinates (continue Example 5 in p. 165).

At a point $p \in S$, it is possible to choose a coordinate system so that T_pS is the xy -plane, N is pointing in the positive z direction, p is the origin $(0, 0, 0)$, and near p , the surface S is the graph of a function $z = h(x, y)$, $(x, y) \in U$, where h satisfies

$$h(0, 0) = 0, \quad h_x(0, 0) = 0, \quad h_y(0, 0) = 0. \quad (159)$$

One can see Exercise 26 in p. 93 for the above properties.

Now if we use the above-mentioned parametrization $\mathbf{x}(x, y) = (x, y, h(x, y))$, $(x, y) \in U$, we have

$$\begin{cases} E(0, 0) = \langle \mathbf{x}_x(0, 0), \mathbf{x}_x(0, 0) \rangle = 1, \\ F(0, 0) = \langle \mathbf{x}_x(0, 0), \mathbf{x}_y(0, 0) \rangle = 0, \\ G(0, 0) = \langle \mathbf{x}_y(0, 0), \mathbf{x}_y(0, 0) \rangle = 1 \end{cases} \quad (160)$$

and

$$e(0, 0) = h_{xx}(0, 0), \quad f(0, 0) = h_{xy}(0, 0), \quad g(0, 0) = h_{yy}(0, 0). \quad (161)$$

By the equations of Weingarten, we have

$$\begin{aligned} \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} &= - \begin{pmatrix} e & f \\ f & g \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \\ &= - \begin{pmatrix} e & f \\ f & g \end{pmatrix} = - \begin{pmatrix} h_{xx}(0, 0) & h_{xy}(0, 0) \\ h_{xy}(0, 0) & h_{yy}(0, 0) \end{pmatrix}. \end{aligned} \quad (162)$$

So the map $dN_p : T_p S \rightarrow T_p S$ with respect to the **orthonormal basis** $\{\mathbf{x}_x, \mathbf{x}_y\} = \{(1, 0), (0, 1)\}$ of the xy -plane is given by

$$-dN_p \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} h_{xx}(0, 0) & h_{xy}(0, 0) \\ h_{xy}(0, 0) & h_{yy}(0, 0) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad (163)$$

and the **second fundamental form** $II_p(v) = -\langle dN_p(v), v \rangle : T_p S \rightarrow \mathbb{R}$ is given by

$$II_p(v) = h_{xx}(0, 0)x^2 + 2h_{xy}(0, 0)xy + h_{yy}(0, 0)y^2, \quad v = (x, y). \quad (164)$$

The matrix in (163) is known as the **Hessian matrix** of h at $(0, 0)$. **Note that at this moment we may not have $h_{xy}(0, 0) = 0$. However, we can make $h_{xy}(0, 0) = 0$ by a rotation in the xy -plane.**

If $k_1 = k_2$ (denote the common number as k) at p , then we must have $-dN_p = kI$ and we have

$$h_{xx}(0, 0) = k, \quad h_{xy}(0, 0) = 0, \quad h_{yy}(0, 0) = k$$

for the above **orthonormal basis** $\{\mathbf{x}_x, \mathbf{x}_y\} = \{(1, 0), (0, 1)\}$.

If $k_1 \neq k_2$, the two **principal directions** e_1 and e_2 must be perpendicular to each other and we can **rotate** the above xy -plane so that the x and y axes are directed along the **principal directions** e_1 and e_2 . That is, after rotation, we have $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Since they are eigenvectors of $-dN_p : T_p S \rightarrow T_p S$, by (163), we have

$$-dN_p \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} h_{xx}(0, 0) & h_{xy}(0, 0) \\ h_{xy}(0, 0) & h_{yy}(0, 0) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} h_{xx}(0, 0) \\ h_{xy}(0, 0) \end{pmatrix} = k_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and

$$-dN_p \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} h_{xy}(0, 0) \\ h_{yy}(0, 0) \end{pmatrix} = k_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Hence we must have

$$h_{xx}(0, 0) = k_1, \quad h_{xy}(0, 0) = 0, \quad h_{yy}(0, 0) = k_2. \quad (165)$$

and by (161) we conclude

$$e(0, 0) = h_{xx}(0, 0) = k_1, \quad f(0, 0) = h_{xy}(0, 0) = 0, \quad g(0, 0) = h_{yy}(0, 0) = k_2. \quad (166)$$

The matrix for $-dN_p$ with respect to $\{\mathbf{x}_x, \mathbf{x}_y\} = \{(1, 0), (0, 1)\}$ is now diagonal, given by

$$\begin{pmatrix} h_{xx}(0, 0) & h_{xy}(0, 0) \\ h_{xy}(0, 0) & h_{yy}(0, 0) \end{pmatrix} = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}.$$

By Taylor series expansion, the function $z = h(x, y)$ near $(0, 0)$ has the form (note that we have (159))

$$\begin{aligned} h(x, y) &= \frac{1}{2} (h_{xx}(0, 0)x^2 + 2h_{xy}(0, 0)xy + h_{yy}(0, 0)y^2) + R(x, y) \\ &= \frac{1}{2} (k_1x^2 + k_2y^2) + R(x, y), \end{aligned} \quad (167)$$

where $R(x, y)$ satisfies

$$\lim_{(x, y) \rightarrow (0, 0)} \frac{R(x, y)}{x^2 + y^2} = 0.$$

2.2.5 Geometric Interpretation of the Gauss Curvature.

Let S and \tilde{S} be two orientable surfaces with orientation N (differentiable unit normal vector field on S) and \tilde{N} (differentiable unit normal vector field on \tilde{S}) respectively. Let $\varphi : S \rightarrow \tilde{S}$ be a differentiable map and assume at $p \in S$ the map $d\varphi_p : T_p S \rightarrow T_{\varphi(p)} \tilde{S}$ is nonsingular.

Definition 2.120 We say the map $\varphi : S \rightarrow \tilde{S}$ is **orientation-preserving** at p if, for any **positive** basis $\{v, w\}$ on $T_p S$ (which means $v \wedge w$ is pointing in the direction of $N(p)$, i.e. $\det(v, w, N(p)) > 0$), the basis $\{d\varphi_p(v), d\varphi_p(w)\}$ on $T_{\varphi(p)} \tilde{S}$ is also **positive** on $T_{\varphi(p)} \tilde{S}$ (which means $d\varphi_p(v) \wedge d\varphi_p(w)$ is pointing in the direction of $\tilde{N}(\varphi(p))$, i.e. $\det(d\varphi_p(v), d\varphi_p(w), \tilde{N}(\varphi(p))) > 0$). Otherwise, the map $\varphi : S \rightarrow \tilde{S}$ is called **orientation-reversing** at p , which means that $d\varphi_p : T_p S \rightarrow T_{\varphi(p)} \tilde{S}$ maps some positive basis on $T_p S$ into negative basis on $T_{\varphi(p)} \tilde{S}$ $\{v, w\}$.

Let S be an orientable surface with orientation $N : S \rightarrow S^2$ and dN_p is **nonsingular** at $p \in S$. By the **inverse function theorem**, there is a **small** neighborhood $V \subset S$ around $p \in S$ such that

$$N : V \subset S \rightarrow N(V) \text{ (denote it as } \tilde{V}) \subset S^2 \quad (168)$$

is a **diffeomorphism** and either $K > 0$ everywhere on V or $K < 0$ everywhere on V .

The Gauss map $N : V \rightarrow \tilde{V}$ will **induce an orientation** on $\tilde{V} \subset S^2$ (since we can **identify** $T_{N(p)} S^2$ as $T_p S$, "**induce**" means we choose the orientation on $T_{N(p)} S^2$ to be the **same** as the orientation on $T_p S$). By the identity

$$dN_q(v) \wedge dN_q(w) = K(q)(v \wedge w), \quad \forall q \in V, \quad v, w \in T_q S, \quad (169)$$

we see that if $K > 0$ everywhere on V , then both $v \wedge w$ and $dN_q(v) \wedge dN_q(w)$ are pointing to the same direction. That is, if $\{v, w\}$ is positive on $T_q S$ (the domain space of dN_q), then $\{dN_q(v), dN_q(w)\}$ is also positive on $T_q S$ (the target space of dN_q). By definition, the Gauss map $N : V \rightarrow \tilde{V}$ is **orientation-preserving** at any $q \in V$. On the other hand, if $K < 0$ everywhere on V , the Gauss map $N : V \rightarrow \tilde{V}$ is **orientation-reversing** at any $q \in V$. Therefore, we conclude:

Lemma 2.121 $N : V \rightarrow \tilde{V}$ is orientation-preserving (orientation-reversing) at all $q \in V$ if and only if $K > 0$ ($K < 0$) everywhere on V .

Remark 2.122 Let $\alpha(s) \in V$ (on V there is an orientation N), $s \in I$, be a small simple closed curve enclosing p in its interior and is **counterclockwise**, which means that when you walk along $\alpha(s) \in V$ in the length-increasing direction, the vector $\alpha'(s_0) \wedge \alpha'(s_0 + \varepsilon)$ ($\varepsilon > 0$ is small) is pointing to the direction of $N(\alpha(s_0))$ for all $s_0 \in I$. Now by (169) we have

$$\begin{aligned} & \left. \frac{d}{ds} \right|_{s=s_0} N(\alpha(s)) \wedge \left. \frac{d}{ds} \right|_{s=s_0+\varepsilon} N(\alpha(s)), \quad N(\alpha(s)) \in S^2 \\ & = [dN_{\alpha(s_0)}(\alpha'(s_0)) \wedge dN_{\alpha(s_0+\varepsilon)}(\alpha'(s_0+\varepsilon))] \approx K(\alpha(s_0)) [\alpha'(s_0) \wedge \alpha'(s_0+\varepsilon)]. \end{aligned} \quad (170)$$

If we have $K > 0$ on V , then the vector $dN_{\alpha(s_0)}(\alpha'(s_0)) \wedge dN_{\alpha(s_0+\varepsilon)}(\alpha'(s_0+\varepsilon))$ is also pointing to the direction of $N(\alpha(s_0))$ (note that $N(\alpha(s_0)) \in S^2$ and we choose the normal N on S^2 at $N(\alpha(s_0))$ as $N(\alpha(s_0))$, i.e. $N(N(\alpha(s_0))) = N(\alpha(s_0))$). Therefore, the curve $N(\alpha(s))$ on S^2 also has **counterclockwise** orientation as s is increasing. On the other hand, if $K < 0$ on V , the curve $N(\alpha(s))$ on S^2 has the property that $\left. \frac{d}{ds} \right|_{s=s_0} N(\alpha(s)) \wedge \left. \frac{d}{ds} \right|_{s=s_0+\varepsilon} N(\alpha(s))$ is pointing to the direction of $-N(\alpha(s_0))$. Therefore, the curve $N(\alpha(s))$ on S^2 has **clockwise** orientation as s is increasing.

We use the convention that if $K > 0$ in V , then the area of the set $N(V)$ in S^2 has a **positive sign**. While if $K < 0$ in V , then the area of the set $N(V)$ in S^2 has a **negative sign**. Under this convention, $N(V)$ has a **signed area**.

With the above convention, we can state the following:

Proposition 2.123 (*This is Proposition 2 in p. 169.*) (**Geometric meaning of the Gauss curvature.**) Let $p \in S$ (with Gauss map $N : S \rightarrow S^2$) such that $K(p) \neq 0$, and V be a connected neighborhood around $p \in S$ such that either $K > 0$ in V or $K < 0$ in V (here $N : V \rightarrow N(V)$ is a diffeomorphism). Then

$$K(p) = \lim_{A \rightarrow 0} \frac{A'_{\text{sign}}}{A}, \quad (171)$$

where A is the area of the region $B \subset V$ containing p , A'_{sign} is the **signed area** of $N(B)$ in S^2 , and the limit is taken through a sequence of regions B_n that converges to p in the sense that any sphere around p contains all B_n , for n sufficiently large.

Remark 2.124 Explain the meaning of B_n that converges to p as $n \rightarrow \infty$.

Proof. The area of $B \subset S$ is given by

$$A = \iint_R |\mathbf{x}_u \wedge \mathbf{x}_v| \, dudv,$$

where $\mathbf{x}(u, v) : R \subset U \subset \mathbb{R}^2 \rightarrow S$ is a parametrization near $p \in S$ with $\mathbf{x}(R) = B$. Also the area of the region $N(B)$ in S^2 is given by

$$A' = \iint_R |N_u \wedge N_v| \, dudv$$

since $N(u, v) = N(\mathbf{x}(u, v)) : R \subset U \subset \mathbb{R}^2 \rightarrow S^2$ is a parametrization near $N(p) \in S^2$ with $N(R) = N(B)$. By the identity (see Lemma 2.79)

$$|N_u \wedge N_v| = |dN(\mathbf{x}_u) \wedge dN(\mathbf{x}_v)| = |K| |\mathbf{x}_u \wedge \mathbf{x}_v|,$$

we have

$$A'_{\text{sign}} = \iint_R K |\mathbf{x}_u \wedge \mathbf{x}_v| \, dudv.$$

Hence

$$\lim_{A \rightarrow 0} \frac{A'_{\text{sign}}}{A} = \lim_{R \rightarrow 0} \frac{\frac{1}{R} \iint_R K |\mathbf{x}_u \wedge \mathbf{x}_v| \, dudv}{\frac{1}{R} \iint_R |\mathbf{x}_u \wedge \mathbf{x}_v| \, dudv} = \frac{K(p) |\mathbf{x}_u \wedge \mathbf{x}_v|(p)}{|\mathbf{x}_u \wedge \mathbf{x}_v|(p)} = K(p).$$

□

Remark 2.125 We can compare the above lemma with the curve case. Recall that for a regular parametrized curve $C \subset \mathbb{R}^2$ we have

$$\text{signed curvature} = k(p) = \frac{d\theta}{ds}(p) = \lim_{\Delta s \rightarrow 0} \frac{\Delta\theta}{\Delta s}, \quad \theta \text{ is the tangent angle,}$$

where $\Delta\theta$ is the **signed length** on the unit circle S^1 of the image of Δs under the map of the unit tangent vector T ($\Delta\theta > 0$ for counterclockwise orientation curve and $\Delta\theta < 0$ for clockwise orientation curve). But it is the same as the signed length on S^1 of the image of Δs under the map of the unit normal vector N .

2.2.6 Gauss and Mean Curvature for Level Surfaces.

Let $S \subset \mathbb{R}^3$ be a regular surface defined by the implicit form $g(x, y, z) = \text{const.}$, where $g : U \subset \mathbb{R}^3 \rightarrow \mathbb{R}$ is a differentiable function on some open set $U \subset \mathbb{R}^3$ with gradient $\nabla g \neq 0$ everywhere. The vector field $Z = \nabla g$ is **normal** to M (not necessarily unit normal) everywhere. Let V and W

be two tangent vector fields on S such that $V \wedge W = Z$ everywhere (or on some open set of S). By Lemma 10.9 we have

$$\begin{cases} K(V \wedge W) = dN(V) \wedge dN(W) \\ -2H(V \wedge W) = dN(V) \wedge W + V \wedge dN(W) \end{cases} \quad (172)$$

and so

$$K = \frac{(dN(V) \wedge dN(W)) \cdot (V \wedge W)}{\|V \wedge W\|^2} = \frac{(dN(V) \wedge dN(W)) \cdot Z}{\|Z\|^2}, \quad Z = V \wedge W \quad (173)$$

where at a fixed point $p = \alpha(0)$ of S we have

$$\begin{aligned} dN_p(V_p) &= \left. \frac{d}{dt} \right|_0 N(\alpha(t)) = \left. \frac{d}{dt} \right|_{t=0} \left(\frac{Z}{\|Z\|} \right) (\alpha(t)) \\ &= \frac{dZ(V)}{\|Z\|} \text{ (tangential part) } + \text{ (normal part) }, \quad \alpha'(0) = V_p. \end{aligned} \quad (174)$$

Here $dZ(V) = \left. \frac{d}{dt} \right|_0 (Z(\alpha(t)))$, $Z = \nabla g$ (the gradient of g on \mathbb{R}^3 , evaluated on S). Hence

$$\begin{aligned} &dN(V) \wedge dN(W) \\ &= \left\{ \frac{dZ(V)}{\|Z\|} + \text{(normal component)} \right\} \wedge \left\{ \frac{dZ(W)}{\|Z\|} + \text{(normal component)} \right\} \\ &= \left(\frac{dZ(V)}{\|Z\|} \wedge \frac{dZ(W)}{\|Z\|} \right) + \text{(tangential component)} + \text{(tangential component)} + \text{(zero)} \end{aligned} \quad (175)$$

and we conclude the formula for the Gauss curvature

$$K = \frac{(dN(V) \wedge dN(W)) \cdot Z}{\|Z\|^2} = \frac{(dZ(V) \wedge dZ(W)) \cdot Z}{\|Z\|^4}, \quad (176)$$

where we note that $D_V Z$ is tangent to the surface S . In a similar way, one can derive the following for the mean curvature

$$H = -\frac{(dZ(V) \wedge W + V \wedge dZ(W)) \cdot Z}{2\|Z\|^3}, \quad Z = V \wedge W. \quad (177)$$

Example 2.126 Consider the ellipsoid $S \subset \mathbb{R}^3$

$$S : g(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

and we choose $Z = \frac{1}{2} \nabla g = \left(\frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2} \right)$, which is **normal** to S everywhere. If $V = (v_1, v_2, v_3)$ is a vector field tangent to S everywhere, we would have

$$dZ(V) = \left(\frac{v_1}{a^2}, \frac{v_2}{b^2}, \frac{v_3}{c^2} \right)$$

and so for $V \wedge W = Z$, $W = (w_1, w_2, w_3)$, the following holds

$$Z \cdot (dZ(V) \wedge dZ(W)) = \begin{vmatrix} \frac{x}{a^2} & \frac{y}{b^2} & \frac{z}{c^2} \\ \frac{v_1}{a^2} & \frac{v_2}{b^2} & \frac{v_3}{c^2} \\ \frac{w_1}{a^2} & \frac{w_2}{b^2} & \frac{w_3}{c^2} \end{vmatrix} = \frac{1}{a^2 b^2 c^2} \begin{vmatrix} x & y & z \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \frac{1}{a^2 b^2 c^2}.$$

Hence

$$K = \frac{Z \cdot (dZ(V) \wedge dZ(W))}{\|Z\|^4} = \frac{1}{a^2 b^2 c^2} \cdot \frac{1}{\|Z\|^4} \quad (178)$$

where for the ellipsoid we have $(x, y, z) \cdot Z = 1$ and so

$$\begin{aligned} \frac{1}{\|Z\|} &= (x, y, z) \cdot \frac{Z}{\|Z\|} \text{ (this is the unit outward normal)} \\ &= \text{the support function of the ellipse.} \end{aligned} \quad (179)$$

For the mean curvature, we have

$$\begin{aligned} &(dZ(V) \wedge W + V \wedge dZ(W)) \cdot Z \\ &= \begin{vmatrix} \frac{x}{a^2} & \frac{y}{b^2} & \frac{z}{c^2} \\ \frac{v_1}{a^2} & \frac{v_2}{b^2} & \frac{v_3}{c^2} \\ w_1 & w_2 & w_3 \end{vmatrix} + \begin{vmatrix} \frac{x}{a^2} & \frac{y}{b^2} & \frac{z}{c^2} \\ v_1 & v_2 & v_3 \\ \frac{w_1}{a^2} & \frac{w_2}{b^2} & \frac{w_3}{c^2} \end{vmatrix} \\ &= \begin{cases} \frac{x}{a^2} \left(\frac{v_2}{b^2} w_3 - \frac{v_3}{c^2} w_2 + v_2 \frac{w_3}{c^2} - v_3 \frac{w_2}{b^2} \right) + \frac{y}{b^2} \left(\frac{v_3}{c^2} w_1 - \frac{v_1}{a^2} w_3 + v_3 \frac{w_1}{a^2} - v_1 \frac{w_3}{c^2} \right) \\ + \frac{z}{c^2} \left(\frac{v_1}{a^2} w_2 - \frac{v_2}{b^2} w_1 + v_1 \frac{w_2}{b^2} - v_2 \frac{w_1}{a^2} \right) \end{cases} \\ &= \frac{x}{a^2} \left(\frac{1}{b^2} + \frac{1}{c^2} \right) (v_2 w_3 - v_3 w_2) + \frac{y}{b^2} \left(\frac{1}{c^2} + \frac{1}{a^2} \right) (v_3 w_1 - v_1 w_3) + \frac{z}{c^2} \left(\frac{1}{a^2} + \frac{1}{b^2} \right) (v_1 w_2 - v_2 w_1). \end{aligned}$$

Also by $V \wedge W = Z$, we have

$$(v_1, v_2, v_3) \wedge (w_1, w_2, w_3) = \left(\frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2} \right)$$

i.e.,

$$v_2 w_3 - v_3 w_2 = \frac{x}{a^2}, \quad v_3 w_1 - v_1 w_3 = \frac{y}{b^2}, \quad v_1 w_2 - v_2 w_1 = \frac{z}{c^2}.$$

Hence the above becomes

$$\begin{aligned} &(dZ(V) \wedge W + V \wedge dZ(W)) \cdot Z \\ &= \frac{x^2}{a^2} \frac{1}{a^2} \left(\frac{1}{b^2} + \frac{1}{c^2} \right) + \frac{y^2}{b^2} \frac{1}{b^2} \left(\frac{1}{c^2} + \frac{1}{a^2} \right) + \frac{z^2}{c^2} \frac{1}{c^2} \left(\frac{1}{a^2} + \frac{1}{b^2} \right) \end{aligned}$$

and then

$$H = -\frac{1}{2\|Z\|^3} \left\{ \frac{x^2}{a^2} \frac{1}{a^2} \left(\frac{1}{b^2} + \frac{1}{c^2} \right) + \frac{y^2}{b^2} \frac{1}{b^2} \left(\frac{1}{c^2} + \frac{1}{a^2} \right) + \frac{z^2}{c^2} \frac{1}{c^2} \left(\frac{1}{a^2} + \frac{1}{b^2} \right) \right\}, \quad (180)$$

where

$$\frac{1}{\|Z\|} = \text{the support function of the ellipse.} \quad (181)$$

Example 2.127 Assume that $S : z = h(x, y)$ is the graph of a differentiable function defined on some open set $U \subset \mathbb{R}^2$. We can take tangent vector fields $V = (1, 0, h_x)$, $W = (0, 1, h_y)$ to get $V \wedge W = Z = (-h_x, -h_y, 1) = \nabla g$, $g(x, y, z) = z - h(x, y)$. By the above example we have

$$K = \frac{Z \cdot (dZ(V) \wedge dZ(W))}{\|Z\|^4}$$

and

$$dZ(V) = \frac{d}{dt} (Z(\alpha(t))) = (-h_{xx}, -h_{xy}, 0), \quad \alpha'(t) = V$$

and

$$dZ(W) = (-h_{xy}, -h_{yy}, 0)$$

and then

$$K = \frac{Z \cdot (dZ(V) \wedge dZ(W))}{\|Z\|^4} = \frac{h_{xx} h_{yy} - h_{xy}^2}{(1 + h_x^2 + h_y^2)^2}. \quad (182)$$

Also note that

$$\begin{aligned}
 & (dZ(V) \wedge W + V \wedge dZ(W)) \cdot Z \\
 &= \begin{vmatrix} -h_x & -h_y & 1 \\ -h_{xx} & -h_{xy} & 0 \\ 0 & 1 & h_y \end{vmatrix} + \begin{vmatrix} -h_x & -h_y & 1 \\ 1 & 0 & h_x \\ -h_{xy} & -h_{yy} & 0 \end{vmatrix} = -(1+h_x^2)h_{yy} + 2h_xh_yh_{xy} - (1+h_y^2)h_{xx}
 \end{aligned}$$

and then

$$H = -\frac{(dZ(V) \wedge W + V \wedge dZ(W)) \cdot Z}{2\|Z\|^3} = \frac{(1+h_x^2)h_{yy} - 2h_xh_yh_{xy} + (1+h_y^2)h_{xx}}{2(1+h_x^2+h_y^2)^{3/2}}. \quad (183)$$

We see that (182) and (183) are the same as (157) and (158).

This is the end of the course lecture notes, 2024-1-4.

The following is the supplementary material for your later reference.

3 Normal and Geodesic Curvature for a Curve in Surface.

Given¹ a smooth surface M in \mathbb{R}^3 with a choice of unit normal \mathbf{n} . We shall always give M the induced metric from \mathbb{R}^3 . Let $\gamma(s) : I = (a, b) \rightarrow M$ be a unit speed curve² whose image is contained in a coordinate patch \mathbf{x} with domain an open set in \mathbb{R}^2 . Let $\mathbf{T}(s) = \gamma'(s)$ and let

$$\mathbf{S} = \mathbf{n} \times \mathbf{T}. \quad (184)$$

We call it the *intrinsic normal*³ of γ . \mathbf{S} is tangent to the surface M and normal to the curve γ . We have

$$\gamma''(s) = \frac{d\mathbf{T}}{ds} = (\text{tangential part}) + (\text{normal part}), \quad (185)$$

where the tangential part is spanned by $\{\mathbf{S}, \mathbf{T}\}$ and since $\langle \gamma''(s), \mathbf{T}(s) \rangle = 0$, we obtain

$$\gamma''(s) = k_g(s) \mathbf{S}(s) + k_n(s) \mathbf{n}(s), \quad s \in I, \quad (186)$$

where $k_g(s)$ is called the *geodesic curvature* of γ ; and $k_n(s)$ is called the *normal curvature* of γ . **We will see that the normal curvature measures how M is curving in \mathbb{R}^3 ; and the geodesic curvature measures how γ is curving in M .**

Example 3.1 Let $\gamma(s)$ be a curve on the sphere S^2 . Then $k_n(s) = \pm 1$ is a constant. To see this, compute

$$k_n(s) = \langle \mathbf{T}'(s), \mathbf{n}(s) \rangle = -\langle \mathbf{T}(s), \mathbf{n}'(s) \rangle = \pm 1,$$

dep on the choice of the unit normal \mathbf{n} . Also, if $\gamma(s)$ is a curve on the plane $\mathbb{R}^2 \subset \mathbb{R}^3$, then $k_n(s) \equiv 0$ and $k_g(s) = k(s)$, where $k(s)$ is the signed plane curvature of γ .

There are other *extrinsic ways* to express $k_n(s)$ and $k_g(s)$ ($k_n(s)$ is an extrinsic quantity while $k_g(s)$ is an intrinsic quantity). Assume the Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ of γ exists. We have

$$\begin{aligned} k_g(s) &= \langle \gamma''(s), \mathbf{S}(s) \rangle = \langle \mathbf{T}'(s), \mathbf{n} \times \mathbf{T} \rangle = \langle k\mathbf{N}, \mathbf{n} \times \mathbf{T} \rangle \\ &= k \langle \mathbf{N}, \mathbf{n} \times \mathbf{T} \rangle = k \langle \mathbf{n}, \mathbf{T} \times \mathbf{N} \rangle = k(s) \langle \mathbf{n}(s), \mathbf{B}(s) \rangle, \end{aligned} \quad (187)$$

where $k(s) = |\gamma''(s)|$ is the curvature of γ in \mathbb{R}^3 . We also have

$$k_n(s) = \langle \gamma''(s), \mathbf{n}(s) \rangle = \langle \mathbf{T}'(s), \mathbf{n}(s) \rangle = k(s) \langle \mathbf{n}(s), \mathbf{N}(s) \rangle. \quad (188)$$

By (185), we have

$$\gamma''(s) = k_g(s) \mathbf{S}(s) + k_n(s) \mathbf{n}(s) \quad (189)$$

and

$$0 \leq k^2(s) = |\gamma''(s)|^2 = k_g^2(s) + k_n^2(s), \quad s \in I. \quad (190)$$

In general, $k_g(s)$ and $k_n(s)$ can be negative since change the orientation of \mathbf{n} will change the sign of $k_g(s)$ and $k_n(s)$ by (187) and (188).

¹Filename: geom-note1.tex. Revised on 2002-1-22.

This note is about *2nd fund form, Weingarten map, normal & geodesic curvatures* and related topics.

²We shall always assume γ is para by arclength s unless otherwise stated.

³Let X_N be the tangential projection of the normal vector N of γ (γ has T, N, B). Then

$$\mathbf{X}_N = \langle \mathbf{N}, \mathbf{T} \rangle \mathbf{T} + \langle \mathbf{N}, \mathbf{S} \rangle \mathbf{S} = \langle \mathbf{N}, \mathbf{S} \rangle \mathbf{S}.$$

Note that $S = n \times T$ is always defined even if N is undefined.

Definition 3.2 The coefficients of the 2nd fundamental form of M are the functions L_{ij} (defined on the domain U of the parametrization \mathbf{x}), where

$$L_{ij} = \langle \mathbf{x}_{ij}, \mathbf{n} \rangle, \quad 1 \leq i, j \leq 2. \quad (191)$$

The Christoffel symbols are the functions Γ_{ij}^k , $1 \leq i, j, k \leq 2$ defined on U by

$$\Gamma_{ij}^k = \langle \mathbf{x}_{ij}, \mathbf{x}_l \rangle g^{lk}, \quad (192)$$

where we have summed over the index l .

Since we have $\mathbf{x}_{ij} = \mathbf{x}_{ji}$, we know $L_{ij} = L_{ji}$ and $\Gamma_{ij}^k = \Gamma_{ji}^k$. Clearly, L_{ij} measures the normal comp of \mathbf{x}_{ij} ; and Γ_{ij}^k measures the tangential comp of \mathbf{x}_{ij} .

Lemma 3.3 (Gauss formula) We have

$$\mathbf{x}_{ij} = L_{ij} \mathbf{n} + \Gamma_{ij}^k \mathbf{x}_k, \quad (193)$$

and the normal curvature of the unit-speed curve $\gamma(s) = \mathbf{x}(\gamma^1(s), \gamma^2(s))$ can be expressed as

$$k_n(s) = L_{ij} (\gamma^i)' (\gamma^j)', \quad (194)$$

and since $k_g \mathbf{S}$ is the tangential part of γ'' , we have

$$k_g \mathbf{S} = \left[(\gamma^k)'' + \Gamma_{ij}^k (\gamma^i)' (\gamma^j)' \right] \mathbf{x}_k. \quad (195)$$

Proof. (193) is clear. We have

$$\gamma'(s) = \mathbf{x}_1 (\gamma^1)' + \mathbf{x}_2 (\gamma^2)', \quad \gamma''(s) = \mathbf{x}_i (\gamma^i)'' + \mathbf{x}_{ij} (\gamma^i)' (\gamma^j)'$$

and so

$$k_n(s) = L_{ij} (\gamma^i)' (\gamma^j)'.$$

Next we have

$$\begin{aligned} k_g \mathbf{S} &= \langle \gamma'', \mathbf{x}_l \rangle g^{lk} \mathbf{x}_k = \left\langle \mathbf{x}_i (\gamma^i)'' + \mathbf{x}_{ij} (\gamma^i)' (\gamma^j)', \mathbf{x}_l \right\rangle g^{lk} \mathbf{x}_k \\ &= (\gamma^i)'' \mathbf{x}_i + \Gamma_{ij}^k (\gamma^i)' (\gamma^j)' \mathbf{x}_k = \left[(\gamma^k)'' + \Gamma_{ij}^k (\gamma^i)' (\gamma^j)' \right] \mathbf{x}_k. \end{aligned}$$

The proof is done. ■

Lemma 3.4 We have

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}). \quad (196)$$

Clearly we have $\Gamma_{ij}^k = \Gamma_{ji}^k$. By (196) and (195), we know that *the geodesic curvature is intrinsic*. Moreover, $\gamma(s)$ is a geodesic iff the eq is satisfied

$$(\gamma^k)''(s) + \Gamma_{ij}^k (\gamma^i)'(s) (\gamma^j)'(s) = 0, \quad k = 1, 2, \quad (197)$$

for all $s \in I$.

4 The 2nd Fundamental Form and the Weingarten Map.

Definition 4.1 The 2nd fundamental form h of a smooth surface M in \mathbb{R}^3 (with a choice of unit normal \mathbf{n}) is given by

$$h(\mathbf{X}, \mathbf{Y}) = L_{ij}X^iY^j,$$

where $\mathbf{X} = X^i\mathbf{x}_i$, $\mathbf{Y} = Y^j\mathbf{x}_j$ are vector fields on M .

Easy to see that h is a symmetric (since $L_{ij} = L_{ji}$) bilinear map from the tangent sp of M to \mathbb{R} . If γ is a unit speed curve in M , then by (194),

$$h(\mathbf{T}, \mathbf{T}) = k_n = \text{the normal curvature of } \gamma, \quad \mathbf{T} = \gamma'(s). \quad (198)$$

In particular, we see that if two unit-speed curves in M which are tangent to each other at a pt, then they have the same normal curvature at that pt.

Definition 4.2 The Weingarten map \mathbf{L} is, for each $p \in M$, the function $\mathbf{L} : T_pM \rightarrow \mathbb{R}^3$ given by

$$\mathbf{L}(\mathbf{X}) = -\mathbf{X}\mathbf{n}. \quad (199)$$

(The minus sign is for future convenience.) That is, $\mathbf{L}(\mathbf{X})$ is the directional derivative of the normal \mathbf{n} in the direction \mathbf{X} .

Because \mathbf{n} is only defined up to sign, the same is true for \mathbf{L} . Note that

$$\langle \mathbf{L}(\mathbf{x}_i), \mathbf{x}_j \rangle = \langle -\mathbf{x}_i\mathbf{n}, \mathbf{x}_j \rangle = \langle \mathbf{n}, \mathbf{x}_{ij} \rangle = L_{ij}. \quad (200)$$

Lemma 4.3 (i) $\mathbf{L} : T_pM \rightarrow T_pM$ is a linear map.

(ii) If $\mathbf{L}(\mathbf{x}_k) = L_k^l\mathbf{x}_l$, then

$$L_k^l = L_{kj}g^{jl}, \quad (201)$$

or equivalently

$$L_k^l g_{lm} = L_{km}. \quad (202)$$

In matrix form, we have

$$\begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} = \begin{pmatrix} L_1^1 & L_1^2 \\ L_2^1 & L_2^2 \end{pmatrix} \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}, \quad (203)$$

which gives the equations of Weingarten:

$$L_1^1 = \frac{L_{11}g_{22} - L_{12}g_{21}}{g}, \quad L_1^2 = \frac{L_{12}g_{11} - L_{11}g_{12}}{g}, \quad (204)$$

$$L_2^1 = \frac{L_{21}g_{22} - L_{22}g_{21}}{g}, \quad L_2^2 = \frac{L_{22}g_{11} - L_{21}g_{12}}{g}, \quad (205)$$

where $g = \det(g_{ij})$.

Remark 4.4 Although we have (see below)

$$L_{12} = L_{21}, \quad g_{12} = g_{21},$$

however the matrix

$$\begin{pmatrix} L_1^1 & L_1^2 \\ L_2^1 & L_2^2 \end{pmatrix}^T = \begin{pmatrix} L_1^1 & L_2^1 \\ L_1^2 & L_2^2 \end{pmatrix},$$

which is the matrix rep of the Weingarten map \mathbf{L} , is NOT necessarily symmetric unless the coordinate frame is orthonormal! By the identity $\langle \mathbf{L}(\mathbf{x}_1), \mathbf{x}_2 \rangle = \langle \mathbf{L}(\mathbf{x}_2), \mathbf{x}_1 \rangle$, we get

$$L_1^1g_{12} + L_1^2g_{22} = L_2^1g_{11} + L_2^2g_{21},$$

and only if the coordinate frame is orthonormal ($g_{ij} = \delta_{ij}$) do we have

$$L_1^2 = L_2^1. \quad (206)$$

Proof. (i) is trivial. For (ii), note that

$$\langle \mathbf{L}(\mathbf{x}_k), \mathbf{x}_i \rangle = L_k^j g_{ji} = \langle -\mathbf{x}_k \mathbf{n}, \mathbf{x}_i \rangle = \langle \mathbf{n}, \mathbf{x}_{ki} \rangle = L_{ki},$$

which is equi to (201). \blacksquare

In conclusion, we have the following important formulas

$$\begin{cases} \mathbf{x}_{ij} = L_{ij} \mathbf{n} + \Gamma_{ij}^k \mathbf{x}_k, \\ \mathbf{n}_j = -\mathbf{L}(\mathbf{x}_j) = -L_j^k \mathbf{x}_k. \end{cases} \quad (207)$$

The following says that the map \mathbf{L} is a *symmetric (self-adj)* linear map from the tangent space to itself. Again note that its matrix rep is NOT necesse symmetric unless the coordinate frames is orthonormal!

Lemma 4.5

$$\langle \mathbf{L}(\mathbf{X}), \mathbf{Y} \rangle = \langle \mathbf{L}(\mathbf{Y}), \mathbf{X} \rangle = h(\mathbf{X}, \mathbf{Y}). \quad (208)$$

Proof. By linearity, it suffices to prove the formula on coordinate frames $\{\mathbf{x}_i\}$. We have

$$\langle \mathbf{L}(\mathbf{x}_i), \mathbf{x}_j \rangle = L_{ij} = h(\mathbf{x}_i, \mathbf{x}_j) = L_{ji} = \langle \mathbf{L}(\mathbf{x}_j), \mathbf{x}_i \rangle.$$

The proof is done. \blacksquare

Definition 4.6 *The Gauss curvature K of M is defined as the determinant of the linear map \mathbf{L} , i.e.,*

$$K = \det(L_i^j) = \frac{L_{11}L_{22} - L_{12}L_{21}}{g}. \quad (209)$$

The mean curvature H of M is def as the trace of \mathbf{L} , which is

$$H = L_1^1 + L_2^2 = \frac{L_{11}g_{22} - L_{12}g_{21}}{g} + \frac{L_{22}g_{11} - L_{21}g_{12}}{g} = \frac{L_{11}g_{22} - 2L_{12}g_{12} + L_{22}g_{11}}{g}.$$

Lemma 4.7 *We have*

$$\mathbf{n}_1 \times \mathbf{n}_2 = K \sqrt{g} \mathbf{n}, \quad (210)$$

where $g = \det(g_{ij})$ and \mathbf{n}_1 is the directional derivative of \mathbf{n} along the \mathbf{x}_1 direction. Same for \mathbf{n}_2 .

Proof.

$$\begin{aligned} \mathbf{n}_1 \times \mathbf{n}_2 &= \mathbf{L}(\mathbf{x}_1) \times \mathbf{L}(\mathbf{x}_2) = (L_1^l \mathbf{x}_l) \times (L_2^l \mathbf{x}_l) \\ &= (L_1^1 \mathbf{x}_1 + L_1^2 \mathbf{x}_2) \times (L_2^1 \mathbf{x}_1 + L_2^2 \mathbf{x}_2) = (L_1^1 L_2^2 - L_1^2 L_2^1) \mathbf{x}_1 \times \mathbf{x}_2 \\ &= \det(\mathbf{L}) \sqrt{g} \mathbf{n} = K \sqrt{g} \mathbf{n}, \end{aligned}$$

since $|\mathbf{x}_1 \times \mathbf{x}_2| = \sqrt{g}$ and $\mathbf{n} = \frac{\mathbf{x}_1 \times \mathbf{x}_2}{|\mathbf{x}_1 \times \mathbf{x}_2|}$. \blacksquare

4.1 Darboux Trihedron and Related Formulas for curves γ in M .

For a unit-speed curve $\gamma(s) = (\gamma^1(s), \gamma^2(s))$ in a surface M , we have formulas similar to the Frenet frame. Note that $\{\mathbf{T}, \mathbf{S}, \mathbf{n}\}$ gives a right-handed orthonormal basis of \mathbb{R}^3 at each pt of γ . We have

Lemma 4.8 *We have*

$$\begin{cases} \mathbf{T}'(s) = k_g \mathbf{S} + h(\mathbf{T}, \mathbf{T}) \mathbf{n}, \\ \mathbf{S}'(s) = -k_g \mathbf{T} + h(\mathbf{T}, \mathbf{S}) \mathbf{n}, \\ \mathbf{n}'(s) = -h(\mathbf{T}, \mathbf{T}) \mathbf{T} - h(\mathbf{T}, \mathbf{S}) \mathbf{S}, \end{cases} \quad (211)$$

which is same as (note that $h(\mathbf{T}, \mathbf{T})$ is the normal curvature k_n of γ)

$$\begin{pmatrix} \mathbf{T}'(s) \\ \mathbf{S}'(s) \\ \mathbf{n}'(s) \end{pmatrix} = \begin{pmatrix} 0 & k_g & k_n \\ -k_g & 0 & h(\mathbf{T}, \mathbf{S}) \\ -k_n & -h(\mathbf{T}, \mathbf{S}) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{S} \\ \mathbf{n} \end{pmatrix}, \quad (212)$$

where the coefficients matrix is skew-symmetric.

Proof. By (198), we have

$$\mathbf{T}'(s) = \gamma''(s) = k_g(s) \mathbf{S}(s) + k_n(s) \mathbf{n}(s) = k_g \mathbf{S} + h(\mathbf{T}, \mathbf{T}) \mathbf{n}.$$

As for the 2nd formula, we note

$$\begin{aligned} \mathbf{S}'(s) &= \mathbf{n}'(s) \times \mathbf{T} + \mathbf{n}(s) \times \mathbf{T}'(s) = -\mathbf{L}(\mathbf{T}) \times \mathbf{T} + \mathbf{n}(s) \times (k_g \mathbf{S} + k_n \mathbf{n}) \\ &= (\mathbf{S} \times \mathbf{n}) \times \mathbf{L}(\mathbf{T}) - k_g \mathbf{T}. \end{aligned}$$

Using the formula

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \langle \mathbf{a}, \mathbf{c} \rangle \mathbf{b} - \langle \mathbf{b}, \mathbf{c} \rangle \mathbf{a},$$

we get

$$\begin{aligned} \mathbf{S}'(s) &= (\mathbf{S} \times \mathbf{n}) \times \mathbf{L}(\mathbf{T}) - k_g \mathbf{T} = \langle \mathbf{S}, \mathbf{L}(\mathbf{T}) \rangle \mathbf{n} - \langle \mathbf{n}, \mathbf{L}(\mathbf{T}) \rangle \mathbf{S} - k_g \mathbf{T} \\ &= h(\mathbf{T}, \mathbf{S}) \mathbf{n} - k_g \mathbf{T}, \end{aligned}$$

since $\langle \mathbf{n}, \mathbf{L}(\mathbf{T}) \rangle = 0$. The last formula is a consequence of the fact that the coefficient matrix must be skew-symmetric. The proof is done. \blacksquare

Definition 4.9 *The number*

$$\tau_g := h(\mathbf{T}, \mathbf{S}) = -\langle \mathbf{n}'(s), \mathbf{S} \rangle = \langle \mathbf{S}'(s), \mathbf{n} \rangle \quad (213)$$

is called the geodesic torsion of γ in M .

We have

Lemma 4.10 (i) $\tau_g \equiv 0$ iff γ is a line of curvature. (ii) If γ is a geodesic and its normal curvature $k_n > 0$ everywhere, then $\tau = \tau_g$, where τ is the torsion of γ in \mathbb{R}^3 .

Proof. Assume γ is a line of curvature, then \mathbf{T} is in the principal direction and so $\mathbf{L}(\mathbf{T}) = -\mathbf{T}\mathbf{n} = -\mathbf{n}'(s)$ is a multiple of \mathbf{T} , hence we know

$$\tau_g = h(\mathbf{T}, \mathbf{S}) = \langle \mathbf{L}(\mathbf{T}), \mathbf{S} \rangle \equiv 0.$$

Conversely, if $\tau_g \equiv 0$, then

$$\langle \mathbf{L}(\mathbf{T}), \mathbf{S} \rangle = \langle \mathbf{L}(\mathbf{T}), \mathbf{n} \times \mathbf{T} \rangle \equiv 0,$$

since $\mathbf{L}(\mathbf{T})$ is a multiple of \mathbf{T} . (i) is done. For (ii), if γ is a geodesic, then its geodesic curvature $k_g \equiv 0$ and so (212) becomes

$$\begin{pmatrix} \mathbf{T}'(s) \\ \mathbf{S}'(s) \\ \mathbf{n}'(s) \end{pmatrix} = \begin{pmatrix} 0 & 0 & k_n \\ \mathbf{0} & 0 & \tau_g \\ -k_n & -\tau_g & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{S} \\ \mathbf{n} \end{pmatrix}, \quad k_n > 0.$$

Therefore the frame $\{\mathbf{T}, \mathbf{n}, \mathbf{B}\}$, $\mathbf{B} = -\mathbf{S}$, is the Frenet frame of γ and so $\mathbf{B}'(s) = -\mathbf{S}'(s) = -\tau_g \mathbf{n}$. On the other hand, for a Frenet frame we have $\mathbf{B}' = -\tau \mathbf{n}$ and so $\tau_g = \tau$. (ii) is proved. \blacksquare

Note that for a plane curve γ , we have $\tau_g = \tau \equiv 0$, but γ may not be a geodesic.

The following three lemmas are also important:

Lemma 4.11 *Let $\gamma(s)$ be a unit-speed curve on M and let \mathbf{S} be its intrinsic normal. Then \mathbf{S} is parallel along γ iff γ is a geodesic.*

Proof. We know that γ is a geodesic iff its geodesic curvature $k_g \equiv 0$. We also have

$$\mathbf{S}'(s) = -k_g \mathbf{T} + h(\mathbf{T}, \mathbf{S}) \mathbf{n}$$

and so γ is a geodesic iff $\mathbf{S}'(s) = h(\mathbf{T}, \mathbf{S}) \mathbf{n}$, which means that $\mathbf{S}(s)$ is parallel along $\gamma(s)$. The proof is done. \blacksquare

Lemma 4.12 *Let $\gamma(s)$ be a unit-speed curve on M and let \mathbf{X}_N be the tangential component of its normal vector \mathbf{N} (assume it is defined, i.e., $k > 0$). Then the following are equi: (i) $\mathbf{X}_N \equiv \mathbf{0}$, (ii) γ is a geodesic, (iii) \mathbf{X}_N is parallel along γ .*

Proof. We have

$$\mathbf{X}_N = \langle \mathbf{N}, \mathbf{S} \rangle \mathbf{S} = \mathbf{N} - \langle \mathbf{N}, \mathbf{n} \rangle \mathbf{n}.$$

If $\mathbf{X}_N \equiv \mathbf{0}$, then $\langle \mathbf{N}, \mathbf{S} \rangle \equiv 0$ and hence

$$\langle \mathbf{T}'(s), \mathbf{S}(s) \rangle = \langle k \mathbf{N}, \mathbf{S} \rangle = 0,$$

for all s , which means that $\gamma''(s)$ has no tangential component; so it is a geodesic. The converse is also clear and (i) is equi to (ii). For (iii), if γ is a geodesic, then $\mathbf{X}_N \equiv \mathbf{0}$ and so \mathbf{X}_N is parallel along γ . Conversely, \mathbf{X}_N is parallel along γ , then $\frac{d}{ds} \mathbf{X}_N(s)$ has no tangential component, which implies that

$$\begin{aligned} \frac{d}{ds} \mathbf{X}_N(s) &= \frac{d}{ds} [\langle \mathbf{N}(s), \mathbf{S}(s) \rangle \cdot \mathbf{S}(s)] = \langle \mathbf{N}, \mathbf{S}'(s) \rangle \cdot \mathbf{S}(s) + \langle \mathbf{N}, \mathbf{S} \rangle \cdot \mathbf{S}'(s) \\ &= \langle \mathbf{N}, \mathbf{S}'(s) \rangle \cdot \mathbf{S}(s) + \langle \mathbf{N}, \mathbf{S} \rangle \cdot [-k_g \mathbf{T} + h(\mathbf{T}, \mathbf{S}) \mathbf{n}] \end{aligned}$$

has no tangential component. Since \mathbf{T} and \mathbf{S} are indep, this can happen only when

$$\langle \mathbf{N}, \mathbf{S} \rangle(s) k_g(s) \equiv 0 \quad \text{and} \quad \langle \mathbf{N}, \mathbf{S}'(s) \rangle \equiv 0.$$

We either have $k_g(s) \equiv 0$ or $\langle \mathbf{N}, \mathbf{S} \rangle(s) \equiv 0$. The pf is done. \blacksquare

Lemma 4.13 *Let $\gamma(s)$ be a unit-speed curve on M and let \mathbf{X}_B be the tangential component of its binormal vector \mathbf{B} (assume it is defined, i.e., $k > 0$). Then*

$$\mathbf{X}_B(s) = -\frac{k_n(s)}{k(s)} \mathbf{S}(s) \tag{214}$$

and the following are equi: (i) $\mathbf{X}_B = \mathbf{B}$, (ii) γ is a geodesic, (iii) \mathbf{X}_B is parallel along γ and $\mathbf{X}_B(s)$ is not identically equal to zero.

Proof. We have

$$\mathbf{X}_B = \langle \mathbf{B}, \mathbf{T} \rangle \mathbf{T} + \langle \mathbf{B}, \mathbf{S} \rangle \mathbf{S} = \langle \mathbf{B}, \mathbf{S} \rangle \mathbf{S}.$$

So we need to evaluate $\langle \mathbf{B}, \mathbf{S} \rangle$. We have $\langle \mathbf{B}, \mathbf{S} \rangle = \langle \mathbf{T} \times \mathbf{N}, \mathbf{S} \rangle$ and by

$$\mathbf{N} = \frac{k_g \mathbf{S} + k_n \mathbf{n}}{k},$$

we know

$$\langle \mathbf{B}, \mathbf{S} \rangle = \left\langle \mathbf{T} \times \frac{k_n \mathbf{n}}{k}, \mathbf{S} \right\rangle = -\frac{k_n}{k}.$$

(214) is proved. If $\mathbf{X}_B = \mathbf{B}$, then \mathbf{B} has no normal component, i.e., $\langle \mathbf{B}, \mathbf{n} \rangle = 0$ and

$$\mathbf{B} = \langle \mathbf{B}, \mathbf{S} \rangle \mathbf{S}, \quad \langle \mathbf{B}, \mathbf{S} \rangle = \pm 1$$

which means $\mathbf{N} = \frac{\gamma''}{k} = \pm \mathbf{n}$ and so γ is a geodesic. Conversely, if γ is a geodesic, then $\mathbf{N} = \pm \mathbf{n}$ and hence $\langle \mathbf{B}, \mathbf{n} \rangle = \pm \langle \mathbf{B}, \mathbf{N} \rangle = 0$ so $\mathbf{X}_B = \mathbf{B}$. Finally, if γ is a geodesic, then

$$\mathbf{X}_B = \mathbf{B} = -\frac{k_n}{k} \mathbf{S}(s), \quad \mathbf{B} = \pm \mathbf{S}.$$

By Lemma 4.11, we know $\mathbf{X}_B = \mathbf{B}$ is parallel along γ . Conversely, if $\mathbf{X}_B = -\frac{k_n(s)}{k(s)} \mathbf{S}(s)$ is parallel along γ , then

$$\begin{aligned} \mathbf{X}'_B(s) &= -\frac{k_n(s)}{k(s)} \mathbf{S}'(s) - \left(\frac{k_n(s)}{k(s)} \right)' (s) \cdot \mathbf{S}(s) \\ &= -\frac{k_n(s)}{k(s)} [-k_g \mathbf{T} + h(\mathbf{T}, \mathbf{S}) \mathbf{n}] - \left(\frac{k_n(s)}{k(s)} \right)' (s) \cdot \mathbf{S}(s) \end{aligned}$$

has no tangential component, i.e.,

$$\frac{k_n(s)}{k(s)} k_g(s) \equiv 0 \quad \text{and} \quad \left(\frac{k_n(s)}{k(s)} \right)' (s) \equiv 0.$$

We either have $k_g(s) \equiv 0$ or $\frac{k_n(s)}{k(s)} \equiv 0$. In the first case, γ is a geodesic and in the second case, we have $k_n(s) \equiv 0$, which will imply $\mathbf{X}_B(s) = 0$ for all s , but this is impossible by our assumption.

The proof is done. \square

5 Riemann Curvature Tensor; Gauss and Codazzi-Mainardi Equations.

Definition 5.1 We define the Riemann Curvature Tensor as (different from Ben's notation)

$$R^l_{ijk} = \partial_j \Gamma^l_{ik} - \partial_k \Gamma^l_{ij} + \Gamma^p_{ik} \Gamma^l_{pj} - \Gamma^p_{ij} \Gamma^l_{pk} \quad (215)$$

where $1 \leq i, j, k, l, p \leq 2$.

From the def, we know that the Riemann Curvature Tensor is an *intrinsic* object. The following is an *extrinsic* way to express it.

Proposition 5.2 (i) (*Gauss equation*) We have

$$R^l_{ijk} = L_{ik} L^l_j - L_{ij} L^l_k \quad (216)$$

or equivalently

$$R_{imjk} = L_{ik}L_{mj} - L_{ij}L_{mk} \quad (217)$$

where

$$R_{imjk} := R_{ijk}^l g_{lm}. \quad (218)$$

(ii) (*Codazzi-Mainardi equation*)

$$\partial_k L_{ij} - \partial_j L_{ik} = \Gamma_{ik}^l L_{lj} - \Gamma_{ij}^l L_{lk}. \quad (219)$$

Remark 5.3 (important) Note the difference: in the Gauss equation, we differentiate the Christoffel symbol Γ_{ij}^k ; while in the Codazzi-Mainardi equation, we differentiate the second fundamental form L_{ij} .

Remark 5.4 (important) The Gauss equation says that the expression $L_{ik}L_{mj} - L_{ij}L_{mk}$ is *intrinsic* !

Proof. The key point of the proof is to use the identities

$$\mathbf{x}_{ijk} = \mathbf{x}_{ikj} \quad \text{and} \quad \mathbf{x}_{ij} = L_{ij}\mathbf{n} + \Gamma_{ij}^k \mathbf{x}_k \quad (220)$$

and compare the tangential and normal components of the id $\mathbf{x}_{ijk} = \mathbf{x}_{ikj}$. First we have

$$\begin{aligned} \mathbf{x}_{ijk} &= \partial_k (\mathbf{x}_{ij}) = \partial_k (L_{ij}\mathbf{n} + \Gamma_{ij}^l \mathbf{x}_l) \\ &= (\partial_k L_{ij})\mathbf{n} + L_{ij}(\partial_k \mathbf{n}) + (\partial_k \Gamma_{ij}^l) \mathbf{x}_l + \Gamma_{ij}^l (\partial_k \mathbf{x}_l) \\ &= (\partial_k L_{ij})\mathbf{n} + L_{ij}(-L_k^l \mathbf{x}_l) + (\partial_k \Gamma_{ij}^l) \mathbf{x}_l + \Gamma_{ij}^l (L_{kl}\mathbf{n} + \Gamma_{kl}^m \mathbf{x}_m) \\ &= (\partial_k L_{ij} + \Gamma_{ij}^l L_{kl})\mathbf{n} + (\partial_k \Gamma_{ij}^l - L_{ij}L_k^l + \Gamma_{ij}^p \Gamma_{pk}^l) \mathbf{x}_l, \end{aligned}$$

and similarly

$$\mathbf{x}_{ikj} = (\partial_j L_{ik} + \Gamma_{ik}^l L_{jl})\mathbf{n} + (\partial_j \Gamma_{ik}^l - L_{ik}L_j^l + \Gamma_{ik}^p \Gamma_{pj}^l) \mathbf{x}_l.$$

Comparing the coefficients of \mathbf{n} gives

$$\partial_k L_{ij} + \Gamma_{ij}^l L_{kl} = \partial_j L_{ik} + \Gamma_{ik}^l L_{jl} \quad (221)$$

which is same as (219). Comparing the coefficients of \mathbf{x}_l gives

$$\partial_k \Gamma_{ij}^l - L_{ij}L_k^l + \Gamma_{ij}^p \Gamma_{pk}^l = \partial_j \Gamma_{ik}^l - L_{ik}L_j^l + \Gamma_{ik}^p \Gamma_{pj}^l. \quad (222)$$

Rewrite (222) as

$$\partial_j \Gamma_{ik}^l - \partial_k \Gamma_{ij}^l + \Gamma_{ik}^p \Gamma_{pj}^l - \Gamma_{ij}^p \Gamma_{pk}^l = L_{ik}L_j^l - L_{ij}L_k^l \quad (223)$$

and by definition, the LHS is R_{ijk}^l , we obtain

$$R_{ijk}^l = L_{ik}L_j^l - L_{ij}L_k^l. \quad (224)$$

The proof is done. \square

Remark 5.5 The Codazzi-Mainardi equation will imply the following more familiar form in differential geometry book:

$$\partial_k L_{ij} - \Gamma_{ki}^s h_{sj} - \Gamma_{kj}^s h_{si} = \boxed{\nabla_k L_{ij} = \nabla_j L_{ik}} = \partial_j L_{ik} - \Gamma_{ji}^s h_{sk} - \Gamma_{jk}^s h_{si}. \quad (225)$$

To see this, cancel the term $\Gamma_{kj}^s h_{si}$ in both sides to get

$$LHS = \partial_k L_{ij} - \Gamma_{ki}^s h_{sj}, \quad RHS = \partial_j L_{ik} - \Gamma_{ji}^s h_{sk}.$$

We see that $LHS = RHS$ by (219).

Note that if $j = k$, then $R_{ijj}^l = 0$; and also one can check that $R_{ikj}^l = -R_{ijk}^l$. Therefore, there are essentially 4 terms to evaluate: R_{112}^1 , R_{112}^2 , R_{212}^1 , R_{212}^2 . By (224), (204), (205) and $L_{ij} = L_{ji}$, we obtain

$$\begin{aligned} R_{112}^1 &= L_{12}L_1^1 - L_{11}L_2^1 = L_{12} \left(\frac{L_{11}g_{22} - L_{12}g_{21}}{g} \right) - L_{11} \left(\frac{L_{21}g_{22} - L_{22}g_{21}}{g} \right) \\ &= \left(\frac{L_{11}L_{22} - L_{12}L_{12}}{g} \right) g_{21} = Kg_{21}. \end{aligned}$$

Similarly

$$\begin{aligned} R_{112}^2 &= L_{12}L_1^2 - L_{11}L_2^2 = L_{12} \left(\frac{L_{12}g_{11} - L_{11}g_{12}}{g} \right) - L_{11} \left(\frac{L_{22}g_{11} - L_{21}g_{12}}{g} \right) \\ &= - \left(\frac{L_{11}L_{22} - L_{12}L_{12}}{g} \right) g_{11} = -Kg_{11}. \end{aligned}$$

Also

$$\begin{aligned} R_{121}^1 &= L_{11}L_2^1 - L_{12}L_1^1 = L_{11} \left(\frac{L_{21}g_{22} - L_{22}g_{21}}{g} \right) - L_{12} \left(\frac{L_{11}g_{22} - L_{12}g_{21}}{g} \right) \\ &= - \left(\frac{L_{11}L_{22} - L_{12}L_{12}}{g} \right) g_{21} = -Kg_{21} = -R_{112}^1, \end{aligned}$$

and similarly,

$$R_{121}^2 = Kg_{11} = -R_{112}^2.$$

Finally

$$\begin{aligned} R_{212}^1 &= L_{22}L_1^1 - L_{21}L_2^1 = L_{22} \left(\frac{L_{11}g_{22} - L_{12}g_{21}}{g} \right) - L_{21} \left(\frac{L_{21}g_{22} - L_{22}g_{21}}{g} \right) \\ &= Kg_{22} \end{aligned}$$

and

$$\begin{aligned} R_{212}^2 &= L_{22}L_1^2 - L_{21}L_2^2 = L_{22} \left(\frac{L_{12}g_{11} - L_{11}g_{12}}{g} \right) - L_{21} \left(\frac{L_{22}g_{11} - L_{21}g_{12}}{g} \right) \\ &= -Kg_{12}. \end{aligned}$$

We summarize as

$$R_{112}^1 = Kg_{21}, \quad R_{112}^2 = -Kg_{11}, \quad R_{212}^1 = Kg_{22}, \quad R_{212}^2 = -Kg_{12}. \quad (226)$$

We have

$$R_{121}^1 g_{12} + R_{121}^2 g_{22} = (-Kg_{21}) g_{12} + (Kg_{11}) g_{22} = Kg$$

and so we obtain the expression for Gauss curvature

$$K = \frac{R_{121}^m g_{m2}}{g} = \frac{R_{1221}}{g}.$$

Theorem 5.6 (*Gauss theorem*) *The Gauss curvature K is intrinsic and we have the formula*

$$K = \frac{R_{121}^l g_{l2}}{g} = \frac{R_{1221}}{g}. \quad (227)$$

Proof. By (203), we have

$$\begin{aligned} R_{121}^l g_{l2} &= (L_{11}L_2^l - L_{12}L_1^l) g_{l2} = L_{11}L_{22} - L_{12}L_{12} \\ &= \det(L_{ij}) = \det(L_i^j) \cdot g = Kg. \end{aligned}$$

The proof is done. □

5.1 Orthogonal & Isothermal Parametrization.

Assume $g_{12} = g_{21} \equiv 0$ in this section. By the formula

$$\Gamma_{ij}^k = \frac{1}{2}g^{kl} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}), \quad \Gamma_{ij}^k = \Gamma_{ji}^k,$$

there are essentially the following 6 terms to decide:

$$\Gamma_{11}^1, \quad \Gamma_{12}^1 = \Gamma_{21}^1, \quad \Gamma_{22}^1, \quad \Gamma_{11}^2, \quad \Gamma_{12}^2 = \Gamma_{21}^2, \quad \Gamma_{22}^2.$$

We have

$$\begin{cases} \Gamma_{11}^1 = \frac{1}{2}g^{11} (2\partial_1 g_{11} - \partial_l g_{11}) = \frac{1}{2}g^{11} (\partial_1 g_{11}) = \frac{1}{2} \frac{\partial_1 g_{11}}{g_{11}} \\ \Gamma_{12}^1 = \frac{1}{2}g^{11} (\partial_1 g_{21} + \partial_2 g_{11} - \partial_1 g_{12}) = \frac{1}{2} \frac{\partial_2 g_{11}}{g_{11}} = \Gamma_{21}^1 \\ \Gamma_{22}^1 = -\frac{1}{2} \frac{\partial_1 g_{22}}{g_{11}} \\ \Gamma_{11}^2 = \frac{1}{2}g^{22} (-\partial_2 g_{11}) = -\frac{1}{2} \frac{\partial_2 g_{11}}{g_{22}} \\ \Gamma_{12}^2 = \frac{1}{2}g^{22} (\partial_1 g_{22} + \partial_2 g_{12} - \partial_2 g_{12}) = \frac{1}{2} \frac{\partial_1 g_{22}}{g_{22}} = \Gamma_{21}^2 \\ \Gamma_{22}^2 = \frac{1}{2}g^{22} (\partial_2 g_{22}) = \frac{1}{2} \frac{\partial_2 g_{22}}{g_{22}} \end{cases} \quad (228)$$

and hence

$$\begin{cases} \partial_1 g_{11} = 2\Gamma_{11}^1 g_{11} \\ \partial_1 g_{22} = 2\Gamma_{21}^2 g_{22} = -2\Gamma_{22}^1 g_{11} \\ \partial_2 g_{11} = 2\Gamma_{21}^1 g_{11} = -2\Gamma_{11}^2 g_{22} \\ \partial_2 g_{22} = 2\Gamma_{22}^2 g_{22}. \end{cases} \quad (229)$$

Since it is always possible to give a surface *orthogonal parametrization* and *isothermal parametrization* locally, the following is important:

Lemma 5.7 *Under orthogonal parametrization, i.e., $g_{12} \equiv 0$, we have*

$$K = -\frac{1}{2\sqrt{g}} \left[\partial_2 \left(\frac{\partial_2 g_{11}}{\sqrt{g}} \right) + \partial_1 \left(\frac{\partial_1 g_{22}}{\sqrt{g}} \right) \right]. \quad (230)$$

Proof. We shall employ (228) and the formula

$$\begin{aligned} -K g_{11} &= R_{112}^2 = \partial_1 \Gamma_{12}^2 - \partial_2 \Gamma_{11}^2 + \Gamma_{12}^p \Gamma_{p1}^2 - \Gamma_{11}^p \Gamma_{p2}^2 \\ &= \partial_1 \Gamma_{12}^2 - \partial_2 \Gamma_{11}^2 + (\Gamma_{12}^1 \Gamma_{11}^2 + \Gamma_{12}^2 \Gamma_{21}^2) - (\Gamma_{11}^1 \Gamma_{12}^2 + \Gamma_{11}^2 \Gamma_{22}^2). \end{aligned}$$

Hence

$$\begin{aligned} -K g_{11} &= \left\{ \partial_1 \left(\frac{1}{2} \frac{\partial_1 g_{22}}{g_{22}} \right) - \partial_2 \left(-\frac{1}{2} \frac{\partial_2 g_{11}}{g_{22}} \right) \right\} \\ &+ \left\{ \begin{aligned} &\left(\frac{1}{2} \frac{\partial_2 g_{11}}{g_{11}} \right) \left(-\frac{1}{2} \frac{\partial_2 g_{11}}{g_{22}} \right) + \left(\frac{1}{2} \frac{\partial_1 g_{22}}{g_{22}} \right) \left(\frac{1}{2} \frac{\partial_1 g_{22}}{g_{22}} \right) \\ &- \left(\frac{1}{2} \frac{\partial_1 g_{11}}{g_{11}} \right) \left(\frac{1}{2} \frac{\partial_1 g_{22}}{g_{22}} \right) - \left(-\frac{1}{2} \frac{\partial_2 g_{11}}{g_{22}} \right) \left(\frac{1}{2} \frac{\partial_2 g_{22}}{g_{22}} \right) \end{aligned} \right\} \\ &= \left\{ \begin{aligned} &\frac{1}{2} \left(\frac{\partial_1^2 g_{22}}{g_{22}} - \frac{(\partial_1 g_{22})^2}{g_{22}^2} \right) + \frac{1}{2} \left(\frac{\partial_2^2 g_{11}}{g_{22}} - \frac{(\partial_2 g_{11})(\partial_2 g_{22})}{g_{22}^2} \right) \\ &- \frac{1}{4} \frac{(\partial_2 g_{11})^2}{g} + \frac{1}{4} \frac{(\partial_1 g_{22})^2}{g_{22}^2} - \frac{1}{4} \frac{(\partial_1 g_{11})(\partial_1 g_{22})}{g} + \frac{1}{4} \frac{(\partial_2 g_{11})(\partial_2 g_{22})}{g_{22}^2} \end{aligned} \right\} \\ &= \left\{ \begin{aligned} &\frac{1}{2} \frac{\partial_1^2 g_{22}}{g_{22}} + \frac{1}{2} \frac{\partial_2^2 g_{11}}{g_{22}} \\ &- \frac{1}{4} \frac{(\partial_1 g_{22})^2}{g_{22}^2} - \frac{1}{4} \frac{(\partial_2 g_{11})(\partial_2 g_{22})}{g_{22}^2} - \frac{1}{4} \frac{(\partial_2 g_{11})^2}{g} - \frac{1}{4} \frac{(\partial_1 g_{11})(\partial_1 g_{22})}{g} \end{aligned} \right\} \end{aligned}$$

and so

$$K = \left\{ \begin{aligned} &-\frac{1}{2} \frac{\partial_1^2 g_{22}}{g} - \frac{1}{2} \frac{\partial_2^2 g_{11}}{g} + \frac{1}{4} \frac{(\partial_1 g_{22})^2}{g g_{22}} \\ &+ \frac{1}{4} \frac{(\partial_2 g_{11})(\partial_2 g_{22})}{g g_{22}} + \frac{1}{4} \frac{(\partial_2 g_{11})^2}{g g_{11}} + \frac{1}{4} \frac{(\partial_1 g_{11})(\partial_1 g_{22})}{g g_{11}} \end{aligned} \right\}. \quad (231)$$

On the other hand,

$$\begin{aligned}
& -\frac{1}{2\sqrt{g}} \left[\partial_2 \left(\frac{\partial_2 g_{11}}{\sqrt{g}} \right) + \partial_1 \left(\frac{\partial_1 g_{22}}{\sqrt{g}} \right) \right] \\
&= -\frac{1}{2\sqrt{g}} \left[\frac{\partial_2(\partial_2 g_{11})}{\sqrt{g}} - \frac{1}{2} \frac{\partial_2 g_{11}}{g^{3/2}} [(\partial_2 g_{11}) g_{22} + g_{11} (\partial_2 g_{22})] \right. \\
&\quad \left. + \frac{\partial_1(\partial_1 g_{22})}{\sqrt{g}} - \frac{1}{2} \frac{\partial_1 g_{22}}{g^{3/2}} [(\partial_1 g_{11}) g_{22} + g_{11} (\partial_1 g_{22})] \right] \\
&= \left\{ \begin{aligned} & -\frac{\partial_2^2 g_{11}}{2g} - \frac{\partial_1^2 g_{22}}{2g} + \frac{1}{4} \frac{\partial_2 g_{11}}{g^2} [(\partial_2 g_{11}) g_{22} + g_{11} (\partial_2 g_{22})] \\ & + \frac{1}{4} \frac{\partial_1 g_{22}}{g^2} [(\partial_1 g_{11}) g_{22} + g_{11} (\partial_1 g_{22})] \end{aligned} \right\} \\
&= \left\{ \begin{aligned} & -\frac{1}{2} \frac{\partial_2^2 g_{11}}{g} - \frac{1}{2} \frac{\partial_1^2 g_{22}}{g} \\ & + \frac{1}{4} \frac{(\partial_2 g_{11})^2}{g g_{11}} + \frac{1}{4} \frac{(\partial_2 g_{11})(\partial_2 g_{22})}{g g_{22}} + \frac{1}{4} \frac{(\partial_1 g_{22})(\partial_1 g_{11})}{g g_{11}} + \frac{1}{4} \frac{(\partial_1 g_{22})^2}{g g_{22}} \end{aligned} \right\},
\end{aligned}$$

which is the same as (231). The proof is done. \square

Lemma 5.8 *Assume that M is parametrized by an isothermal parametrization, i.e. $g_{11} = g_{22} = \lambda(x^1, x^2)$, $g_{12} = g_{21} \equiv 0$, then*

$$K = -\frac{1}{2\lambda} \Delta (\log \lambda), \quad \Delta = \frac{\partial^2}{\partial (x^1)^2} + \frac{\partial^2}{\partial (x^2)^2} \quad (232)$$

where Δ is the Laplace operator.

Proof. By the above lemma, we have

$$-\frac{1}{2\lambda} \Delta (\log \lambda) = -\frac{1}{2\lambda} \left[\partial_1 \left(\frac{\partial_1 \lambda}{\lambda} \right) + \partial_2 \left(\frac{\partial_2 \lambda}{\lambda} \right) \right] = K.$$

The proof is done. \square

Corollary 5.9 *Assume that M is parametrized by an isothermal parametrization with $g_{11} = g_{22} = \lambda(x^1, x^2)$. Then $K \equiv 0$ iff $\log \lambda$ is a harmonic function.*

Another corresponding result is

Lemma 5.10 *Assume that M is parametrized by an isothermal parametrization $\mathbf{x}(x^1, x^2)$. Then*

$$\Delta \mathbf{x} := \mathbf{x}_{11} + \mathbf{x}_{22} = \lambda H \mathbf{n} \quad (233)$$

where H is the mean curvature of M and \mathbf{n} is its unit normal.

Proof. We know that

$$\langle \mathbf{x}_1, \mathbf{x}_1 \rangle = \langle \mathbf{x}_2, \mathbf{x}_2 \rangle = \lambda(x^1, x^2), \quad \langle \mathbf{x}_1, \mathbf{x}_2 \rangle \equiv 0.$$

By the formula

$$\mathbf{x}_{ij} = L_{ij} \mathbf{n} + \Gamma_{ij}^k \mathbf{x}_k,$$

we get

$$\begin{aligned}
\mathbf{x}_{11} + \mathbf{x}_{22} &= (L_{11} \mathbf{n} + \Gamma_{11}^k \mathbf{x}_k) + (L_{22} \mathbf{n} + \Gamma_{22}^k \mathbf{x}_k) \\
&= (L_{11} + L_{22}) \mathbf{n} + (\Gamma_{11}^k + \Gamma_{22}^k) \mathbf{x}_k.
\end{aligned}$$

But by $L_k^l g_{lm} = L_{km}$ and (228), we have

$$(L_{11} + L_{22}) \mathbf{n} = (L_1^k g_{k1} + L_2^k g_{k2}) \mathbf{n} = (L_1^1 \lambda + L_2^2) \lambda \mathbf{n} = \lambda H \mathbf{n}$$

and

$$\begin{aligned} (\Gamma_{11}^k + \Gamma_{22}^k) \mathbf{x}_k &= (\Gamma_{11}^1 + \Gamma_{22}^1) \mathbf{x}_1 + (\Gamma_{11}^2 + \Gamma_{22}^2) \mathbf{x}_2 \\ &= \left(\frac{1}{2} \frac{\partial_1 g_{11}}{g_{11}} - \frac{1}{2} \frac{\partial_1 g_{22}}{g_{11}} \right) \mathbf{x}_1 + \left(-\frac{1}{2} \frac{\partial_2 g_{11}}{g_{22}} + \frac{1}{2} \frac{\partial_2 g_{22}}{g_{22}} \right) \mathbf{x}_2 = 0. \end{aligned}$$

The proof is done. \square

Corollary 5.11 *For a minimal surface M , if \mathbf{x} is an isothermal parametrization, then its components are harmonic functions. The converse is also true.*

6 Geometry of surfaces in \mathbb{R}^3 .

Definition 6.1 *A subset $S \subset \mathbb{R}^3$ is called a **regular surface** if, for each $p \in S$, there exists a neighborhood V in \mathbb{R}^3 and a map $\mathbf{x} : U \rightarrow V \cap S$ of an open set $U \subset \mathbb{R}^2$ onto $V \cap S \subset \mathbb{R}^3$ such that the following hold:*

1. \mathbf{x} is differentiable. This means that if we write

$$\mathbf{x}(u, v) = (x(u, v), y(u, v), z(u, v)), \quad (u, v) \in U \quad (234)$$

the functions $x(u, v)$, $y(u, v)$, $z(u, v)$ have continuous partial derivatives of all orders in U .

2. $\mathbf{x} : U \rightarrow V \cap S$ is a homeomorphism. This means that $\mathbf{x} : U \rightarrow V \cap S$ is 1-1, onto, and the inverse $\mathbf{x}^{-1} : V \cap S \rightarrow U$ is the restriction of a continuous map $F : W \subset \mathbb{R}^3 \rightarrow \mathbb{R}^2$ defined on some open set W in \mathbb{R}^3 containing $V \cap S$.
3. (**regularity condition**) For each $q \in U$, the differential $d\mathbf{x}_q : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ has rank 2 (same as $d\mathbf{x}_q : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is 1-1).

Remark 6.2 *Regarding condition 2 of the definition, imagine the situation of "figure 8" in \mathbb{R}^2 . It does not satisfy condition 2. As a result, the change of coordinates $\mathbf{x}^{-1} \circ \mathbf{y}$ is not even continuous. In view of this, one must require $\mathbf{x} : U \rightarrow V \cap S$ to be a homeomorphism. With it, the coordinates change $\mathbf{x}^{-1} \circ \mathbf{y}$ will be a **diffeomorphism** and a regular surface $S \subset \mathbb{R}^3$ is locally diffeomorphic to an open set of \mathbb{R}^2 .*

Definition 6.3 *In above, the map $\mathbf{x} : U \rightarrow V \cap S$ is called a **parametrization** of the regular surface S near p . The neighborhood $V \cap S$ of p in S is called a **coordinate neighborhood**.*

Let $q = (u_0, v_0) \in U \subset \mathbb{R}^2$. The differential $d\mathbf{x}_q : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ has rank 2 at q means that the two vectors

$$d\mathbf{x}_q(e_1) = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right) (u_0, v_0) \quad \text{and} \quad d\mathbf{x}_q(e_2) = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right) (u_0, v_0)$$

are linearly independent. Hence at least one of the following holds

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \neq 0, \quad \frac{\partial(y, z)}{\partial(u, v)} \neq 0, \quad \frac{\partial(x, z)}{\partial(u, v)} \neq 0 \quad (235)$$

at the point q .

The most typical example of a regular surface is:

Lemma 6.4 If $f(x, y) : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ is a differentiable function defined on some open set U of \mathbb{R}^2 , then its graph

$$S = \{(x, y, f(x, y)) : (x, y) \in U\} \subset \mathbb{R}^3$$

is a regular surface.

Proof. Routine check. □

Conversely every regular surface looks like a graph locally. That is:

Lemma 6.5 Let $S \subset \mathbb{R}^3$ be a regular surface and $p \in S$. Then there exists a neighborhood V of p in S such that V is the graph of a differentiable function of one of the forms: $z = f(x, y)$, $y = g(x, z)$, $x = h(y, z)$.

Proof. Use the implicit function theorem and (235). □

To determine if a set S in \mathbb{R}^3 is a regular surface or not, the simplest way is to use the concept of regular value.

Definition 6.6 Given a differentiable function $F : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$, where U is an open set in \mathbb{R}^n , we say $p \in U$ is a **critical point** of F if the differential $dF_p : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is **not** an onto map. The image $F(p) \in \mathbb{R}^m$ of a critical point is called a **critical value** of F . A point of \mathbb{R}^m which is not a critical value is called a **regular value** of F .

Remark 6.7 In the above definition, the interesting case is when $n \geq m$. Also if $a \notin F(U)$, then it is trivially a regular value.

Let $f : U \subset \mathbb{R}^3 \rightarrow \mathbb{R}$ be a differentiable map and $a \in f(U)$ be a regular value. It would imply that the set

$$f^{-1}(a) = \{(x, y, z) \in U : f(x, y, z) = a\} \neq \emptyset$$

and the gradient vector $\nabla f(p) \neq 0$ for any $p \in f^{-1}(a)$.

We can now state the following:

Lemma 6.8 Assume $U \subset \mathbb{R}^3$ is an open set. Let $f : U \subset \mathbb{R}^3 \rightarrow \mathbb{R}$ be a differentiable map and $a \in f(U)$ be a regular value. Then the set $f^{-1}(a)$ is a regular surface in \mathbb{R}^3 .

Proof. Let $p = (x_0, y_0, z_0) \in f^{-1}(a)$, where we know that $\nabla f(p) \neq 0$. Without loss of generality we may assume that $f_z \neq 0$ at p . Define a map $F : U \subset \mathbb{R}^3$ (with coordinates (x, y, z)) $\rightarrow \mathbb{R}^3$ (with coordinates (u, v, t)) by

$$F(x, y, z) = (x, y, f(x, y, z))$$

then use the inverse function theorem to F at the point p . The set $f^{-1}(a)$ near p looks like (via the map F) the flat one $\{(u, v, t) : t = a\}$. Hence it is a regular surface. □

Since we want to do calculus on a regular surface, we need to know the meaning of a "differentiable function" on a surface.

Definition 6.9 Let $f : V \subset S \rightarrow \mathbb{R}$ be a function defined in an open subset V of a regular surface S . We say f is **differentiable** at $p \in V$ if for some parametrization $\mathbf{x} : U \subset \mathbb{R}^2 \rightarrow S$ with $p \in \mathbf{x}(U) \subset V$, the composition $f \circ \mathbf{x} : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ is differentiable at $\mathbf{x}^{-1}(p)$. We say f is differentiable in V if it is differentiable at every $p \in V$.

Remark 6.10 One can check that the definition does **not** depend on the choice of the parametrization \mathbf{x} .

Remark 6.11 The definition of differentiability can be easily extended to a map between two surfaces. Hence one can define the

Definition 6.12 A **parametrized surface** $\mathbf{x} : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is a differentiable map \mathbf{x} from an open set $U \subset \mathbb{R}^2$ into \mathbb{R}^3 . The set $\mathbf{x}(U)$ is called the **trace** of \mathbf{x} . \mathbf{x} is **regular** if $d\mathbf{x}_q : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is 1-1 for all $q \in U$ (same as $d\mathbf{x}_q$ has rank 2). If $d\mathbf{x}_q$ is not 1-1, then we say $q \in U$ is a **singular point** of \mathbf{x} .

Remark 6.13 A regular parametrized surface may have self-intersections. Moreover, even if $\mathbf{x} : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is 1-1, $\mathbf{x} : U \rightarrow \mathbf{x}(U)$ (with subspace topology of \mathbb{R}^3) may not be a homeomorphism.

Locally, a regular surface is obviously a regular parametrized surface. The converse is also correct:

Lemma 6.14 Let $\mathbf{x} : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be a regular parametrized surface and let $q \in U$. Then there exists a neighborhood V of q in \mathbb{R}^2 such that $\mathbf{x}(V) \subset \mathbb{R}^3$ is a regular surface.

Proof. Exercise. □

Definition 6.15 By a **tangent vector** to S at a point $p \in S$, we mean the tangent vector $\alpha'(0)$ of a differentiable parametrized curve $\alpha : (-\varepsilon, \varepsilon) \rightarrow S$ with $\alpha(0) = p$.

Lemma 6.16 Let $\mathbf{x} : U \subset \mathbb{R}^2 \rightarrow S$ be a parametrization of a regular surface S and let $q \in U$. Then the vector space of dimension 2, given by

$$d\mathbf{x}_q(\mathbb{R}^2) \subset \mathbb{R}^3 \tag{236}$$

coincides with the set of all tangent vectors to S at $\mathbf{x}(q)$.

Remark 6.17 In particular, the plane $d\mathbf{x}_q(\mathbb{R}^2)$, which passes through $\mathbf{x}(q) = p \in S$, does **not** depend on the parametrization \mathbf{x} . We call it the **tangent plane** to S at p and denote it by $T_p(S)$.

Proof. Exercise. □

With the notion of a tangent plane we can define the **differential** of a map between surfaces. Let S_1 and S_2 be two regular surfaces and let $\varphi : V \subset S_1 \rightarrow S_2$ be a differentiable map of an open set V of S_1 into S_2 . If $p \in V$, we know that every tangent vector $w \in T_p(S_1)$ is the velocity vector $\alpha'(0)$ of a differentiable parametrized curve $\alpha : (-\varepsilon, \varepsilon) \rightarrow V$ with $\alpha(0) = p$. The curve $\beta = \varphi \circ \alpha$ will be a curve in S_2 with $\beta(0) = \varphi(p)$, and therefore $\beta'(0)$ is a vector in $T_{\varphi(p)}(S_2)$.

Proposition 6.18 In the discussion above, given $w \in T_p(S_1)$, the vector $\beta'(0)$ does not depend on the choice of α . The map $d\varphi_p : T_p(S_1) \rightarrow T_{\varphi(p)}(S_2)$ defined by $d\varphi_p(w) = \beta'(0)$ is linear.

Proof. Exercise. □

Definition 6.19 The linear map $d\varphi_p : T_p(S_1) \rightarrow T_{\varphi(p)}(S_2)$ is called the **differential** of φ at $p \in S_1$. Similarly we can define the **differential** of a differentiable function $f : U \subset S \rightarrow \mathbb{R}$ at $p \in U$ as a linear map $df_p : T_p(S) \rightarrow \mathbb{R}$.

Proposition 6.20 (inverse function theorem) Let $\varphi : V \subset S_1 \rightarrow S_2$ be a differentiable map of an open set V of S_1 into S_2 . If $p \in V$ and the map $d\varphi_p : T_p(S_1) \rightarrow T_{\varphi(p)}(S_2)$ is an isomorphism, then φ is a local diffeomorphism at p .

Proof. Exercise (apply the inverse function theorem in the plane case and lift it to the surfaces via coordinate maps). □

7 First fundamental forms.

Since our regular surfaces, by definition, are embedded in \mathbb{R}^3 , the natural inner product $\langle \cdot, \cdot \rangle$ of $\mathbb{R}^3 \supset S$ induces on each tangent plane $T_p(S)$ an inner product, to be denoted by $\langle \cdot, \cdot \rangle_p : T_p(S) \times T_p(S) \rightarrow \mathbb{R}$.

Definition 7.1 The quadratic form $I_p : T_p(S) \rightarrow \mathbb{R}$ given by

$$I_p(w) = \langle w, w \rangle_p = |w|^2 \geq 0, \quad w \in T_p(S) \quad (237)$$

is called **the first fundamental form** of the regular surface $S \subset \mathbb{R}^3$ at $p \in S$.

If $\mathbf{x} : U \subset \mathbb{R}^2 \rightarrow S$ is a parametrization with local coordinates (u, v) , then for any curve $\alpha(t) \in S$ with $\alpha(0) = p \in \mathbf{x}(U)$, $\alpha'(0) = w$, we have $\alpha(t) = \mathbf{x}(u(t), v(t))$ and

$$\begin{aligned} I_p(w) &= \langle \alpha'(0), \alpha'(0) \rangle_p = \left\langle \underbrace{u'(0) \mathbf{x}_u(u_0, v_0) + v'(0) \mathbf{x}_v(u_0, v_0)}_{\text{...}}, \dots \right\rangle \\ &= E(u_0, v_0) (u'(0))^2 + 2F(u_0, v_0) u'(0) v'(0) + G(u_0, v_0) (v'(0))^2, \quad (u_0, v_0) = (u(0), v(0)) \end{aligned}$$

where

$$\begin{cases} E(u_0, v_0) = \langle \mathbf{x}_u(u_0, v_0), \mathbf{x}_u(u_0, v_0) \rangle \\ F(u_0, v_0) = \langle \mathbf{x}_u(u_0, v_0), \mathbf{x}_v(u_0, v_0) \rangle \\ G(u_0, v_0) = \langle \mathbf{x}_v(u_0, v_0), \mathbf{x}_v(u_0, v_0) \rangle. \end{cases}$$

We call them the **coefficients** of the first fundamental form in the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$ of $T_p(S)$. As $p \in \mathbf{x}(U)$ varies, we get $E(u, v)$, $F(u, v)$ and $G(u, v)$. They are differentiable functions of $(u, v) \in U$.

With the information of $E(u, v)$, $F(u, v)$ and $G(u, v)$ on surfaces, we can treat metric questions on a surface without furthermore references to the ambient space \mathbb{R}^3 . In terms of them, the arclength of a parametrized curve $\alpha : (-\varepsilon, \varepsilon) \rightarrow S$ is given by

$$s(t) = \int_0^t \sqrt{E(u')^2 + 2F u' v' + G(v')^2} dt$$

where $E = E(u(t), v(t))$, $u' = u'(t)$, $v' = v'(t)$, ..., etc.

Also the area of a bounded region $R \subset \mathbf{x}(U) \subset S$ is given by

$$A(R) = \iint_Q |\mathbf{x}_u \times \mathbf{x}_v| dudv = \iint_Q \sqrt{EG - F^2} dudv, \quad Q = \mathbf{x}^{-1}(R).$$

8 Orientation of surfaces in \mathbb{R}^3 .

Definition 8.1 A regular surface is called **orientable** if it is possible to cover it with a family of coordinate neighborhoods in such a way that if a point $p \in S$ belongs to two neighborhoods of this family, then the change of coordinates has **positive Jacobian** at p . The choice of such a family is called an **orientation** of S , and S , in this case, is called **oriented**. If such a choice is not possible, then S is called **nonorientable**.

Remark 8.2 Orientation is a **global** property, in the sense that it involves the whole surface. **Locally every surface is orientable**. It is not easy to find a nonorientable surface in \mathbb{R}^3 (one example is the *Mobius band*).

Lemma 8.3 A regular surface $S \subset \mathbb{R}^3$ is orientable if and only if there exists a differentiable (or just continuous) field of unit normal vectors $\mathbf{N} : S \rightarrow \mathbb{R}^3$ on S .

Remark 8.4 Note that here \mathbf{N} must be globally defined on S .

Proof. Exercise. □

Lemma 8.5 If a regular surface is given by $S = \{(x, y, z) : f(x, y, z) = b\}$, where $f : U \subset \mathbb{R}^3 \rightarrow \mathbb{R}$ is differentiable and b is a regular value of f , then S is orientable.

Proof. Since b is a regular value of f , the vector

$$\mathbf{N}(x, y, z) = \left(\frac{f_x}{\sqrt{f_x^2 + f_y^2 + f_z^2}}, \frac{f_y}{\sqrt{f_x^2 + f_y^2 + f_z^2}}, \frac{f_z}{\sqrt{f_x^2 + f_y^2 + f_z^2}} \right) \in \mathbb{R}^3$$

is a differentiable field of unit normal vectors on S . By Lemma 8.3, S is orientable. □

9 The geometry of the Gauss map; second fundamental forms.

From now on, we shall assume $S \subset \mathbb{R}^3$ is an orientable regular surface with the choice of an orientation (i.e., a differentiable field of unit normal vectors $\mathbf{N} : S \rightarrow \mathbb{R}^3$ on S).

Definition 9.1 Let $S \subset \mathbb{R}^3$ be a regular surface with an orientation \mathbf{N} . The map $\mathbf{N} : S \rightarrow \mathbb{R}^3$ has its image in the unit sphere

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}. \quad (238)$$

The map $\mathbf{N} : S \rightarrow S^2$, thus defined, is called the **Gauss map** of S . One can check that $\mathbf{N} : S \rightarrow S^2$ is a differentiable map between the two regular surfaces S and S^2 .

For $p \in S$, the differential $d\mathbf{N}_p : T_p(S) \rightarrow T_{\mathbf{N}_p}(S^2)$ is a linear map from $T_p(S)$ to $T_{\mathbf{N}_p}(S^2)$. Since $T_p(S)$ and $T_{\mathbf{N}_p}(S^2)$ are **parallel planes**, one can view $d\mathbf{N}_p$ as a linear map on $T_p(S)$. Thus $d\mathbf{N}_p : T_p(S) \rightarrow T_p(S)$.

For a vector $v \in T_p(S)$,

$$d\mathbf{N}_p(v) = \left. \frac{d}{dt} \mathbf{N}(\alpha(t)) \right|_{t=0}$$

measures the rate of change of \mathbf{N} along any curve $\alpha(t) : (-\varepsilon, \varepsilon) \rightarrow S$ with $\alpha(0) = p$, $\alpha'(0) = v$. In the case of curves in \mathbb{R}^2 , this measure is given by a number, called the "curvature". In the case of surfaces in \mathbb{R}^3 , **this measure is now given by a linear map**.

Example 9.2 Consider the unit sphere S^2 . The vector (x, y, z) is normal to the sphere at the point $(x, y, z) \in S^2$. Thus $\bar{\mathbf{N}}(x, y, z) = (x, y, z)$ and $\mathbf{N}(x, y, z) = (-x, -y, -z)$ are two fields of unit normal vectors on S . We have

$$d\mathbf{N}(x(t), y(t), z(t)) = (-x'(t), -y'(t), -z'(t))$$

and so $d\mathbf{N}_p(v) = -v$ for all $p \in S$ and $v \in T_p(S)$. Similarly, we have $d\bar{\mathbf{N}}_p(v) = v$ for all $p \in S$ and $v \in T_p(S)$.

The following is of fundamental importance:

Proposition 9.3 The differential $d\mathbf{N}_p : T_p(S) \rightarrow T_p(S)$ of the Gauss map is a **self-adjoint** linear map. That is we have

$$\langle d\mathbf{N}_p(v), w \rangle = \langle v, d\mathbf{N}_p(w) \rangle \quad \text{for all } v, w \in T_p(S). \quad (239)$$

Proof. Since $d\mathbf{N}_p : T_p(S) \rightarrow T_p(S)$ is linear, it suffices to show that $\langle d\mathbf{N}_p(w_1), w_2 \rangle = \langle w_1, d\mathbf{N}_p(w_2) \rangle$ for a basis $\{w_1, w_2\}$ of $T_p(S)$. Let $\mathbf{x}(u, v)$ be a parametrization of S near p with the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$ of $T_p(S)$. If $\alpha(t) = \mathbf{x}(u(t), v(t))$ is a parametrized curve in S with $\alpha(0) = p$, then

$$\begin{aligned} d\mathbf{N}_p(\alpha'(0)) &= \left. \frac{d}{dt} \mathbf{N}(\alpha(t)) \right|_{t=0} = \left. \frac{d}{dt} \mathbf{N}(u(t), v(t)) \right|_{t=0}, \quad \mathbf{N}(u, v) = \mathbf{N} \circ \mathbf{x}(u, v) \\ &= \mathbf{N}_u u'(0) + \mathbf{N}_v v'(0). \end{aligned}$$

In particular, $d\mathbf{N}_p(\mathbf{x}_u) = \mathbf{N}_u$ and $d\mathbf{N}_p(\mathbf{x}_v) = \mathbf{N}_v$. Now it suffices to show that

$$\langle \mathbf{N}_u, \mathbf{x}_v \rangle = \langle \mathbf{x}_u, \mathbf{N}_v \rangle. \quad (240)$$

To see this, note that

$$\langle \mathbf{N}(u, v), \mathbf{x}_u(u, v) \rangle = \langle \mathbf{N}(u, v), \mathbf{x}_v(u, v) \rangle = 0 \quad \text{for all } (u, v)$$

and so

$$\langle \mathbf{N}_v(u, v), \mathbf{x}_u(u, v) \rangle + \langle \mathbf{N}(u, v), \mathbf{x}_{uv}(u, v) \rangle = 0.$$

We also have

$$\langle \mathbf{N}_u(u, v), \mathbf{x}_v(u, v) \rangle + \langle \mathbf{N}(u, v), \mathbf{x}_{vu}(u, v) \rangle = 0$$

and therefore

$$\langle \mathbf{N}_u(u, v), \mathbf{x}_v(u, v) \rangle = -\langle \mathbf{N}(u, v), \mathbf{x}_{uv}(u, v) \rangle = \langle \mathbf{N}_v(u, v), \mathbf{x}_u(u, v) \rangle.$$

The proof is done. □

Definition 9.4 The quadratic form $\mathbb{I}_p : T_p(S) \rightarrow \mathbb{R}$ given by

$$\mathbb{I}_p(v) = -\langle d\mathbf{N}_p(v), v \rangle, \quad v \in T_p(S) \quad (241)$$

is called **the second fundamental form** of S at p .

For $v \in T_p(S)$, with $|v| = 1$, the value $\mathbb{I}_p(v) = -\langle d\mathbf{N}_p(v), v \rangle_p$ has a geometric meaning. For any unit speed curve $\alpha(s) \in S$, $s \in (-\varepsilon, \varepsilon)$, with $\alpha(0) = p$, $\alpha'(0) = v \in T_p(S)$, we have $\langle \mathbf{N}(\alpha(s)), \alpha'(s) \rangle = 0$ for all s . Hence

$$\langle d\mathbf{N}_p(v), v \rangle + \langle \mathbf{N}(p), \alpha''(0) \rangle = 0, \quad |v| = 1$$

and so

$$\mathbb{I}_p(v) = \langle \mathbf{N}(p), \alpha''(0) \rangle = k \langle \mathbf{N}(p), \mathbf{n} \rangle, \quad |v| = 1 \quad (242)$$

where k is the curvature of the curve α at p , and \mathbf{n} is the normal vector of the curve.

Definition 9.5 The number $\mathbb{I}_p(v) = k \langle \mathbf{N}(p), \mathbf{n} \rangle$ is called the **normal curvature** of $\alpha \subset S$ at p . It is the normal (normal to the surface) component of $\alpha''(0)$.

Remark 9.6 (be careful) Although the normal curvature is referring to a curve α lying on S , by the identity

$$k \langle \mathbf{N}(p), \mathbf{n} \rangle = \langle \mathbf{N}(p), \alpha''(0) \rangle = -\langle d\mathbf{N}_p(v), v \rangle = \mathbb{I}_p(v) \quad (243)$$

the normal curvature of $\alpha \subset S$ at p is, in fact, a measure of how the surface S is curving in \mathbb{R}^3 . Since we require the curve $\alpha(s)$ to lie on S , we automatically have identity (243).

By the above, we can see that for any unit speed curve $\beta(s) \in S$ with $\beta(0) = p$, $\beta'(0) = v \in T_p(S)$, they have the **same** normal curvature at p . Moreover, since $\mathbb{I}_p(-v) = \mathbb{I}_p(v)$, we have:

Lemma 9.7 (Meusnier) All curves lying on a surface S and having at a given point $p \in S$ the same tangent line have at this point the same normal curvature. In particular, we can talk about the normal curvature along a given direction at $p \in S$.

Definition 9.8 Given a unit vector $v \in T_p(S)$, the intersection of S with the plane spanned by v and $\mathbf{N}(p)$ is called the **normal section** of S at p along v .

In a neighborhood of p , the normal section of S at p is a regular plane curve on S whose normal vector \mathbf{n} at p is $\pm\mathbf{N}(p)$ or zero. Hence its curvature $k \geq 0$ is equal to the absolute value of the normal curvature along v at p . Hence we have

$$|\mathbb{I}_p(v)| = |k \langle \mathbf{N}(p), \mathbf{n} \rangle| = k(p). \quad (244)$$

That is:

Lemma 9.9 Given a unit vector $v \in T_p(S)$, the absolute value of the normal curvature at $p \in S$ along v is equal to the curvature $k \geq 0$ of the normal section of S at p along v .

We shall need the following fact from linear algebra, whose proof is elementary:

Lemma 9.10 Let V be an inner product space with $\dim V = 2$, and let $A : V \rightarrow V$ be a **self-adjoint** linear map. Then there exists an **orthonormal** basis $\{e_1, e_2\}$ of V such that $A(e_1) = \lambda_1 e_1$, $A(e_2) = \lambda_2 e_2$, where the eigenvalues $\lambda_1 \geq \lambda_2$ are the **maximum** and the **minimum**, respectively, of the quadratic form $Q(v) = \langle A(v), v \rangle$ on the unit circle of V .

Remark 9.11 The above lemma is also true in the higher dimensional case.

By Lemma 9.10, there exists an orthonormal basis $\{e_1, e_2\}$ of $T_p(S)$ such that

$$-d\mathbf{N}_p(e_1) = k_1 e_1 \quad \text{and} \quad -d\mathbf{N}_p(e_2) = k_2 e_2 \quad (245)$$

where $k_1 \geq k_2$ are the maximum and minimum of the second fundamental form $\mathbb{I}_p(v)$ restricted to the unit circle of $T_p(S)$.

We are now ready to define the important curvature quantities of a surface: **mean curvature** and **Gaussian curvature**.

Definition 9.12 The maximum normal curvature k_1 and the minimum normal curvature k_2 are called the **principal curvatures** at p ; the corresponding directions, that is, the directions given by the eigenvectors e_1, e_2 , are called **principal directions** at p .

Definition 9.13 Let $p \in S$ and let $d\mathbf{N}_p : T_p(S) \rightarrow T_p(S)$ be the differential of the Gauss map. The **determinant** of $d\mathbf{N}_p$ is called the **Gaussian curvature** K of S at p . The **negative of half of the trace** of $d\mathbf{N}_p$ is called the **mean curvature** H of S at p . In terms of the principal curvatures we have

$$K = k_1 k_2, \quad H = \frac{k_1 + k_2}{2}. \quad (246)$$

Remark 9.14 Note that K and $2H$ are the determinant and trace of the map $d\mathbf{N}_p$.

Assuming that $\{e_1, e_2\}$ are from (245) (we can also make it to have the same orientation as $T_p(S)$), for any unit vector $v \in T_p(S)$ we can express it as

$$v = e_1 \cos \theta + e_2 \sin \theta$$

where θ is the angle from e_1 to v in the orientation of $T_p(S)$. Hence

$$\begin{aligned}\Pi_p(v) &= -\langle d\mathbf{N}_p(v), v \rangle = -\langle d\mathbf{N}_p(e_1 \cos \theta + e_2 \sin \theta), e_1 \cos \theta + e_2 \sin \theta \rangle \\ &= \langle e_1 k_1 \cos \theta + e_2 k_2 \sin \theta, e_1 \cos \theta + e_2 \sin \theta \rangle \\ &= k_1 \cos^2 \theta + k_2 \sin^2 \theta, \quad 0 \leq \theta \leq 2\pi,\end{aligned}\tag{247}$$

where the last expression is known classically as the **Euler formula**. It is the expression of the second fundamental form in the basis $\{e_1, e_2\}$.

By Euler formula we can obtain the **average formula** for the mean curvature

$$H = \frac{1}{2\pi} \int_0^{2\pi} \Pi_p(v) d\theta, \quad |v| = 1, \quad v = (\cos \theta, \sin \theta).\tag{248}$$

Also one can see that the sum of the normal curvatures for any pair of orthogonal directions, at $p \in S$, is a constant, given by $k_1 + k_2$, due to the identity

$$k_1 \cos^2 \theta_0 + k_2 \sin^2 \theta_0 + k_1 \cos^2 \left(\theta_0 + \frac{\pi}{2} \right) + k_2 \sin^2 \left(\theta_0 + \frac{\pi}{2} \right) = k_1 + k_2 \quad \text{for all } \theta_0.$$

As for the Gaussian curvature, the most elementary formula is

$$d\mathbf{N}_p(w_1) \wedge d\mathbf{N}_p(w_2) = K(p)(w_1 \wedge w_2), \quad w_1, w_2 \in T_p(S)\tag{249}$$

and so the Gauss map \mathbf{N} will be orientation-preserving at $p \in S$ if $K(p) > 0$ and orientation-reversing if $K(p) < 0$.

Remark 9.15 *One can also write as (249)*

$$d\mathbf{N}_p(\mathbf{x}_u) \wedge d\mathbf{N}_p(\mathbf{x}_v) = K(p) \left(\|\mathbf{x}_u \wedge \mathbf{x}_v\| \frac{\mathbf{x}_u \wedge \mathbf{x}_v}{\|\mathbf{x}_u \wedge \mathbf{x}_v\|} \right) = K(p) \sqrt{EG - F^2} \mathbf{N}(p)\tag{250}$$

where $(\mathbf{x}_u \wedge \mathbf{x}_v) / \|\mathbf{x}_u \wedge \mathbf{x}_v\| = \mathbf{N}(p)$.

We make the convention that the area of a region contained in a connected neighborhood V of S , where $K \neq 0$, and the area of its image by \mathbf{N} have the same sign if $K > 0$ in V , and opposite signs if $K < 0$ in V . With this convention, we can state the following geometric interpretation of the Gaussian curvature K , for $K \neq 0$:

Lemma 9.16 *Let $p \in S$ such that $K(p) \neq 0$ and let V be a connected neighborhood of p where K does not change sign. Then*

$$K(p) = \lim_{A \rightarrow 0} \frac{A'}{A}\tag{251}$$

where A is the area of a region $B \subset V$ containing p , A' is the area of the image of B by the Gauss map $\mathbf{N} : S \rightarrow S^2$, and the limit is taken through a sequence of region B_n that converges to p , in the sense that any sphere around p contains all B_n , for n sufficiently large.

Proof. The area A of B is given by

$$A = \iint_R |\mathbf{x}_u \times \mathbf{x}_v| dudv,$$

where $\mathbf{x}(u, v)$ is a parametrization in p , whose coordinate neighborhood contains V and R is the region in uv -plane corresponding to B . The area A' of $\mathbf{N}(B)$ is given by

$$A' = \iint_R |\mathbf{N}_u \times \mathbf{N}_v| dudv.$$

By formula (249) and the above convention, we can write

$$A' = \iint_R K |\mathbf{x}_u \times \mathbf{x}_v| dudv.$$

Hence

$$\lim_{A \rightarrow 0} \frac{A'}{A} = \lim_{A \rightarrow 0} \frac{A'/R}{A/R} = \lim_{R \rightarrow 0} \frac{\frac{1}{R} \iint_R K |\mathbf{x}_u \times \mathbf{x}_v| dudv}{\frac{1}{R} \iint_R |\mathbf{x}_u \times \mathbf{x}_v| dudv} = \frac{K |\mathbf{x}_u \times \mathbf{x}_v|}{|\mathbf{x}_u \times \mathbf{x}_v|} = K.$$

The proof is done. \square

Another interesting fact is the following:

Lemma 9.17 *If $H \equiv 0$ everywhere (i.e., S is a **minimal surface**) and S has no **planar points** (a point $p \in S$ such that $d\mathbf{N}_p \equiv 0$ is called a planar point), then the Gauss map $\mathbf{N} : S \rightarrow S^2$ satisfies*

$$\langle d\mathbf{N}_p(w_1), d\mathbf{N}_p(w_2) \rangle = -K(p) \langle w_1, w_2 \rangle \quad (252)$$

for all $p \in S$ and $w_1, w_2 \in T_p(S)$. In particular, (252) implies that $d\mathbf{N}_p$ is a conformal map and the angle of two intersecting curves on S and the angle of their spherical images are equal up to a sign (orientation).

Proof. S has no planar points means that both k_1 and k_2 are nonzero with $k_1 = -k_2$ (since $H \equiv 0$). Hence

$$K = -k_1^2 = -k_2^2 < 0 \quad \text{everywhere.}$$

Let e_1 and e_2 be the orthonormal eigenvectors with $d\mathbf{N}_p(e_1) = -k_1 e_1$, $d\mathbf{N}_p(e_2) = -k_2 e_2$, and write $w_1 = a_1 e_1 + a_2 e_2$, $w_2 = b_1 e_1 + b_2 e_2$. Then

$$\begin{aligned} \langle d\mathbf{N}_p(w_1), d\mathbf{N}_p(w_2) \rangle &= \langle d\mathbf{N}_p(a_1 e_1 + a_2 e_2), d\mathbf{N}_p(b_1 e_1 + b_2 e_2) \rangle \\ &= \langle -a_1 k_1 e_1 - a_2 k_2 e_2, -b_1 k_1 e_1 - b_2 k_2 e_2 \rangle = a_1 b_1 k_1^2 + a_2 b_2 k_2^2 \\ &= -(a_1 b_1 + a_2 b_2) K = -K \langle w_1, w_2 \rangle \end{aligned}$$

where K is the Gauss curvature of S at p . (252) is proved. It also means that $d\mathbf{N}_p$ is a conformal map, hence it preserves angle. \square

10 The Gauss map in local coordinates.

The following fact from linear algebra will be used:

Lemma 10.1 *Let V be a vector space of dimension n with inner product \langle, \rangle and let $\{e_i\}_{i=1}^n$ be a basis (arbitrary). Let $g_{ij} = \langle e_i, e_j \rangle$ and let g^{ij} be the (i, j) component of the inverse matrix of $[g_{ij}]_{n \times n}$ (i.e., $[g^{ij}] = [g_{ij}]^{-1}$). Then for any $v \in V$ we have (we use tensor notation below)*

$$v = \langle v, e_i \rangle g^{ij} e_j \quad (\text{sum over repeated index}) \quad (253)$$

and if $A : V \rightarrow V$ is a linear map, we have

$$\text{Tr } A = g^{ij} h_{ij}, \quad \text{where } h_{ij} = \langle Ae_i, e_j \rangle \quad (254)$$

and

$$\det A = \frac{\det h_{ij}}{\det g_{ij}}. \quad (255)$$

Now let $\mathbf{x}(u, v)$ be a parametrization around $p \in S$ and consider the self-adjoint linear map $-d\mathbf{N}$. Denote

$$\begin{cases} e = -\langle \mathbf{N}_u, \mathbf{x}_u \rangle = \langle \mathbf{N}, \mathbf{x}_{uu} \rangle \\ f = -\langle \mathbf{N}_u, \mathbf{x}_v \rangle = -\langle \mathbf{N}_v, \mathbf{x}_u \rangle = \langle \mathbf{N}, \mathbf{x}_{uv} \rangle \\ g = -\langle \mathbf{N}_v, \mathbf{x}_v \rangle = \langle \mathbf{N}, \mathbf{x}_{vv} \rangle \end{cases}$$

where $\mathbf{N}_u = d\mathbf{N}(\mathbf{x}_u)$, $\mathbf{N}_v = d\mathbf{N}(\mathbf{x}_v)$. By (255) we have the expression of Gaussian curvature

$$K = \frac{\det \langle Ae_i, e_j \rangle}{\det g_{ij}} = \frac{eg - f^2}{EG - F^2}, \quad \text{where } A = -d\mathbf{N} \quad (256)$$

and the expression of mean curvature

$$\begin{aligned} H &= \frac{1}{2} \text{Tr } A = \frac{1}{2} g^{ij} h_{ij} \\ &= \frac{1}{2(EG - F^2)} \text{Trace} \left(\begin{bmatrix} G & -F \\ -F & E \end{bmatrix} \begin{bmatrix} e & f \\ f & g \end{bmatrix} \right) = \frac{eG - 2fF + gE}{2(EG - F^2)}. \end{aligned} \quad (257)$$

Finally the principal curvatures $k_1 \geq k_2$ are given by

$$k_1 = H + \sqrt{H^2 - K}, \quad k_2 = H - \sqrt{H^2 - K} \quad (258)$$

where we know that $H^2 - K \geq 0$ everywhere. From it we see that k_1 and k_2 are differentiable functions on S except at those points where $H^2 = K$ (i.e., where $k_1 = k_2$).

Example 10.2 We shall derive the formulas for K and H for a graph surface S given by $z = h(x, y)$, $(x, y) \in U \subset \mathbb{R}^2$. We have $\mathbf{x}(u, v) = (u, v, h(u, v))$ and

$$\begin{cases} \mathbf{x}_u = (1, 0, h_u), & \mathbf{x}_v = (0, 1, h_v) \\ \mathbf{x}_{uu} = (0, 0, h_{uu}), & \mathbf{x}_{uv} = (0, 0, h_{uv}), & \mathbf{x}_{vv} = (0, 0, h_{vv}) \end{cases}$$

and so

$$\mathbf{N}(x, y) = \frac{(-h_x, -h_y, 1)}{(1 + h_x^2 + h_y^2)^{1/2}}$$

is a unit normal vector field on the surface. The coefficients of the second fundamental form in this orientation is given by

$$e = \frac{h_{xx}}{(1 + h_x^2 + h_y^2)^{1/2}}, \quad f = \frac{h_{xy}}{(1 + h_x^2 + h_y^2)^{1/2}}, \quad g = \frac{h_{yy}}{(1 + h_x^2 + h_y^2)^{1/2}}.$$

We also have

$$E = 1 + h_x^2, \quad F = h_{xy}, \quad G = 1 + h_y^2, \quad EG - F^2 = 1 + h_x^2 + h_y^2$$

From those expressions we can obtain

$$K = \frac{h_{xx}h_{yy} - h_{xy}^2}{(1 + h_x^2 + h_y^2)^2} \quad (259)$$

and

$$2H = \frac{(1 + h_y^2)h_{xx} - 2h_xh_yh_{xy} + (1 + h_x^2)h_{yy}}{(1 + h_x^2 + h_y^2)^{3/2}}. \quad (260)$$

For $p \in S$ we may assume that $p = (0, 0, 0)$ with $T_p(S)$ being the xy -plane. In doing so, we have $h(0, 0) = 0$, $h_x(0, 0) = 0$, $h_y(0, 0) = 0$. The differential $d\mathbf{N}_p$ applied to a vector $(x, y) \in \mathbb{R}^2$ becomes, in this case,

$$d\mathbf{N}_p \begin{pmatrix} x \\ y \end{pmatrix} = - \begin{bmatrix} h_{xx}(0, 0) & h_{xy}(0, 0) \\ h_{xy}(0, 0) & h_{yy}(0, 0) \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad (x, y) \in \mathbb{R}^2 = T_p(S). \quad (261)$$

Therefore the quadratic form (second fundamental form) $\Pi_p(v) = -\langle d\mathbf{N}_p(v), v \rangle$, $v \in T_p(S)$, is precisely the **Hessian** of h at $(0, 0)$. By a further rotation of the coordinate axis (x -axis points in the principal direction for k_1 , y -axis points in the principal direction for k_2), we can even make $h_{xy}(0, 0) = 0$. In such a situation, we obtain the nice representation

$$d\mathbf{N}_p \begin{pmatrix} x \\ y \end{pmatrix} = - \begin{bmatrix} h_{xx}(0, 0) & 0 \\ 0 & h_{yy}(0, 0) \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad (262)$$

where $h_{xx}(0, 0) = k_1(p)$, $h_{yy}(0, 0) = k_2(p)$. By developing $h(x, y)$ into a Taylor's expansion about $(0, 0)$, and noting that $h(0, 0) = 0$, $h_x(0, 0) = 0$, $h_y(0, 0) = 0$, we obtain

$$\begin{aligned} h(x, y) &= \frac{1}{2} [h_{xx}(0, 0)x^2 + 2h_{xy}(0, 0)xy + h_{yy}(0, 0)y^2] + R \\ &= \frac{1}{2} (k_1x^2 + k_2y^2) + R \end{aligned}$$

where $\lim_{(x,y) \rightarrow (0,0)} R/(x^2 + y^2) = 0$. Hence if at least one of k_1 and k_2 is nonzero, one can view the quadratic surface $z = \frac{1}{2}(k_1x^2 + k_2y^2)$ as a first-order approximation of $z = h(x, y)$ near $(0, 0)$.

10.1 Curvature of Level Surfaces in \mathbb{R}^3 .

Let S be a level surface given by

$$\Psi(x, y, z) = 0$$

where we assume $\Psi : \mathbb{R}^3 \rightarrow \mathbb{R}$ is a smooth function and 0 is a regular value of Ψ . Hence S is smooth everywhere. Motivated by the case of level curves, we compute the following

$$\operatorname{div} \left(\frac{\nabla \Psi}{|\nabla \Psi|} \right) = \frac{\Delta \Psi}{|\nabla \Psi|} + \nabla \left(\frac{1}{|\nabla \Psi|} \right) \cdot \nabla \Psi$$

where

$$\nabla \left(\frac{1}{|\nabla \Psi|} \right) = -\frac{1}{2} |\nabla \Psi|^{-3} \nabla (\Psi_x^2 + \Psi_y^2 + \Psi_z^2)$$

and so

$$\operatorname{div} \left(\frac{\nabla \Psi}{|\nabla \Psi|} \right) = \frac{\Delta \Psi}{|\nabla \Psi|} - \frac{1}{2} |\nabla \Psi|^{-3} \underbrace{\nabla (\Psi_x^2 + \Psi_y^2 + \Psi_z^2) \cdot \nabla \Psi}_{(263)}. \quad (263)$$

In the special case when $\Psi(x, y, z) = z - h(x, y)$, then

$$|\nabla \Psi| = (1 + h_x^2 + h_y^2)^{1/2}$$

and

$$\begin{aligned} &\underbrace{\nabla (\Psi_x^2 + \Psi_y^2 + \Psi_z^2) \cdot \nabla \Psi}_{(263)} \\ &= \nabla (h_x^2 + h_y^2 + 1) \cdot (-h_x, -h_y, 1) = -h_x [2h_x h_{xx} + 2h_y h_{xy}] - h_y [2h_x h_{xy} + 2h_y h_{yy}] \\ &= -2h_x^2 h_{xx} - 4h_x h_y h_{xy} - 2h_y^2 h_{yy} \end{aligned}$$

and (263) becomes

$$\begin{aligned}
\operatorname{div} \left(\frac{\nabla \Psi}{|\nabla \Psi|} \right) &= \frac{1}{|\nabla \Psi|^3} \left\{ |\nabla \Psi|^2 \Delta \Psi - \frac{1}{2} [-2h_x^2 h_{xx} - 4h_x h_y h_{xy} - 2h_y^2 h_{yy}] \right\} \\
&= \frac{1}{|\nabla \Psi|^3} \left\{ (1 + h_x^2 + h_y^2) (-h_{xx} - h_{yy}) + h_x^2 h_{xx} + 2h_x h_y h_{xy} + h_y^2 h_{yy} \right\} \\
&= \frac{1}{|\nabla \Psi|^3} \left\{ -h_{xx} - h_{yy} - h_x^2 h_{yy} - h_y^2 h_{xx} + 2h_x h_y h_{xy} \right\}
\end{aligned}$$

which gives

$$-\operatorname{div} \left(\frac{\nabla \Psi}{|\nabla \Psi|} \right) = \frac{1}{(1 + h_x^2 + h_y^2)^{3/2}} \left\{ (1 + h_y^2) h_{xx} - 2h_x h_y h_{xy} + (1 + h_x^2) h_{yy} \right\} = 2H. \quad (264)$$

This is twice of the **mean curvature** of the surface S (see (260)). We can write it as

$$\begin{aligned}
&-\operatorname{div} \left(\frac{\nabla \Psi}{|\nabla \Psi|} \right) \quad (\text{gradient \& divergence in } \mathbb{R}^3) \\
&= \operatorname{div} \left(\frac{\nabla h}{\sqrt{1 + |\nabla h|^2}} \right) \quad (\text{gradient \& divergence in } \mathbb{R}^2)
\end{aligned}$$

where $\Psi(x, y, z) = z - h(x, y)$. In general we can also conclude the validity of the following:

$$-\operatorname{div} \left(\frac{\nabla \Psi}{|\nabla \Psi|} \right) = 2H, \quad \text{where } H \text{ is the mean curvature.} \quad (265)$$

Hence the well-known **minimal surface equation** is given by

$$\operatorname{div} \left(\frac{\nabla \Psi}{|\nabla \Psi|} \right) = 0, \quad \Psi = \Psi(x, y, z), \quad \text{or} \quad \operatorname{div} \left(\frac{\nabla h}{\sqrt{1 + |\nabla h|^2}} \right) = 0, \quad h = h(x, y). \quad (266)$$

10.2 Orthogonal & Isothermal Parametrization.

In this section we shall investigate the formulas of the Gaussian curvature when the parametrization $\mathbf{x}(u, v)$ is orthogonal or isothermal.

Lemma 10.3 *Let $S \subset \mathbb{R}^3$ be a regular surface and $p \in S$. Then near $p = \mathbf{x}(u_0, v_0)$ one can give the surface a parametrization $\mathbf{x}(u, v)$ such that*

$$\langle \mathbf{x}_u(u, v), \mathbf{x}_v(u, v) \rangle = 0$$

for all (u, v) near (u_0, v_0) . This will imply that $F \equiv 0$ around p .

Remark 10.4 *Such parametrization is called an **orthogonal** parametrization.*

Proof. Omit (see any textbook on Differential Geometry). □

We now have the following:

Lemma 10.5 Under orthogonal parametrization, i.e., $g_{12} = g_{21} \equiv 0$ (in a neighborhood), we have

$$K = -\frac{1}{2\sqrt{g}} \left[\partial_2 \left(\frac{\partial_2 g_{11}}{\sqrt{g}} \right) + \partial_1 \left(\frac{\partial_1 g_{22}}{\sqrt{g}} \right) \right], \quad g = \det(g_{ij}). \quad (267)$$

Or, in terms of the E, F, G, u, v notation,

$$K = \frac{eg - f^2}{EG - F^2} = -\frac{1}{2\sqrt{EG}} \left[\left(\frac{E_v}{\sqrt{EG}} \right)_v + \left(\frac{G_u}{\sqrt{EG}} \right)_u \right]. \quad (268)$$

Remark 10.6 The surprising thing about formula (268) is that it does not involve second fundamental form e, f, g at all. Therefore the Gaussian curvature is **intrinsic** in the sense that **it is invariant under local isometry**. This is an important observation due to Gauss.

Proof. We shall not prove (268) as it is essentially based on straightforward but long computations. \square

It is even possible to locally parametrize a regular surface S so that **locally** we have

$$E(u, v) = G(u, v) \quad \text{and} \quad F(u, v) \equiv 0 \quad (269)$$

for all (u, v) in some neighborhood of uv -plane. Such a parametrization $\mathbf{x}(u, v)$ is called **isothermal**.

As an immediate consequence of the above lemma, under **isothermal parametrization**, we have:

Lemma 10.7 Under isothermal parametrization, we have the formula for Gaussian curvature

$$K = -\frac{1}{2\lambda} \Delta(\log \lambda) \quad (270)$$

where $\Delta = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}$ is the **Laplace operator** on uv -plane and $\lambda = \lambda(u, v) = E(u, v) = G(u, v)$.

Proof. By the above lemma, we have

$$-\frac{1}{2\lambda} \Delta(\log \lambda) = -\frac{1}{2\lambda} \left[\left(\frac{\lambda_u}{\lambda} \right)_u + \left(\frac{\lambda_v}{\lambda} \right)_v \right] = K.$$

The proof is done. \square

Corollary 10.8 Assume that S is parametrized by an isothermal parametrization $\mathbf{x}(u, v)$. Then $K \equiv 0$ if and only if $\log \lambda(u, v)$ is a harmonic function.

Another important result is:

Lemma 10.9 Assume that S is parametrized by an isothermal parametrization $\mathbf{x}(u, v)$. Then

$$\Delta \mathbf{x} := \mathbf{x}_{uu} + \mathbf{x}_{vv} = 2\lambda H \cdot \mathbf{N} \quad (271)$$

where H is the **mean curvature** of S , \mathbf{N} is its surface normal, $\lambda = \lambda(u, v) = E(u, v) = G(u, v)$, and

$$\Delta = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}$$

is the usual Laplace operator on uv -plane. In the Euclidean components of $\mathbf{x}(u, v) = (x(u, v), y(u, v), z(u, v))$ we have

$$\begin{pmatrix} \Delta x(u, v) \\ \Delta y(u, v) \\ \Delta z(u, v) \end{pmatrix} = 2\lambda H \cdot \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}.$$

Proof. We first note that under the isothermal parametrization $\mathbf{x}(u, v)$ the mean curvature H is given by

$$H = \frac{eG - 2fF + gE}{2(EG - F^2)} = \frac{e + g}{2\lambda}. \quad (272)$$

Applying differentiation (∂_u to the first equation, ∂_v to the second equation) to the identities

$$\langle \mathbf{x}_u, \mathbf{x}_u \rangle = \langle \mathbf{x}_v, \mathbf{x}_v \rangle = \lambda(u, v) \quad \text{and} \quad \langle \mathbf{x}_u, \mathbf{x}_v \rangle \equiv 0$$

we get

$$\langle \mathbf{x}_{uu}, \mathbf{x}_u \rangle = \langle \mathbf{x}_{vu}, \mathbf{x}_v \rangle = -\langle \mathbf{x}_{vv}, \mathbf{x}_u \rangle$$

and so

$$\langle \mathbf{x}_{uu} + \mathbf{x}_{vv}, \mathbf{x}_u \rangle = 0. \quad (273)$$

Similarly we have

$$\langle \mathbf{x}_{uu} + \mathbf{x}_{vv}, \mathbf{x}_v \rangle = 0. \quad (274)$$

The above two identities imply that $\Delta \mathbf{x}$ is parallel to \mathbf{N} . Note that we can write \mathbf{x}_{uu} as

$$\mathbf{x}_{uu} = A\mathbf{x}_u + B\mathbf{x}_v + e\mathbf{N}$$

and write \mathbf{x}_{vv} as

$$\mathbf{x}_{vv} = C\mathbf{x}_u + D\mathbf{x}_v + g\mathbf{N}$$

and so

$$\mathbf{x}_{uu} + \mathbf{x}_{vv} = (e + g)\mathbf{N} = 2\lambda H \cdot \mathbf{N}.$$

The proof is done. □

Remark 10.10 *The Laplace operator Δ on the surface S is given by (assume $f : S \rightarrow \mathbb{R}$ is a smooth function)*

$$\begin{aligned} \Delta f &= \operatorname{div}(\operatorname{grad} f) \\ &= \frac{1}{\sqrt{g}} \partial_j (\partial_i f \cdot g^{ij} \cdot \sqrt{g}) = \frac{1}{\sqrt{g}} \partial_i (g^{ij} \sqrt{g} \partial_j f), \quad g = \det(g_{ij}). \end{aligned} \quad (275)$$

In the situation of Lemma 10.9 we have $1 \leq i, j \leq 2$, and

$$(g_{ij}) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \quad (g^{ij}) = \begin{pmatrix} \frac{1}{\lambda} & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}$$

and

$$\begin{aligned} \Delta_{\text{surface}} f &= \frac{1}{\sqrt{g}} \partial_i (g^{ij} \sqrt{g} \partial_j f) = \frac{1}{\lambda} \partial_1 (g^{11} \lambda \partial_1 f) + \frac{1}{\lambda} \partial_2 (g^{22} \lambda \partial_2 f), \quad \partial_1 = \partial_u, \quad \partial_2 = \partial_v \\ &= \frac{1}{\lambda} \left(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2} \right) = \frac{1}{\lambda(u, v)} \Delta_{\text{Euclidean}}(f \circ \mathbf{x})(u, v), \quad f : S \rightarrow \mathbb{R}. \end{aligned}$$

If we take $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ to be the identity map, $f(x, y, z) = (x, y, z)$, then the restriction of f to the surface is a smooth vector map. We have

$$\frac{1}{\lambda(u, v)} \Delta_{\text{Euclidean}}(f \circ \mathbf{x})(u, v) = \frac{1}{\lambda(u, v)} \Delta_{\text{Euclidean}}(x(u, v), y(u, v), z(u, v)) = 2H \cdot \mathbf{N}$$

and

$$\Delta_{\text{surface}} f = \Delta_{\text{surface}}(x, y, z) = \Delta_{\text{surface}}(\text{position vector of } S)$$

where each x, y, z is a smooth function on S . Hence we obtain

$$\Delta_{\text{surface}}(\text{position vector of } S) = 2H \cdot \mathbf{N}. \quad (276)$$

Corollary 10.11 For a *minimal surface* S (by definition, a *minimal surface* S has $H \equiv 0$ everywhere), if \mathbf{x} is an *isothermal parametrization*, then its components are *harmonic functions*. The converse is also true.

Example 10.12 We shall look at an example to demonstrate formulas (270) and (271). One can parametrize the unit sphere $S^2 \subset \mathbb{R}^3$ by the familiar one

$$\mathbf{x}(\theta, \varphi) = (\sin \varphi \cos \theta, \sin \varphi \sin \theta, \cos \varphi), \quad 0 < \theta < 2\pi, \quad 0 < \varphi < \pi$$

and see that it is an orthogonal parametrization since

$$\mathbf{x}_\theta = (-\sin \varphi \sin \theta, \sin \varphi \cos \theta, 0), \quad \mathbf{x}_\varphi = (\cos \varphi \cos \theta, \cos \varphi \sin \theta, -\sin \varphi)$$

are perpendicular everywhere. However, it is not isothermal since $\langle \mathbf{x}_\theta, \mathbf{x}_\theta \rangle = \sin^2 \varphi$, $\langle \mathbf{x}_\varphi, \mathbf{x}_\varphi \rangle = 1$. Instead we consider the following **Mercator projection parametrization** (this is one of the methods of drawing the world's map), given by

$$\mathbf{x}(u, v) = (\operatorname{sech} u \cos v, \operatorname{sech} u \sin v, \tanh u), \quad -\infty < u < \infty, \quad 0 < v < 2\pi.$$

We have

$$\begin{aligned} \mathbf{x}_u &= (-\operatorname{sech} u \tanh u \cos v, -\operatorname{sech} u \tanh u \sin v, \operatorname{sech}^2 u) \\ \mathbf{x}_v &= (-\operatorname{sech} u \sin v, \operatorname{sech} u \cos v, 0) \end{aligned}$$

and (note that $\lim_{u \rightarrow \pm\infty} \operatorname{sech}^2 u = 0$)

$$\lambda(u, v) = \langle \mathbf{x}_u, \mathbf{x}_u \rangle = \langle \mathbf{x}_v, \mathbf{x}_v \rangle = \operatorname{sech}^2 u, \quad \langle \mathbf{x}_u, \mathbf{x}_v \rangle = 0.$$

Hence it is an isothermal parametrization. By formula (270) we have

$$K = -\frac{1}{2\lambda} \Delta(\log \lambda) = -\frac{1}{2 \operatorname{sech}^2 u} \Delta(2 \log \operatorname{sech} u) \equiv 1 \quad \text{everywhere.}$$

Also

$$\begin{aligned} \mathbf{x}_{uu} + \mathbf{x}_{vv} &= (-2 \operatorname{sech}^3 u \cos v, -2 \operatorname{sech}^3 u \sin v, -2 \operatorname{sech}^2 u \tanh u) \\ &= -2 \operatorname{sech}^2 u \cdot (\operatorname{sech} u \cos v, \operatorname{sech} u \sin v, \tanh u) \end{aligned}$$

On the other hand, one can compute

$$\frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|} = -(\operatorname{sech} u \cos v, \operatorname{sech} u \sin v, \tanh u)$$

and so

$$2\lambda H \cdot \mathbf{N} = 2\lambda H \cdot \mathbf{N} = -2H \operatorname{sech}^2 u \cdot (\operatorname{sech} u \cos v, \operatorname{sech} u \sin v, \tanh u).$$

By comparison, we see that

$$H \equiv 1 \quad \text{everywhere.}$$