Applied Mathematics E-Notes, 26(2026), 316-331 © ISSN 1607-2510
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/

Transmuted Power Inverse Rayleigh Distribution: Statistical Characteristics, Estimation And Applications*

Mohammad Izhar Khan†, Abdelfattah Mustafa‡

Received 2 February 2026

Abstract

In this study, we introduce a new probability distribution model called the Transmuted Power Inverse Rayleigh Distribution (TPIRD), which exhibits distinctive and flexible statistical properties. A comprehensive analysis of this distribution is carried out, encompassing key statistical and structural characteristics such as moments, reliability function, quantile function, incomplete moments, and order statistics. To estimate the unknown parameters of the proposed model, the maximum likelihood (ML) estimation method is employed. Furthermore, to assess its practical applicability, three real-world datasets are analyzed, where model fitting and parameter estimation demonstrate the TPIRD's effectiveness in modeling and predicting empirical data. Overall, this work not only presents a novel probability model but also offers researchers and practitioners a robust analytical framework for real-life data analysis.

1 Introduction

The inverse Rayleigh distribution (IRD) was first introduced by Voda [1] and has since become a widely used lifetime model, particularly in reliability analysis and survival data applications. A substantial body of research has been devoted to the IRD and its properties. Gharraph [2] obtained closed-form expressions for several fundamental characteristics of the distribution, while Soliman et al. [3] explored different methods for estimating its parameters. In recent years, numerous generalizations of the IRD have been proposed. Notable among these are the exponentiated inverse Rayleigh (EIR) distribution introduced by Rehman and Dar [4], which extends the IRD by incorporating an additional shape parameter, and the Kumaraswamy exponentiated IRD suggested by Haq [5]. Fatima and Ahmad [6] investigated the estimation of the scale parameter of the generalized IRD using informative priors. Furthermore, Shala and Merovci [7] discussed a three-parameter extension of the IRD, and Asif et al. [8] analyzed the parameters of the exponentiated IRD within a Bayesian framework.

Nashat [9] introduced the powered inverse Rayleigh distribution (PIRD), an important statistical model with wide-ranging applications in reliability testing, stock market analysis, survival studies, communication systems, and several other domains. The PIRD represents a specialized form of distribution obtained by fixing one of the parameters of the three-parameter Burr–XII distribution, underscoring its theoretical relevance. Due to its flexibility and practical usefulness, extensive research has been conducted on statistical inference related to the PIRD. For example, Mustafa and Khan [10] examined the length-biased version of the PIRD and explored its applications. Khan [11] derived its moment properties using dual generalized order statistics, while Khan and Mustafa [12] studied the PIRD under the DUS transformation. More recently, Khan [13] developed expressions for order statistics and actuarial measures associated with the PIRD.

A random variable (R.V.) X is said to have PIRD, if its probability density function (PDF) is given by

$$f(x,\alpha,\theta)=\frac{2\alpha}{\theta x^{2\alpha+1}}e^{-\frac{1}{\theta x^{2\alpha}}},\ x>0,\ \alpha,\theta>0.$$

*Mathematics Subject Classifications: 60E05, 62E10, 62F10.

†Department of Mathematics, Faculty of Science, Islamic University of Madinah, Madinah 42351, Saudi Arabia.

‡Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt.

The cumulative distribution function (CDF) is given by

$$F(x,\alpha,\theta)=e^{-\frac{1}{\theta x^{2\alpha}}},\ x>0,\ \alpha,\ \theta>0.$$

As researchers from various disciplines continue to address increasingly diverse problems, relying solely on traditional probability distributions has become less effective for modeling modern data. This has motivated the modification and extension of classical distributions. The literature on distribution theory is rich with methods for expanding existing models, and numerous approaches have been proposed to generate new distributions from established baseline forms. Among the most notable techniques are the power transformation introduced by Gupta et al. [14], the DUS transformation proposed by Kumar et al. [15], and the alpha power transformation developed by Mahdavi and Kundu [16].

Beyond the aforementioned developments, many applied fields–such as lifetime analysis and insurance modeling–require extended distributions that offer greater flexibility for fitting real-world data, which often exhibit substantial skewness and kurtosis. One effective strategy for achieving such flexibility is to introduce an additional parameter to an existing distribution, thereby generating a transmuted distribution. A notable example is the quadratic rank transmutation map (QRTM) proposed by Shaw and Buckley [17]. In this work, we apply the QRTM to the power inverse Rayleigh distribution, resulting in a more adaptable model. Motivated by the widespread practical utility of transmuted distributions, this study focuses on developing the transmuted power inverse Rayleigh distribution (TPIRD).

The remainder of the paper is organized as follows. Section 2 introduces the transmuted version of the PIRD. In Section 3, we derive key statistical properties, including the moments, variance, skewness, kurtosis, moment-generating function, quantile function, and reliability measures. Section 4 discusses the incomplete moments, conditional moments, and inequality measures of the TPIRD. Order statistics are presented in Section 5, while Section 6 outlines the parameter estimation procedures for the TPIRD. Section 7 illustrates the applicability of the proposed distribution through three real-life datasets. Finally, Section 8 concludes the paper and offers directions for future research.

2 Transmuted Model of PIRD

Shaw and Buckley [17] introduced an elegant method for incorporating an additional parameter into an existing distribution, referred to as the quadratic transmuted family (QTF) of distributions. The cumulative distribution function (CDF) of the QTF is defined using the following simple quadratic form

$$F(x;\lambda,\Theta)=(1+\lambda)\,G\left(x;\Theta\right)-\lambda\ G^{2}\left(x;\Theta\right),\quad x\in\mathbb{R},\tag{1}$$

where $\lambda\in[-1,1]$ and $G(x;\Theta)$ is the CDF of the baseline distribution, and the corresponding PDF is

$$f(x;\lambda,\Theta)=\left(1+\lambda-2\lambda G\left(x;\Theta\right)\right)g\left(x;\Theta\right),\tag{2}$$

Aryal and Tsokos [18, 19] were the first to emphasize the technique presented in (1) and introduced several transmuted probability distributions aimed at providing greater flexibility for modeling reliability and environmental data. Since then, transmuted distributions have become widely studied in statistical literature. Comprehensive properties and results for various members of the transmuted family have been examined by several authors, including Ahmad et al. [20], Das [21], Bourguignon et al. [22], Abdullahi and Ieren [23], Gharaibeh and Al-Omari [24], Rehman et al. [25], Khan and Mustafa [26], and Khan [27].

The CDF and PDF of TPIRD are derived from (1) and (2) as follows

$$F(x,\lambda,\alpha,\theta)=\left(1+\lambda-\lambda e^{-\frac{1}{\theta x^{2\alpha}}}\right)e^{-\frac{1}{\theta x^{2\alpha}}},\quad\alpha,\theta>0,\ |\lambda|\le1,\ x>0,\tag{3}$$

$$f(x,\lambda,\alpha,\theta)=\frac{2\alpha}{\theta\ x^{2\alpha+1}}\left(1+\lambda-2\lambda e^{-\frac{1}{\theta x^{2\alpha}}}\right)e^{-\frac{1}{\theta x^{2\alpha}}},\quad\alpha,\theta>0,\ |\lambda|\le1,\ x>0.$$

Figure 1 shows the pdf for the TPIRD at different values of the parameters (λ,α,θ). From Figure 1, the TPIRD has only one mode for the different values of parameters.

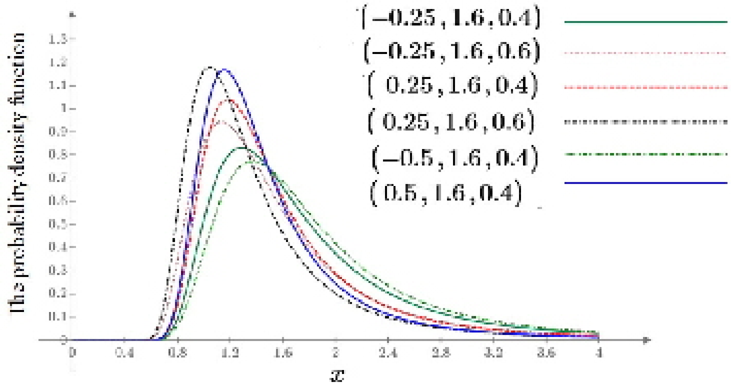

Figure 1: The PDF of the TPIRD for different parameter values λ, α, θ.

3 Statistical Characteristics

3.1 Moments

Assuming that the random variable (R.V.) X is distributed with TPIRD, the following theorem obtains the kth moment of X.

Theorem 1 *The kth moments for the TPIRD with the parameters λ, α and θ is given by*

$$\mu_k' = \left(\frac{1}{\theta}\right)^{\frac{k}{2\alpha}} \left[1 + \lambda\left(1 - 2^{\frac{k}{2\alpha}}\right)\right] \Gamma\left(1 - \frac{k}{2\alpha}\right), \qquad 2\alpha > k. \tag{4}$$

Proof. By the definition of kth moments and Eq. (3), we have

$$\begin{aligned} \mu_k' &= E\left(X^k\right) = \int_0^\infty x^k f(x;\lambda,\alpha,\theta)dx = \frac{2\alpha}{\theta}\int_0^\infty x^{k-2\alpha-1}\left(1+\lambda-2\lambda e^{-\frac{1}{\theta x^{2\alpha}}}\right)e^{-\frac{1}{\theta x^{2\alpha}}}dx \\ &= \frac{2\alpha}{\theta}\left[(1+\lambda) I_1 - 2\lambda I_2\right], \end{aligned} \tag{5}$$

where

$$I_1 = \int_0^\infty x^{-2\alpha+k-1} e^{-\frac{1}{\theta x^{2\alpha}}} dx.$$

Let $u = \frac{1}{\theta}x^{-2\alpha}$. Then

$$I_1 = \frac{(\theta)^{\frac{2\alpha-k}{2\alpha}}}{2\alpha}\int_0^\infty u^{\frac{2\alpha-k}{2\alpha}-1} e^{-u}\, du = \frac{\theta^{1-\frac{k}{2\alpha}}}{2\alpha}\Gamma\left(1-\frac{k}{2\alpha}\right), \tag{6}$$

and

$$I_2 = \int_0^\infty x^{-2\alpha+k-1} e^{-\frac{2}{\theta x^{2\alpha}}} dx.$$

Let $u = \frac{2}{\theta}x^{-2\alpha}$. Then

$$I_2 = \frac{(\theta/2)^{\frac{2\alpha-k}{2\alpha}}}{2\alpha}\int_0^\infty u^{\frac{2\alpha-k}{2\alpha}-1} e^{-u}\, du = \frac{\theta^{1-\frac{k}{2\alpha}}}{2^{1-\frac{k}{2\alpha}}(2\alpha)}\Gamma\left(1-\frac{k}{2\alpha}\right). \tag{7}$$

Substituting from (6) and (7) into (5), we have (4), this completes the proof. ■

From Theorem 1, we can obtain the following measures:

- The mean is given by

$$\mu = \mu_1' = \left(\frac{1}{\theta}\right)^{\frac{1}{2\alpha}} \left[1 + \lambda\left(1 - 2^{\frac{1}{2\alpha}}\right)\right] \Gamma\left(1 - \frac{1}{2\alpha}\right), \quad 2\alpha > 1.$$

- The variance is given by

$$\sigma_X^2 = \mu_2' - {\mu_1'}^2 = \left(\frac{1}{\theta}\right)^{\frac{1}{\alpha}} \left[1 + \lambda\left(1 - 2^{\frac{1}{\alpha}}\right)\right] \Gamma\left(1 - \frac{1}{\alpha}\right) - \left(\frac{1}{\theta}\right)^{\frac{1}{\alpha}} \left[1 + \lambda\left(1 - 2^{\frac{1}{2\alpha}}\right)\right]^2 \Gamma^2\left(1 - \frac{1}{2\alpha}\right).$$

- The skewness is given by

$$sk = E\left(\frac{X - \mu}{\sigma}\right)^3 = \frac{\mu_3' - 3\mu_2'\mu_1' + 2{\mu_1'}^3}{\left(\mu_2' - {\mu_1'}^2\right)^{\frac{3}{2}}}.$$

- The kurtosis is given by

$$ku = E\left(\frac{X - \mu}{\sigma}\right)^4 = \frac{\mu_4' - 4\mu_3'\mu_1' + 6\mu_2'{\mu_1'}^2 - 3{\mu_1'}^4}{\left(\mu_2' - {\mu_1'}^2\right)^2}.$$

Table 1: Moment characteristic of the TPIRD for $\alpha = 2.95, \theta = 0.6$ and $\lambda \in [-1, 1]$.

λ	Mean	Variance	CV	Skewness	Kurtosis
-1.00	1.388	0.126	25.5738	2.857	25.846
-0.75	1.349	0.124	26.1035	2.759	24.719
-0.50	1.311	0.119	26.313	2.734	24.353
-0.25	1.272	0.111	26.1923	2.770	24.710
0.00	1.234	0.100	25.6262	2.857	25.846
0.25	1.196	0.086	24.5199	2.987	27.896
0.50	1.157	0.069	22.7034	3.126	30.934
0.75	1.119	0.049	19.7819	3.105	33.504
1.00	1.080	0.026	14.9301	1.355	7.059

From Table 1, one can conclude that:

- The mean decreases when the parameter λ increases.
- The variance decreases when the parameter λ increases.
- The coefficient of variation decreases when the parameter λ increases.
- The TPIRD is positively and extremely skewed for all values of λ.
- The TPIRD is Leptokurtic or heavy-tailed distribution ($ku > 3$) for all λ.

The harmonic mean of TPIRD is obtained as follows

$$H = E\left(\frac{1}{X}\right) = \theta^{\frac{1}{2\alpha}} \left[1 + \lambda\left(1 - 2^{-\frac{1}{2\alpha}}\right)\right] \Gamma\left(1 + \frac{1}{2\alpha}\right).$$

3.2 Moment Generating Function

Theorem 2 *The moment generating function (MGF) for TPIRD is*

$$M_X(t)=\sum_{k=0}^{\infty}\frac{t^k}{k!}\left(\frac{1}{\theta}\right)^{\frac{k}{2\alpha}}\left[1+\lambda-\lambda\left(2\right)^{\frac{k}{2}}\right]\Gamma\left(1-\frac{k}{2\alpha}\right),\quad k<2. \tag{8}$$

Proof. The MGF derives using the relation

$$\begin{aligned}M_X(t) &= E\left[e^{tX}\right]=\int_0^{\infty}e^{tX}f(x;\lambda,\alpha,\theta)dx=\int_0^{\infty}\sum_{k=0}^{\infty}\frac{(tx)^k}{k!}f\left(x,\lambda,\alpha,\theta\right)dx\\ &= \sum_{k=0}^{\infty}\frac{t^k}{k!}\int_0^{\infty}x^k f\left(x,\lambda,\alpha,\theta\right)dx=\sum_{k=0}^{\infty}\mu_k'\frac{t^k}{k!}.\end{aligned} \tag{9}$$

Substituting from (4) into (9), we obtain the MGF in (8). ■

3.3 Quantile Function

Theorem 3 *The quantile function (QF) of TPIRD with parameters* (λ,α,θ) *is given by*

$$Q_p=\begin{cases}\left[-\theta\ln\left(\frac{1+\lambda\pm\sqrt{(1+\lambda)^2-4\lambda p}}{2\lambda}\right)\right]^{-\frac{1}{2\alpha}}, & if\ \ \lambda\neq 0,\\ \left[-\theta\ln\left(p\right)\ \right]^{-\frac{1}{2\alpha}}, & if\ \ \lambda=0.\end{cases}$$

Proof. From the CDF and the following equation

$$F\left(x_p,\alpha,\theta,\lambda\right)=p.$$

We have

$$\left(1+\lambda-\lambda e^{-\frac{1}{\theta x_p^{2\alpha}}}\right)e^{-\frac{1}{\theta x_p^{2\alpha}}}=p. \tag{10}$$

For $\lambda\neq 0$, the equation simplifies as

$$\lambda e^{-\frac{2}{\theta x_p^{2\alpha}}}-\left(1+\lambda\right)e^{-\frac{1}{\theta x_p^{2\alpha}}}+p=0.$$

Let

$$e^{-\frac{1}{\theta x^{2\alpha}}}=\frac{(1+\lambda)\pm\ \sqrt{\left(1+\lambda\right)^2-4\lambda p}}{2\lambda}.$$

Therefore

$$x_p=\left[-\theta\ln\left(\frac{1+\lambda\pm\sqrt{\left(1+\lambda\right)^2-4\lambda p}}{2\lambda}\right)\right]^{-\frac{1}{2\alpha}}.$$

For $\lambda=0$, the Equation (10), simplifies

$$e^{-\frac{1}{\theta x_p^{2\alpha}}}=p\Longrightarrow x_p=\left[-\theta\ln\left(p\right)\right]^{-\frac{1}{2\alpha}}.$$

■

3.4 Reliability Analysis

The reliability function (RF) of the TPIRD is

$$R\left(x;\lambda,\alpha,\theta\ \right)=1-F\left(x;\lambda,\alpha,\theta\right)=1-\left(1+\lambda-\lambda e^{-\frac{1}{\theta x^{2\alpha}}}\right)e^{-\frac{1}{\theta x^{2\alpha}}}.$$

The hazard rate function (HRF) is

$$h\left(x;\lambda,\alpha,\theta\right)=\frac{f\left(x;\lambda,\alpha,\theta\right)}{1-F\left(x;\lambda,\alpha,\theta\right)}=\frac{2\alpha x^{-2\alpha-1}\left(1+\lambda-2\lambda e^{-\frac{1}{\theta}x^{-2\alpha}}\right)}{\theta\left[e^{\frac{1}{\theta}x^{-2\alpha}}-(1+\lambda)+\lambda e^{-\frac{1}{\theta}x^{-2\alpha}}\right]}.$$

Figure 2 shows the HRF for the TPIRD at different values of the parameters (λ,α,θ). From Figure 2, the

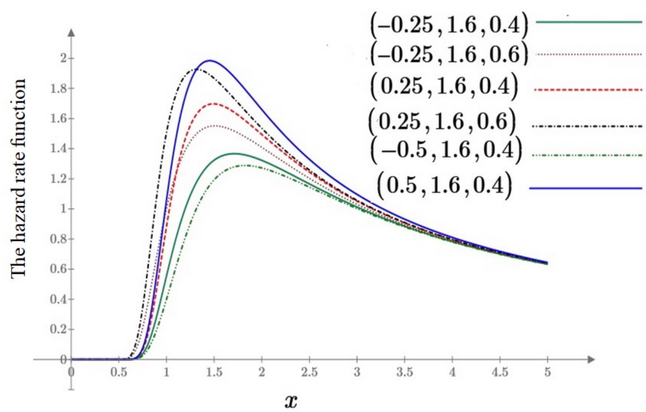

Figure 2: The HRF of the TPIRD for different parameter values $(\lambda, \alpha,\ \theta)$.

HRF of the TPIRD is concave downward for the different values of parameters.

4 Incomplete Moments and Inequality Curves

The incomplete moments of TPIRD is as follows

$$\begin{aligned}I_k(t) &= \int_0^t x^k f(x;\lambda,\alpha,\theta)dx\\ &= \left(\frac{1}{\theta}\right)^{\frac{k}{2\alpha}}\left[(1+\lambda)\,\Gamma\left(1-\frac{k}{2\alpha},\frac{1}{\theta t^{2\alpha}}\right)-\lambda(2^{\frac{k}{2\alpha}}\ )\Gamma\left(1-\frac{k}{2\alpha},\frac{2}{\theta t^{2\alpha}}\right)\ \right],\qquad 2\alpha>k.\end{aligned}$$

where $\Gamma(k,t)$ is the upper incomplete gamma function (IGF).

$$\Gamma\left(k,t\right)=\int_t^\infty y^{k-1}e^{-y}dy.$$

The conditional moments of TPIRD is as follows

$$\begin{aligned}C_k(t) &= \int_t^\infty x^k f(x:\lambda,\alpha,\theta)dx\\ &= \left(\frac{1}{\theta}\right)^{\frac{k}{2\alpha}}\left[(1+\lambda)\,\gamma\left(1-\frac{k}{2\alpha},\frac{1}{\theta t^{2\alpha}}\right)-\lambda(2^{\frac{k}{2\alpha}}\ )\gamma\left(1-\frac{k}{2\alpha},\frac{2}{\theta t^{2\alpha}}\right)\ \right],\quad 2\alpha>k,\end{aligned}$$

where $\gamma(k,t)$ is the lower IGF.

$$\gamma\left(k,t\right)=\int_0^t y^{k-1}e^{-y}dy.$$

A non-negative continuous R.V. X has PDF $f(x)$ and CDF $F(x)$, then Lorenz and Bonferroni curves are defined by

$$L_p=\frac{1}{\mu}\int_0^q xf\left(x\right)dx \quad \text{and} \quad B_p=\frac{1}{p\mu}\int_0^q xf\left(x\right)dx,$$

where $\mu=E(x)$ and $q=F^{-1}\left(p\right)$ for $0<p<1$. So, when X$\sim$ TPIRD(α,θ,λ) then the Lorenz and Bonferroni curves are given by

$$L_p=\frac{(1+\lambda)\,\Gamma\left(1-\frac{1}{2\alpha},\frac{1}{\theta q^{2\alpha}}\right)-\lambda(2^{\frac{1}{2\alpha}}\ )\gamma\left(1-\frac{1}{2\alpha},\frac{2}{\theta q^{2\alpha}}\right)}{\left[1+\lambda\left(1-2^{\frac{1}{2\alpha}}\right)\right]\Gamma\left(1-\frac{1}{2\alpha}\right)},$$

and

$$B_p=\frac{(1+\lambda)\,\Gamma\left(1-\frac{1}{2\alpha},\frac{1}{\theta q^{2\alpha}}\right)-\lambda(2^{\frac{1}{2\alpha}}\ )\Gamma\left(1-\frac{1}{2\alpha},\frac{2}{\theta q^{2\alpha}}\right)}{p\left[1+\lambda\left(1-2^{\frac{1}{2\alpha}}\right)\right]\Gamma\left(1-\frac{1}{2\alpha}\right)}.$$

5 Order Statistics

Let $X_1,X_2,\cdots,\ X_n$ be a random sample of size n from TPIRD. Let $X_{1:n}\leq X_{2:n}\leq\cdots\leq X_{n:n}$ denote the order statistics. Then PDF of r-th O. S. is given by David and Nagaraja [28] for $1\leq r\leq n$

$$f_{r:n}(x)=C_{r:n}\left[F(x)\right]^{r-1}\left[1-F(x)\right]^{n-r}f(x),\quad x\geq 0,$$

where $C_{r:n}=\frac{n!}{(r-1)!(n-r)!}$. Therefore, the PDF of r-th order statistics, smallest and largest order statistics from TPIRD are given by

$$\begin{aligned} f_{r:n}(x) &= C_{r:n}\frac{2\alpha}{\theta\ x^{2\alpha+1}}\left(1+\lambda-\lambda e^{-\frac{1}{\theta x^{2\alpha}}}\right)^{r-1}\left[1-\left(1+\lambda-\lambda e^{-\frac{1}{\theta x^{2\alpha}}}\right)e^{-\frac{1}{\theta x^{2\alpha}}}\right]^{n-r} \\ &\quad \times\left(1+\lambda-2\lambda e^{-\frac{1}{\theta x^{2\alpha}}}\right)e^{-\frac{r}{\theta x^{2\alpha}}} \\ &= \frac{2\alpha C_{r:n}}{\theta\ x^{2\alpha+1}}\sum_{k=0}^{n-r}(-1)^k\left(1+\lambda-\lambda e^{-\frac{1}{\theta x^{2\alpha}}}\right)^{k+r-1}\left(1+\lambda-2\lambda e^{-\frac{1}{\theta x^{2\alpha}}}\right)e^{-\frac{(k+r)}{\theta x^{2\alpha}}}. \end{aligned} \tag{11}$$

From (11), one can derive

- The PDF of the $X_{n:n}=\max\left(X_1,X_2,\cdots,X_n\right)$, when $r=n$, as follows

$$f_{n:n}\left(x\right)=\frac{2n\alpha}{\theta\ x^{2\alpha+1}}\left(1+\lambda-\lambda e^{-\frac{1}{\theta x^{2\alpha}}}\right)^{n-1}\left(1+\lambda-2\lambda e^{-\frac{1}{\theta x^{2\alpha}}}\right)e^{-\frac{n}{\theta x^{2\alpha}}}.$$

- The PDF of the $X_{1:n}=\min\left(X_1,X_2,\cdots,X_n\right)$, when $r=1$, as follows

$$f_{1:n}\left(x\right)=\frac{2n\alpha}{\theta\ x^{2\alpha+1}}\left[1-\left(1+\lambda-\lambda e^{-\frac{1}{\theta x^{2\alpha}}}\right)e^{-\frac{1}{\theta x^{2\alpha}}}\right]^{n-1}\left(1+\lambda-2\lambda e^{-\frac{1}{\theta x^{2\alpha}}}\right)e^{-\frac{1}{\theta x^{2\alpha}}}.$$

The PDF for the first and last order statistics can be used in reliability analysis to describe the lifetimes for the series and parallel systems with n components.

6 Parameters Estimation

In this section, the unknown parameters of the TPIRD are estimated using the maximum likelihood method, with both point and interval estimates based on complete sample.

6.1 Maximum Likelihood Estimation

Let $X_1,\ X_2,\ \cdots,\ X_n$ be a sample of size n from TPIRD. Then, the likelihood function is given as

$$\begin{aligned} L(\lambda,\alpha,\theta,x) &= \prod_{i=1}^{n} f(x_i,\lambda,\alpha,\theta) = \prod_{i=1}^{n} \frac{2\alpha}{\theta\ x_i^{2\alpha+1}}\left[1+\lambda-2\lambda e^{-\frac{1}{\theta x_i^{2\alpha}}}\right] e^{-\frac{1}{\theta x_i^{2\alpha}}} \\ &= \prod_{i=1}^{n} \exp\left\{\ln(2\alpha)-\ln(\theta)-(2\alpha+1)\ln(x_i)-\frac{1}{\theta x_i^{2\alpha}}+\ln\left(1+\lambda-2\lambda e^{-\frac{1}{\theta x_i^{2\alpha}}}\right)\right\}. \quad (12) \end{aligned}$$

The log-likelihood function of Equation (12) is

$$\ell = n\ln(2\alpha)-n\ln(\theta)-(2\alpha+1)\sum_{i=1}^{n}\ln(x_i)-\frac{1}{\theta}\sum_{i=1}^{n}x_i^{-2\alpha}+\sum_{i=1}^{n}\ln\left(1+\lambda-2\lambda e^{-\frac{1}{\theta x_i^{2\alpha}}}\right). \quad (13)$$

The ML estimation of the parameters $\Theta=(\lambda,\alpha,\theta)$ are obtained by differentiating the Equation (13) with respect to the parameters λ,α and θ and setting the result to zero. We have the following normal equations.

$$\frac{\partial\ell}{\partial\lambda} = \sum_{i=1}^{n}\frac{1-2e^{-\frac{1}{\theta}x_i{}^{-2\alpha}}}{1+\lambda-2\lambda e^{-\frac{1}{\theta}x_i^{-2\alpha}}}, \quad (14)$$

$$\frac{\partial\ell}{\partial\alpha} = \frac{n}{\alpha}-2\sum_{i=1}^{n}\log(x_i)+\frac{2}{\theta}\sum_{i=1}^{groupn}x_I^{-2\alpha}\log(x_i)-\frac{4\lambda}{\theta}\sum_{i=1}^{n}\frac{x_i^{-2\alpha}e^{-\frac{1}{\theta}x_i{}^{-2\alpha}}\log(x_i)}{1+\lambda-2\lambda e^{-\frac{1}{\theta}x_i{}^{-2\alpha}}}, \quad (15)$$

$$\frac{\partial\ell}{\partial\theta} = -\frac{n}{\theta}+\frac{1}{\theta^2}\sum_{i=1}^{n}x_i^{-2\alpha}-\frac{2\lambda}{\theta^2}\sum_{i=1}^{n}\frac{x_i{}^{-2\alpha}e^{-\frac{1}{\theta}x_i{}^{-2\alpha}}}{1+\lambda-2\lambda e^{-\frac{1}{\theta}x_i^{-2\alpha}}}. \quad (16)$$

By solving the Equations, (14)–(16), numerically for λ,α and θ, ML estimates can be obtained.

6.2 Asymptotic Confidence Bounds

In this subsection, we derive the asymptotic confidence intervals (ACI) for the parameters under the conditions $|\hat{\lambda}|\le 1$, $\hat{\alpha}>0$, and $\hat{\theta}>0$. Since the ML estimates of λ,α, and θ do not have closed-form expressions, we construct the intervals using the variance-covariance matrix I^{-1}; see Lawless [29]. Here, I^{-1} denotes the inverse of the observed information matrix, defined as follows:

$$I^{-1} = \begin{pmatrix} -\frac{\partial^2\ell}{\partial\lambda^2} & -\frac{\partial^2\ell}{\partial\lambda\partial\alpha} & -\frac{\partial^2\ell}{\partial\lambda\partial\theta} \\ -\frac{\partial^2\ell}{\partial\alpha\partial\lambda} & -\frac{\partial^2\ell}{\partial\alpha^2} & -\frac{\partial^2\ell}{\partial\alpha\partial\theta} \\ -\frac{\partial^2\ell}{\partial\theta\partial\lambda} & -\frac{\partial^2\ell}{\partial\theta\partial\alpha} & -\frac{\partial^2\ell}{\partial\theta^2} \end{pmatrix}^{-1} = \begin{pmatrix} var(\hat{\lambda}) & cov(\hat{\lambda},\hat{\alpha}) & cov(\hat{\lambda},\hat{\theta}) \\ cov(\hat{\alpha},\hat{\lambda}) & var(\hat{\alpha}) & cov(\hat{\alpha},\hat{\theta}) \\ cov(\hat{\theta},\hat{\lambda}) & cov(\hat{\theta},\hat{\alpha}) & var(\hat{\theta}) \end{pmatrix}$$

where

$$\frac{\partial^2\ell}{\partial\lambda^2} = -\sum_{i=1}^{n}\left(\frac{1-2e^{-\frac{1}{\theta}x_i^{-2\alpha}}}{1+\lambda-2\lambda e^{-\frac{1}{\theta}x_i^{-2\alpha}}}\right)^2,$$

$$\frac{\partial^2\ell}{\partial\lambda\partial\alpha} = -\frac{4}{\theta}\sum_{i=1}^{n}\frac{e^{-\frac{1}{\theta}x_i^{-2\alpha}}x_i^{-2\alpha}\log(x_i)}{\left(1+\lambda-2\lambda e^{-\frac{1}{\theta}x_i{}^{-2\alpha}}\right)^2},$$

$$\frac{\partial^2\ell}{\partial\lambda\partial\theta} = -\frac{2}{\theta^2}\sum_{i=1}^{n}\frac{x_i^{-2\alpha}e^{-\frac{1}{\theta}x_i^{-2\alpha}}}{\left(1+\lambda-2\lambda e^{-\frac{1}{\theta}x_i{}^{-2\alpha}}\right)^2},$$

$$\frac{\partial^2\ell}{\partial\alpha^2} = -\frac{n}{\alpha^2}-\frac{4}{\theta}\sum_{i=1}^{n}x_i^{-2\alpha}\log^2(x_i)-\frac{8\lambda(1+\lambda)}{\theta^2}\sum_{i=1}^{n}\frac{x_i^{-4\alpha}e^{-\frac{1}{\theta}x_i^{-2\alpha}}\log^2(groupx_i)}{\left(1+\lambda-2\lambda e^{-\frac{1}{\theta}x_i^{-2\alpha}}\right)^2}$$

$$+\frac{8\lambda}{\theta}\sum_{i=1}^{n}\frac{x_i^{-2\alpha}e^{-\frac{1}{\theta}x_i^{-2\alpha}}\log^2\left(groupx_i\right)}{1+\lambda+2\lambda e^{-\frac{1}{\theta}x_i^{-2\alpha}}},$$

$$\begin{aligned}\frac{\partial^2\ell}{\partial\alpha\partial\theta} &= -\frac{2}{\theta^2}\sum_{i=1}^{n}x_i^{-2\alpha}\log\left(x_i\right)+\frac{8\lambda}{\theta^2}\sum_{i=1}^{n}\frac{x_i^{-2\alpha}e^{-\frac{1}{\theta}x_i^{-2\alpha}}\log\left(x_i\right)}{1+\lambda-2\lambda e^{-\frac{1}{\theta}x_i^{-2\alpha}}}\\ &\quad -\frac{4\lambda\left(1+\lambda\right)}{\theta^3}\times\sum_{i=1}^{n}\frac{x_i^{-4\alpha}e^{-\frac{1}{\theta}x_i^{-2\alpha}}\log\left(x_i\right)}{\left(1+\lambda-2\lambda e^{-\frac{1}{\theta}x_i^{-2\alpha}}\right)^2},\end{aligned}$$

$$\begin{aligned}\frac{\partial^2\ell}{\partial\theta^2} &= \frac{n}{\theta^2}-\frac{2}{\theta^3}\sum_{i=1}^{n}x_i^{-2\alpha}+\frac{4\lambda}{\theta^3}\sum_{i=1}^{n}\frac{x_i^{-2\alpha}e^{-\frac{1}{\theta}x_i^{-2\alpha}}}{1+\lambda-2\lambda e^{-\frac{1}{\theta}x_i^{-2\alpha}}}\\ &\quad -\frac{2\lambda\left(1+\lambda\right)}{\theta^4}\sum_{i=1}^{n}\frac{x_i^{-4\alpha}e^{-\frac{1}{\theta}x_i^{-2\alpha}}}{\left(1+\lambda-2\lambda e^{-\frac{1}{\theta}x_i^{-2\alpha}}\right)^2}.\end{aligned}$$

We can obtain the approximate two-sided $100(1-\delta)\%$ CI of the parameters λ, α and θ as in the following forms:
For λ

$$\hat{\lambda}\pm\ Z_{\frac{\delta}{2}}\sqrt{var\left(\hat{\lambda}\right)}$$

for α

$$\hat{\alpha}\pm\ Z_{\frac{\delta}{2}}\sqrt{var\left(\hat{\alpha}\right)}$$

and for θ

$$\hat{\theta}\pm\ Z_{\frac{\delta}{2}}\sqrt{var\left(\hat{\theta}\right)}$$

where $Z_{\frac{\delta}{2}}$ is the upper $\frac{\delta}{2}$-th percentile of the standard normal distribution.

7 Applications

In this section, we analyze three real datasets using the TPIRD$(\lambda, \alpha, \theta)$ model and compare its performance with other fitted models, including PIRD, TIRD, IRD and RD. The comparison is carried out by several goodness-of-fit measures, such as

- Kolmogorov–Smirnov (K–S) statistic:

$$K-S=sup_x\left|F_m\left(x\right)-\hat{F}\left(x\right)\right|,$$

- Akaike Information Criterion (AIC) [Akaike, [30]]:

$$AIC=2k-2\ell,$$

- Corrected AICC

$$AAIC=AIC+\frac{2k(k+1)}{m-k+1},$$

- Bayesian Information Criterion (BIC)

$$BIC=k\ln\left(m\right)-2\ell,$$

In addition to Hannan–Quinn IC (HQIC) and Schwarz IC (SIC) [Schwarz, [31]]. Further, the coefficient of determination (R^2) and root mean square error (RMSE) are evaluated to assess model adequacy.

$$R^2 = \frac{\sum_{i=1}^{m}\left(\widehat{F}(x_i)-\bar{F}\right)^2}{\sum_{i=1}^{m}\left(\widehat{F}(x_i)-\bar{F}\right)^2+\sum_{i=1}^{m}\left(F_n(x_i)-\widehat{F}(x_i)\right)^2},$$

$$RMSE = \left[\frac{1}{m}\sum_{i=1}^{m}\left(F_m(x_i)-\widehat{F}(x_i)\right)^2\right]^{1/2}.$$

where k and m are the number of parameters and sample size, whereas $\widehat{F}(x)$ and $F_m(x)$ stand for estimated CDF and empirical CDF respectively,

$$\bar{F}(x) = \frac{1}{m}\sum_{i=1}^{m}\widehat{F}(x_i), \qquad F_m(x) = \frac{1}{m}\sum_{i=1}^{m} I\left(x_{(i)} \le x\right),$$

and

$$I\left(x_{(i)} \le x\right) = \begin{cases} 1, & \text{if } x_{(i)} \le x \\ 0, & \text{otherwise} \end{cases}$$

Models with minimum values of the -2ℓ, AIC, AAIC, BIC, HQIC, K-S and RMSE and maximum values of the of R^2 and p are assumed superior in terms of fit to the observed data.

Example 1 *The rapid spread of the coronavirus (COVID-19) from China to the rest of the world has made it critically important to analyze the numbers of confirmed cases and deaths. The following dataset, obtained from the National Vital Statistics in Egypt, records the total number of COVID-19 deaths from the onset of the pandemic in Egypt up to May 27, 2020. The data are presented as follows:*

1	2	2	2	2	2	4	6	6	7	8
10	14	19	21	21	24	30	36	40	41	46
52	52	66	71	78	85	103	118	135	146	159
164	178	183	196	205	224	239	250	264	264	276
294	307	337	359	380	392	406	415	429	436	452
469	482	503	514	525	533	544	556	571	592	612
630	645	659	680	696	707	735	764	783	797	

https://ourworldindata.org/coronavirus/country/egypt.country= EGY. [32].

Table 2 gives MLEs of parameters, K-S Statistics and corresponding p-value for TPIRD, PIRD, TIRD, IRD and RD. The values of the log-likelihood functions (LF), AIC, AICC, BIC, HQIC, RMSE and R^2 are in Table 3. Same pattern for data set 2 and data set 3 are used for comparison.

Table 2: Estimates, K-S Statistics, and p-values for different distributions.

Models	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\theta}$	K-S	p-value
TPIRD	-0.593	0.265	0.20	0.2041449	0.0029656
PIRD		0.242	0.167	0.2102475	0.0019973
TIRD	-0.822		0.034	0.7226885	0.0000000
IRD			0.032	0.7473802	0.0000000
RD			3.03574×10^4	0.9740874	0.0000000

Table 3: The estimates and goodness- of fit measures for different distributions.

Models	ℓ	AIC	AICC	BIC	HQIC	RMSE	R^2
TPIRD	-522.259	1050.52	1050.85	1057.51	1053.31	0.0983110	0.8777030
PIRD	-524.496	1052.99	1053.16	1057.65	1054.86	0.1044234	0.8591370
TIRD	-804.744	1613.49	1613.65	1618.15	1615.35	0.4249275	0.3525204
IRD	-831.743	1665.49	1665.54	1667.82	1666.42	0.4414023	0.3097478
RD	-723.3467	1448.69	1448.75	1451.02	1449.62	0.5712662	0.0002027

On comparing both Tables 2 and 3, we come across TPIRD gives better fit to the data over PIRD, TIRD, IRD and RD.

The variance and covariance matrix is given as

$$I^{-1} = \begin{pmatrix} 0.041 & -2.659 \times 10^{-4} & -0.005 \\ -2.659 & 4.535 \times 10^{-4} & -5.13 \times 10^{-4} \\ -0.005 & -5.13 \times 10^{-4} & 0.002 \end{pmatrix}$$

Then the 95% CI for λ, α, and θ for TPIRD are $(-0.99, -0.197)$, $(0.223, 0.307)$, and $(0.117, 0.283)$, respectively. Figures 3 and 4 show that the LF functions represent unique solutions.

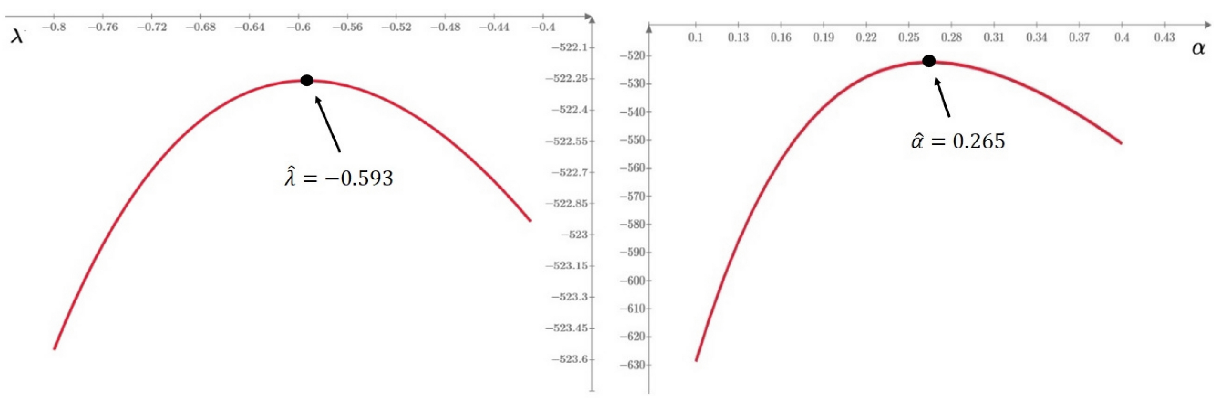

Figure 3: The profile of the log-LF of λ and α.

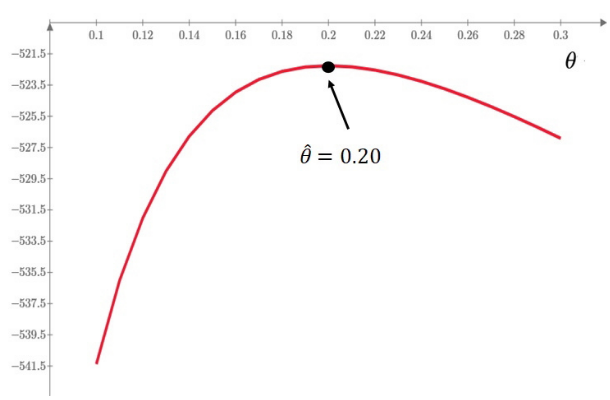

Figure 4: The profile of the log-LF of θ.

Example 2 *The following data represents the remission times (in months) of a random sample of 128 bladder cancer patients, see Lee and Wang [33]:*

0.08	0.2	0.4	0.5	0.51	0.81	0.9	1.05	1.19	1.26	1.35	1.40
1.46	1.76	2.02	2.02	2.07	2.09	2.23	2.26	2.46	2.54	2.62	2.64
2.69	2.69	2.75	2.83	2.87	3.02	3.25	3.31	3.36	3.36	3.48	3.52
3.57	3.64	3.70	3.82	3.88	4.18	4.23	4.26	4.33	4.34	4.40	4.50
4.51	4.87	4.98	5.06	5.09	5.17	5.32	5.32	5.34	5.41	5.41	5.49
5.62	5.71	5.85	6.25	6.54	6.76	6.93	6.94	6.97	7.09	7.26	7.28
7.32	7.39	7.59	7.62	7.63	7.66	7.87	7.93	8.26	8.37	8.53	8.65
8.66	9.02	9.22	9.47	9.74	10.06	10.34	10.66	10.75	11.25	11.64	11.79
11.98	12.02	12.03	12.07	12.63	13.11	13.29	13.80	14.24	14.76	14.77	14.83
15.96	16.62	17.12	17.14	17.36	18.10	19.13	20.28	21.73	22.69	23.63	25.74
25.82	26.31	32.15	34.26	36.66	43.01	46.12	79.05				

Table 4: MLEs of the parameters and the K-S, and p-value.

Models	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\theta}$	K-S	p-value
TPIRD	-0.856	0.418	0.640	0.1243370	0.0352831
PIRD		0.376	0.411	0.1409818	0.0111347
TIRD	-0.954		1.788	0.6810746	0.0000000
IRD			1.620	0.7502318	0.0000000
RD			197.277	0.8282699	0.0000000

Table 5: The log likelihood, AIC, AICC, BIC, RMSE and R^2.

Models	ℓ	AIC	AICC	BIC	HQIC	RMSE	R^2
TPIRD	-436.678	879.356	879.549	887.912	882.832	0.0735509	0.9083769
PIRD	-444.001	892.002	892.097	897.705	894.319	0.0869716	0.8642517
TIRD	-710.198	1424.40	1424.49	1430.1	1426.71	0.4368467	0.2111173
IRD	-774.342	1550.68	1550.71	1553.54	1551.84	0.4723724	0.1537383
RD	-491.266	984.531	984.563	987.383	985.690	0.5239529	0.0078796

On comparing both Tables 4 and 5, we come across TPIRD gives better fit to the data over PIRD, TIRD, IRD and RD. The variance and covariance matrix is given as

$$I^{-1} = \begin{pmatrix} 0.009 & 7.381\times10^{-5} & -0.003 \\ 7.381\times10^{-6} & 5.508\times10^{-4} & -4.854\times10^{-4} \\ -0.003 & -4.854\times10^{-4} & 0.005 \end{pmatrix}$$

Then the 95% CI for λ, α, and θ for TPIRD are $(-1.039, -0.673)$, $(0.372, 0.464)$ and $(0.497, 0.783)$, respectively. Figures 5 and 6 show that the LF has shown unique solutions.

Example 3 *The Wheaton River dataset, presented in Table 6, consists of $n = 72$ observations of flood peak exceedances (in m^3/s) for the Wheaton River, Yukon Territory, Canada, covering the years 1958 to 1984. The values are rounded to one decimal place. This dataset was previously analyzed by Akinsete et al. [34].*

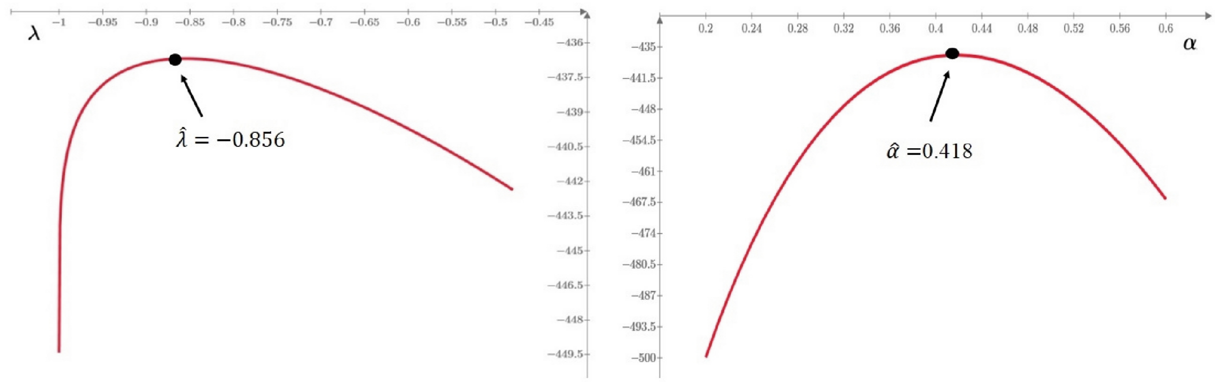

Figure 5: The profile of the log-LF of λ and α.

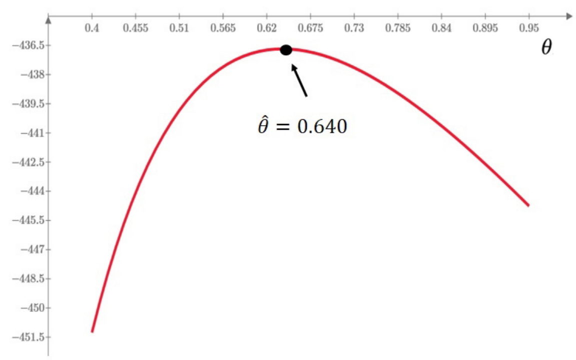

Figure 6: The profile of the log-LF of θ.

1.7	1.4	0.6	9.0	5.6	1.5	2.2	18.7	2.2	1.7	30.8	2.5
14.4	8.5	39.0	7.0	13.3	27.4	1.1	25.5	0.3	20.1	4.2	1.0
0.4	11.6	15.0	0.4	25.5	27.1	20.6	14.1	11.0	2.8	3.4	20.2
5.3	22.1	7.3	14.1	11.9	16.8	0.7	1.1	22.9	9.9	21.5	5.3
1.9	2.5	1.7	10.4	27.6	9.7	13.0	14.4	0.1	10.7	36.4	27.5
12.0	1.7	1.1	30.0	2.7	2.5	9.3	37.6	0.6	3.6	64.0	27.0

Table 6: MLEs of the parameters and the K-S, and p-value.

Models	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\theta}$	K-S	p-value
TPIRD	-0.729	0.355	0.747	0.1496147	0.0724578
PIRD		0.326	0.502	0.1531574	0.0618067
TIRD	-0.913		2.242	0.5947902	0.0000000
IRD			1.931	0.6553894	0.0000000
RD			298.063	0.8634650	0.0000000

Table 7: The log likelihood, AIC, AICC, BIC, RMSE and R^2.

Models	ℓ	AIC	AICC	BIC	HQIC	RMSE	R^2
TPIRD	-264.9920	535.984	536.337	542.814	538.703	0.0692795	0.9386887
PIRD	-267.0189	538.038	538.212	542.591	539.851	0.0738902	0.9266714
TIRD	-428.8753	861.751	861.924	866.304	863.563	0.3841004	0.3615082
IRD	-457.8396	917.679	917.736	919.956	918.586	0.4183843	0.2818210
RD	-302.8378	607.676	607.733	609.952	608.582	0.5321698	0.0051323

On comparing both Tables 6 and 7, we come across TPIRD gives better fit to the data over PIRD, TIRD, IRD and RD. The variance and covariance matrix is given as

$$I^{-1} = \begin{pmatrix} 0.055 & 5.238 \times 10^{-4} & -0.028 \\ 5.238 \times 10^{-4} & 8.668 \times 10^{-4} & -9.479 \times 10^{-4} \\ -0.028 & -9.479 \times 10^{-4} & 0.025 \end{pmatrix}$$

Then the 95% CI for λ, α, and θ for TPIRD are $(-1.187, -0.271)$, $(0.298, 0.413)$ and $(0.440, 1.054)$, respectively. Figures 7 and 8 show that the LF has corresponded to unique solutions.

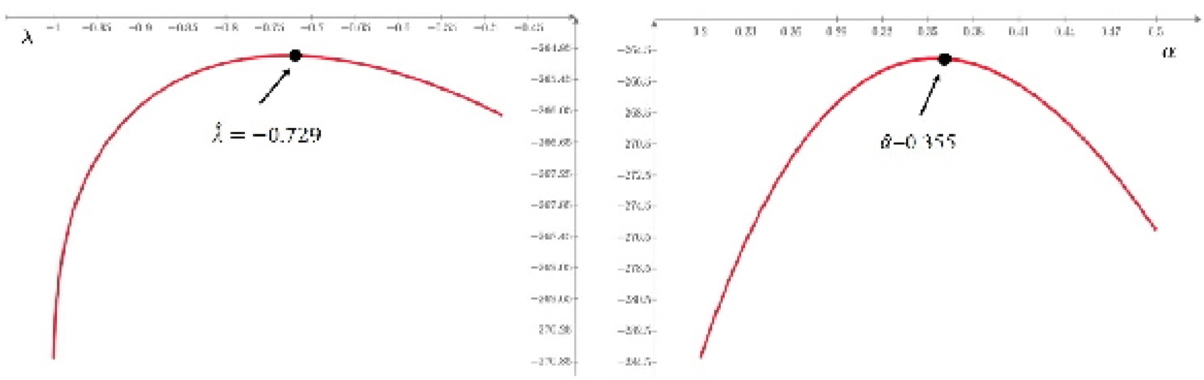

Figure 7: The profile of the log-LF of λ and α.

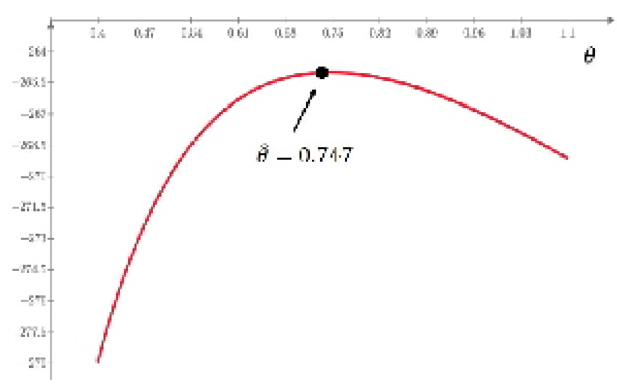

Figure 8: The profile of the log-LF of θ.

8 Conclusion

In this study, we introduced the transmuted PIRD and investigated several of its key statistical properties. The model parameters were estimated using the ML method. To evaluate its practical applicability, three real-life datasets were analyzed, and a range of model selection criteria was employed to compare performance. The findings demonstrate that the proposed distribution consistently provides a superior fit relative to existing probability models in literature. For future research, this distribution may be further extended to enhance its modeling flexibility, for instance, by developing its length-biased, cubic-transmuted, or exponentiated variants, or by exploring Bayesian estimation techniques.

Acknowledgment. The authors extend their appreciation to the Deanship of Scientific Research, Islamic University of Madinah, Saudi Arabia, for funding this research work.

References

[1] V. G. H. Voda, On the inverse Rayleigh distributed random variable, Rep. Stat. Appl. Res., 19(1972), 13–21.

[2] M. Gharraph, Comparison of estimators of location measures of an inverse Rayleigh distribution, Egypt. Statist. J., 37(1993), 295–309.

[3] A. Soliman, E. A. Amin, and A. A. Abd-El Aziz, Estimation and prediction from inverse Rayleigh distribution based on lower record values, Appl. Math. Sci., 4(2010), 3057–3066.

[4] S. Rehman and I. S. Dar, Exponentiated Inverse Rayleigh Distribution, MPhil Thesis, 2015.

[5] M. A. U. Haq, Kumaraswamy exponentiated inverse Rayleigh distribution, Math. Theory Model., 6(2016), 93–104.

[6] K. Fatima and S. P. Ahmad, Preference of priors for the generalized inverse Rayleigh distribution under different loss functions, J. Stat. Appl. Prob. Lett., 2(2017), 73–90.

[7] M. Shala and F. Merovci, A new three-parameter inverse Rayleigh distribution: Simulation and application to real data, Symmetry, 16(2024), 634.

[8] M. Asif, M. Raftab, M. Usman, and A. A. Khan, Analysis of parameters of the exponentiated inverse Rayleigh distribution under the Bayesian framework, Kuwait J. Sci., 52(2025), 100424.

[9] J. M. A. Nashaat, Estimation of two parameter powered inverse Rayleigh distribution, Pak. J. Statist., 36(2020), 117–133.

[10] A. Mustafa and M. I. Khan, The length-biased powered inverse Rayleigh distribution with applications, J. Appl. Math. Inform., 40(2022), 1–13.

[11] M. I. Khan, Dual generalized order statistics with moments properties using powered inverse Rayleigh distribution, J. Stat. Manag. Syst., 25(2022), 2087–2099.

[12] M. I. Khan and A. Mustafa, Powered inverse Rayleigh distribution using DUS transformation, Int. J. Anal. Appl., 21(2023), 61.

[13] M. I. Khan, Order statistics and actuarial measures from powered inverse Rayleigh distribution, Eur. J. Pure Appl. Math., 17(2024), 2182–2195.

[14] R. C. Gupta, P. L. Gupta, and R. D. Gupta, Modeling failure time data by Lehman alternatives, Commun. Statist. Theory Methods, 27(1998), 887–904.

[15] D. Kumar, U. Singh, and S. K. Singh, A method of proposing new distribution and its application to bladder cancer patients' data, J. Stat. Appl. Prob. Lett., 2(2015), 235–245.

[16] A. Mahdavi and D. Kundu, A new method for generating distributions with an application to exponential distribution, Commun. Statist. Theory Methods, 46(2017), 6543–6557.

[17] W. T. Shaw and I. R. C. Buckley, The alchemy of probability distributions: beyond Gram-Charlier expansions, and a skew-kurtotic-normal distribution from a rank transmutation map, Research report, 2007.

[18] G. R. Aryal and C. P. Tsokos, On the transmuted extreme value distribution with application, Nonlinear Anal., 71(2009), 1401–1407.

[19] G. R. Aryal and C. P. Tsokos, Transmuted Weibull distribution: A generalization of the Weibull probability distribution, Eur. J. Pure Appl. Math., 4(2011), 89–102.

[20] A. Ahmad, S. P. Ahmad, and A. Ahmad, Transmuted inverse Rayleigh distribution: A generalization of the inverse Rayleigh distribution, Math. Theory Model., 4(2014), 90–98.

[21] K. K. Das, On some generalized transmuted distributions, Int. J. Sci. Eng. Res., 6(2015), 1686–1691.

[22] M. Bourguignon, I. Ghosh, and G. M. Cordeiro, General results for the transmuted family of distributions and new models, J. Probab. Stat., (2016), Art. ID 7208425.

[23] U. K. Abdullahi and T. G. Ieren, On the inferences and applications of transmuted exponential Lomax distribution, Int. J. Adv. Stat. Prob., 6(2018), 30–36.

[24] M. M. Gharaibeh and A. I. Al-Omari, Transmuted Ishita distribution and its applications, J. Stat. Appl. Prob., 8(2019), 67–81.

[25] M. M. Rahman, B. Al-Zahrani, S. H. Shahbaz, and M. Q. Shahbaz, Transmuted probability distributions: A review, Pak. J. Statist. Oper. Res., 16(2020), 83–94.

[26] M. I. Khan and A. Mustafa, The transmuted power hazard rate distribution and its applications, Int. J. Math. Comput. Sci., 17(2022), 1697–1713.

[27] M. I. Khan, Moments of ordered random variates for transmuted power hazard distribution, J. Appl. Math. Inform., 41(2023), 1047–1056.

[28] H. A. David and H. N. Nagaraja, Order Statistics, Third Edition, John Wiley and Sons, New York, 2003.

[29] J. F. Lawless, Statistical Models and Methods for Lifetime Data, Second Edition, John Wiley and Sons, New York, 20(2003), 1108–1113.

[30] H. Akaike, A new look at the statistical model identification, IEEE Trans. Automat. Control, 19(1974), 716–723.

[31] G. Schwarz, Estimating the dimension of a model, Ann. Statist., 6(1978), 461–464.

[32] Our World in Data, Coronavirus (COVID-19) cases in Egypt, https://ourworldindata.org/coronavirus/country/egypt?country=EGY.

[33] E. T. Lee and J. W. Wang, Statistical Methods for Survival Data Analysis, Third Edition, Wiley, New York, 2003.

[34] A. Akinsete, F. Famoye, and C. Lee, The beta-Pareto distribution, Statistics, 42(2008), 547–563.