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Fuzzy b-Normed Spaces Are Fuzzy Normable*

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Abstract

We introduce the notion of fuzzy b-normed space as a generalization of the concept of fuzzy normed space and also as a fuzzy counterpart of the classical concept of quasi-normed space in the sense of Hyers and Bourgin. We show that every fuzzy b-normed space is fuzzy normable and deduce that every fuzzy b-Banach space admits the structure of a fuzzy Banach space. These facts reveal striking differences with what happens in classical analysis, where there are emblematic examples of quasi-Banach spaces that are not normable.

1 Introduction and Preliminaries

Throughout this paper by $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{N}$ we will denote the set of real numbers, the set of non-negative real numbers and the set of positive integer numbers, respectively. Our notation and terminology is standard. All topologies are assumed to be Hausdorff. As usual, we will say that a topological space (X, τ) is metrizable if there is a metric on X whose induced topology agrees with τ. In this case, we will say that τ is a metrizable topology.

A classical and fundamental theorem in Functional Analysis, due to Kolmogorov [12], states that a topological vector space is normable if and only if it is locally bounded and locally convex. This result admits a nice extension that can be formulated as follows: a topological vector space is quasi-normable if and only if it is locally bounded ([3, p. 445], [10, p. 77], [13, p. 159], [11, p. 245]). Therefore, the spaces $l^p(\mathbb{R})$ and $L^p([0,1])$, $0 < p < 1$, yield classical examples of quasi-normed spaces that are not normable. An interesting reminder on the origins of quasi-normed spaces may be found in [19, p. 66].

Let us recall ([3, 4, 10, 11, 19]) that a quasi-norm on a (real) vector space X is a function $\|.\| : X \to \mathbb{R}^+$ that satisfies the following conditions for all $x, y \in X$ and $\lambda \in \mathbb{R}$:

(qn0) $\|x\| = 0$ if and only if $x = \theta$;

(qn1) $\|\lambda x\| = |\lambda| \, \|x\|$;

(qn2) there is a constant $K \geq 1$ such that $\|x + y\| \leq K(\|x\| + \|y\|)$.

A quasi-normed space is a pair $(X, \|.\|)$ such that X is a (real) vector space and $\|.\|$ is a quasi-norm on X.

It is clear that each norm is a quasi-norm and, hence, each normed space is a quasi-normed space.

The so-called b-metric spaces, as defined and examined by Czerwik in [5, 6], constitute the expected metric extension of the notion of a quasi-normed space.

Let X be a (non-empty) set. We say that a function $d : X \times X \to \mathbb{R}^+$ is a b-metric on X if it satisfies the following conditions for all $x, y, z \in X$:

(bm0) $d(x, y) = 0$ if and only if $x = y$;

(bm1) $d(x, y) = d(y, x)$;

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(bm2) there is a constant $K \geq 1$ such that $d(x, y) \leq K(d(x, z) + d(z, y))$.

By a b-metric space we mean a pair (X, d) such that X is a (non-empty) set and d is a b-metric on X. Obviously, every metric (space) is a b-metric (space).

The b-metric spaces are called quasi-metric spaces by several authors (see [1, 8, 18]). However, since in the realm of general topology the term quasi-metric is used from many time ago to refer a function $d : X \times X \to \mathbb{R}^+$ satisfying conditions (bm0) and (bm2) with $K = 1$ (see [27]), we prefer to use the terms b-metric and b-metric space instead of quasi-metric and quasi-metric space, respectively.

It is well known that, as in the metric case, each b-metric d on X induces a topology τ_d on X defined as follows:

$$\tau_d := \{A \subseteq X : \text{for any } x \in A \text{ there is } \varepsilon > 0 \text{ such that } B_d(x, \varepsilon) \subseteq A\},$$

where $B_d(x, \varepsilon) := \{y \in X : d(x, y) < \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

Of course, the set $B_d(x, \varepsilon)$ is τ_d-open if d is a metric on X. However, Paluszyński and Stempak presented in [18, p. 4310] an example of a b-metric space (X, d) for which there exist $x \in X$ and $\varepsilon > 0$ such that the set $B_d(x, \varepsilon)$ is not τ_d-open.

It is also well known ([1, 8, 15, 18]) that the topology induced by any b-metric is metrizable. If $\| \cdot \|$ is a quasi-norm on X, then the function $d_{\| \cdot \|} : X \times X \to \mathbb{R}^+$ given by $d_{\| \cdot \|}(x, y) = \|y - x\|$ is a b-metric on X and $(X, \tau_{d_{\| \cdot \|}})$ is a topological vector space. In the sequel, the topology $\tau_{d_{\| \cdot \|}}$ will be simply denoted by $\tau_{\| \cdot \|}$ and it will be called the topology induced by $\| \cdot \|$. A quasi-Banach space is a quasi-normed space $(X, \|.\|)$ such that the b-metric space $(X, d_{\| \cdot \|})$ is complete.

A topological vector space (X, τ) is (quasi-)normable provided that there is a (quasi-)norm $\|.\|$ on X such that $\tau = \tau_{\|.\|}$.

In this note we introduce the notions of fuzzy b-normed space and fuzzy b-Banach space as generalizations of the concepts of fuzzy normed space and fuzzy Banach space, respectively, and also as fuzzy counterparts of the concepts of quasi-normed space and quasi-Banach space. In contrast to what happens with respect to the normability of quasi-normal spaces, we will observe that every fuzzy b-normed space is fuzzy normable and deduce that every fuzzy b-Banach space admits the structure of a fuzzy Banach space.

2 Fuzzy b-Normed Spaces

We start this section with the following well-known but fundamental notion to our study.

According to [25], a binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous t-norm if $*$ satisfies the following conditions: (i) $*$ is associative and commutative; (ii) $*$ is continuous; (iii) $a * 1 = a$ for every $a \in [0, 1]$; (iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, with $a, b, c, d \in [0, 1]$.

Let us recall that for each continuous t-norm $*$ it follows $* \leq \wedge$, where $\wedge$ denotes the minimum continuous t-norm, i.e., $a \wedge b = \min\{a, b\}$ for all $a, b \in [0, 1]$.

In [17] (see also [9, 23, 24, 26, 28]), Nădăban introduced the following concepts as natural b-metric generalizations of the classical notions of fuzzy metric and fuzzy metric space as given by Kramosil and Michálek ([14]).

A fuzzy b-metric on a (non-empty) set X is a pair $(M, *)$ such that $*$ is a continuous t-norm and M is a fuzzy set in $X \times X \times \mathbb{R}^+$ satisfying the following conditions for all $x, y, z \in X$:

(fbm0) $M(x, y, 0) = 0$;

(fbm1) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$;

(fbm2) $M(x, y, t) = M(y, x, t)$, for all $t > 0$;

(fbm3) the function $M(x, y, \_) : \mathbb{R}^+ \to [0, 1]$ is left continuous;

(fbm4) there is a constant $K \geq 1$ such that $M(x, y, K(t+s)) \geq M(x, z, t) * M(z, y, s)$, for all $t, s > 0$.

A fuzzy b-metric space is a triple $(X, M, *)$ such that X is a (non-empty) set and $(M, *)$ is a fuzzy b-metric on X. If $K = 1$ we have the notions of fuzzy metric and fuzzy metric space in the sense of Kramosil and Michálek ([14]).

Similarly to the case of fuzzy metric spaces each fuzzy b-metric $(M, *)$ on a set X induces a topology $\tau_{(M,*)}$ defined as follows:

$$\tau_{(M,*)} := \{A \subseteq X : \text{for any } x \in A \text{ there are } \varepsilon \in (0,1) \text{ and } t > 0 \text{ such that } B_M(x, \varepsilon, t) \subseteq A\},$$

where $B_M(x, \varepsilon, t) := \{y \in X : M(x, y, t) > 1 - \varepsilon\}$ for all $x \in X$, $\varepsilon \in (0,1)$ and $t > 0$.

Remark 1 *Given a fuzzy b-metric space $(X, M, *)$, we get that a sequence $(x_n)_n$ in X is $\tau_{(M,*)}$-convergent to $x \in X$ if and only if for each $t > 0$, $\lim_{n\to\infty} M(x, x_n, t) = 1$.*

Remark 2 *It is well known that if $(X, M, *)$ is a fuzzy metric space, then each ball $B_M(x, \varepsilon, t)$ is a $\tau_{(M,*)}$-open set. This fact does not hold for fuzzy b-metric spaces in general (see, e.g., [21, Example 5]).*

Remark 3 *In [21, Theorem 1] it was shown that every fuzzy b-metric space $(X, M, *)$ is metrizable, i.e., there is a metric d on X such that $\tau_{(M,*)} = \tau_d$, where by τ_d we denote the topology induced by d.*

Let us recall that a fuzzy b-metric space $(X, M, *)$ is complete provided that every Cauchy sequence in X is $\tau_{(M,*)}$-convergent, where a sequence $(x_n)_n$ in X is said to be a Cauchy sequence if for each $\varepsilon \in (0,1)$ and $t > 0$ there exists an $n_{\varepsilon,t} \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m \geq n_{\varepsilon,t}$.

The following is a typical example of a fuzzy b-metric space (see, e.g., [17, 23, 26]).

Example 1 *Let (X, d) be a b-metric space with constant K. Denote by M_d the fuzzy set in $X \times X \times \mathbb{R}^+$ given by $M_d(x, y, 0) = 0$ and*

$$M_d(x, y, t) = \frac{t}{t + d(x, y)},$$

for all $t > 0$. Then, for each continuous t-norm $*$, $(X, M_d, *)$ is a fuzzy b-metric space with constant K such that $\tau_{(M_d,*)} = \tau_d$. Furthermore, $(X, M_d, *)$ is complete if and only if (X, d) is complete.

Next, we introduce and discuss our fuzzy counterpart to the notion of quasi-normed space.

Definition 1 *A fuzzy b-norm on a (real) vector space X is a pair $(N, *)$ such that $*$ is a continuous t-norm and N is a fuzzy set in $X \times \mathbb{R}^+$ satisfying the following conditions for all $x, y \in X$ and $\lambda \in \mathbb{R}\backslash\{0\}$:*

(fbn0) $N(x, 0) = 0$;

(fbn1) $N(x, t) = 1$ *for all* $t > 0$ *if and only if* $x = \theta$;

(fbn2) $N(\lambda x, t) = N(x, t/|\lambda|)$, *for all* $t > 0$;

(fbn3) the function $N(x, \_) : \mathbb{R}^+ \to [0,1]$ *is left continuous;*

(fbn4) $\lim_{t\to\infty} N(x, t) = 1$;

(fbn5) there is a constant $K \geq 1$ *such that* $N(x + y, K(t+s)) \geq N(x, t) * N(y, s)$, *for all* $t, s > 0$.

A fuzzy b-normed space is a triple $(X, N, *)$ such that X is a (real) vector space and $(N, *)$ is a fuzzy b-norm on X. If $K = 1$ we have the notions of fuzzy norm and fuzzy normed space, respectively (see, e.g., [2, 7]).

Clearly, every fuzzy b-norm $(N, *)$ on X induces a fuzzy b-metric $(M_N, *)$ on X given by $M_N(x, y, 0) = 0$, and $M_N(x, y, t) = N(y - x, t)$ for all $x, y \in X$ and $t > 0$.

The topology $\tau_{(M_N,*)}$ induced by $(M_N, *)$ will be simply denoted by $\tau_{(N,*)}$.

By a fuzzy b-Banach space we mean a fuzzy b-normed space $(X, N, *)$ such that the fuzzy b-metric space $(X, M_N, *)$ is complete. Then, the notion of fuzzy Banach space is stated in the obvious manner.

Now, we give two paradigmatic examples of fuzzy b-normed spaces.

Example 2 *Let $(X, \|.\|)$ be a quasi-normed space with constant K. Denote by $N_{\|.\|}$ the fuzzy set in $X \times \mathbb{R}^+$ given by $N_{\|.\|}(x, 0) = 0$ for all $x \in X$, and*

$$N_{\|.\|}(x, t) = \frac{t}{t + \|x\|},$$

for all $x \in X$ and $t > 0$. It is straightforward to check that, for any continuous t-norm $*$, the pair $(N_{\|.\|}, *)$ is a fuzzy b-norm on X with constant K, such that $\tau_{(N_{\|.\|},*)} = \tau_{\|.\|}$. Furthermore, $(X, N_{\|.\|}, *)$ is a fuzzy b-Banach space if and only if $(X, \|.\|)$ is a quasi-Banach space.

Example 3 *Let $(X, \|.\|)$ be a quasi-normed space with constant K. Denote by $N_{\|.\|,01}$ the fuzzy set in $X \times \mathbb{R}^+$ given by $N_{\|.\|,01}(x, 0) = 0$ for all $x \in X$, and for all $x \in X$ and $t > 0$, $N_{\|.\|,01}(x, t) = 1$ if $\|x\| < t$ and $N_{\|.\|,01}(x, t) = 0$ if $\|x\| \geq t$. It is straightforward to check that, for any continuous t-norm $*$, the pair $(N_{\|.\|,01}, *)$ is a fuzzy b-norm on X with constant K, such that $\tau_{(N_{\|.\|,01},*)} = \tau_{\|.\|}$. Furthermore, $(X, N_{\|.\|,01}, *)$ is a fuzzy b-Banach space if and only if $(X, \|.\|)$ is a quasi-Banach space.*

In the sequel, we will say that a fuzzy b-normed space $(X, N, *)$ is fuzzy normable if there is a fuzzy norm $(N', *')$ on X such that $\tau_{(N,*)} = \tau_{(N',*')}$ on X, and we will say that a fuzzy b-Banach space $(X, N, *)$ is fuzzy Banach structurable if there is a fuzzy norm $(N', *')$ on X such that $(X, N', *')$ is a fuzzy Banach space and $\tau_{(N,*)} = \tau_{(N',*')}$ on X.

Proposition 1 *Let $(X, N, *)$ be a fuzzy b-normed space. Then, $\tau_{(N,*)}$ is a metrizable topology on X.*

Proof. Consider the fuzzy b-metric space $(X, M_N, *)$ and apply Remark 3. ■

Proposition 2 *Let $(X, N, *)$ be a fuzzy b-normed space. Then, $(X, \tau_{(N,*)})$ is a topological vector space.*

Proof. Let $(\lambda_n)_n$ be a sequence of real numbers convergent to $\lambda \in \mathbb{R}$ with respect to th Euclidean metric on $\mathbb{R}$, and let $(x_n)_n$ be a sequence in X that $\tau_{(N,*)}$-converges to a $x \in X$. Given $\varepsilon \in (0, 1)$ and $t > 0$, there exist $\delta \in (0, 1)$ and $n_0 \in \mathbb{N}$ such that $(1-\delta)*(1-\delta) > 1-\varepsilon$, $|\lambda_n - \lambda| < \varepsilon$, $N(x - x_n, t/2K^2(|\lambda| + \varepsilon)) > 1 - \delta$, and $N(x, t/2K\,|\lambda_n - \lambda|) > 1 - \delta$ for all $n \geq n_0$. Hence,

$$\begin{aligned} N(\lambda x - \lambda_n x_n, t) &= N(\lambda x - \lambda_n x + \lambda_n x - \lambda_n x_n, t) \\ &\geq N(\lambda x - \lambda_n x, t/2K) * N(\lambda_n x - \lambda_n x_n, t/2K) \\ &= N(x, t/2K\,|\lambda_n - \lambda|) * N(x - x_n, t/2K\,|\lambda_n|) \\ &\geq N(x, t/2K\,|\lambda_n - \lambda|) * N(x - x_n, t/2K^2(|\lambda| + \varepsilon)) \\ &\geq (1-\delta)*(1-\delta) > 1 - \varepsilon, \end{aligned}$$

for all $n \geq n_0$. We conclude that $\lim_{n\to\infty} N(\lambda x - \lambda_n x_n, t) = 1$, so, by Remark 1, $(X, \tau_{(N,*)})$ is a topological vector space. ■

Theorem 1 *Every fuzzy b-normed space is fuzzy normable.*

Proof. Let $(X, N, *)$ be a fuzzy b-normed space. It follows from Propositions 1 and 2 that $(X, \tau_{(M,*)})$ is a metrizable topological vector space. Choose, and denote by d_I, any invariant metric on X satisfying $d_I(0, \lambda x) \leq d_I(0, \mu x)$ whenever $\lambda \leq \mu$, and $\tau_{(N,*)} = \tau_{d_I}$ (the existence of such metrics is guaranteed by [22, Theorem 1.24]). Now, following [2, Theorem 8], we have that the triple $(X, N', \cdot)$ is a fuzzy normed space such that $\tau_{(N,*)} = \tau_{(N',\cdot)}$, where by $\cdot$ we denote the product continuous t-norm, and N' is the fuzzy set in $X \times \mathbb{R}^+$ given by $N'(x, 0) = 0$ for all $x \in X$, and

$$N'(x, t) = \frac{1}{1 + d_I(0, x/t)}, \tag{1}$$

for all $x \in X$ and $t > 0$. This finishes the proof. ■

Theorem 2 *Every fuzzy b-Banach space is fuzzy Banach structurable.*

Proof. Let $(X, N, *)$ be a fuzzy b-Banach space and let $(x_n)_n$ be a Cauchy sequence in the fuzzy normed space $(X, N', \cdot)$ as constructed in Theorem 1. By (1), we get that $(x_n)_n$ is a Cauchy sequence in the metric space (X, d_I). Choose an arbitrary $\varepsilon \in (0, 1)$. Since $\tau_{(N,*)} = \tau_{d_I}$ we find a $\delta > 0$ such that $B_d(0, \delta) \subseteq B_{M_N}(0, \varepsilon, \varepsilon)$. Since d_I is invariant, there is $n_0 \in \mathbb{N}$ such that $d_I(0, x_n - x_m) < \delta$ for all $n, m \geq n_0$, so $M_N(x_n, x_m, \varepsilon) > 1 - \varepsilon$ for all $n, m \geq n_0$. Hence, there is $z \in X$ such that $(x_n)_n$ is $\tau_{(N,*)}$-convergent to z, i.e., $\tau_{(N',\cdot)}$-convergent to z. We conclude that $(X, N', \cdot)$ is a fuzzy Banach space. ■

Example 4 *Fix $p \in (0, 1)$. Let $l^p(\mathbb{R})$ be the vector space of all sequences $x := (x_n)_n$ in $\mathbb{R}$ such that $\sum_{n=1}^{\infty} | x_n |^p < \infty$. Then, $(l^p(\mathbb{R}), \|.\|)$ is a quasi-Banach space, with constant $2^{(1/p)-1}$, where*

$$\|x\| = (\sum_{n=1}^{\infty} | x_n |^p)^{1/p},$$

for all $x \in l^p(\mathbb{R})$. Therefore, the triple $(l^p(\mathbb{R}), N_{\|.\|}, *)$ is a fuzzy b-Banach space such that $\tau_{\|.\|} = \tau_{(N_{\|.\|},*)}$, where $(N_{\|.\|}, *)$ is the fuzzy b-norm given in Example 2.

It is well known that the function d_I defined on $l^p(\mathbb{R}) \times l^p(\mathbb{R})$ by $d_I(x, y) = \sum_{n=1}^{\infty} |x_n - y_n|^p$ for all $x, y \in l^p(\mathbb{R})$, is a translation invariant metric on $l^p(\mathbb{R})$ such that $d_I(0, \lambda x) \leq d_I(0, \mu x)$ whenever $\lambda \leq \mu$, and $\tau_{\|.\|} = \tau_{d_I}$. So, $\tau_{(N_{\|.\|},*)} = \tau_{d_I}$. By applying Theorems 1 and 2, we deduce that the triple $(l^p(\mathbb{R}), N', \cdot)$ is a fuzzy Banach space such that $\tau_{(N',\cdot)} = \tau_{(N_{\|.\|},*)}$, where (see the proof of Theorem 1) $N'(x, 0) = 0$ for all $x \in l^p(\mathbb{R})$, and

$$N'(x, t) = \frac{1}{1 + t^{-p} \sum_{n=1}^{\infty} |x_n|^p},$$

for all $x \in l^p(\mathbb{R})$ and $t > 0$.

Remark 4 *It is not difficult to find examples of fuzzy normed spaces that are not quasi-normable. Indeed, choose any metrizable topological vector space (X, τ) that is not locally bounded (see, e.g., [22, Example 1.44]). Then, (X, τ) is not quasi-normable but it admits the structure of a fuzzy normed space ([2, 16, 20]).*

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