

C^* -Algebra Valued $b_{\mathcal{R}}$ -Metric Space, Fixed Point Theorems And Application To Operator Equation*

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Abstract

In the present manuscript, we give the notion of C^* -algebra valued $b_{\mathcal{R}}$ -metric space, which enlarges the class of some metric space presented in the literature. We also give the generalization of the Banach Contraction Principle in the framework of such space and prove fixed point results for some well-known contractions in the existing literature. As an application, we apply our result to examine the existence and uniqueness of the solution of bounded linear operator equation. Moreover, some examples are presented here to illustrate the usability of obtained results.

1 Introduction

In 2014, Shukla [23] extended the class of b -metric space by introducing the concept of partial b -metric space and established some fixed point results. Later in 2015, Ma et al. [26] generalised b -metric space by extending C^* -algebra valued metric space [25] with C^* -algebra valued b -metric space and proved some fixed point results. Recently, Mlaiki et al. [10] gave the notion of C^* -algebra valued partial b -metric space and proved some fixed point result. In the same manuscript the authors checked the existence and uniqueness of solution for Fredholm integral equation. Generalization of Banach contraction principle in C^* -algebra under various structures have been discussed by many researcher for more detail, one can refer to Banerjee and Paul [1], Moeini and Kumar [2], Kumar et al. [3], Massit and Rossafi [5], Asim et al. [7], Kumar et al. [8], Rossafi et al. [9], Dung et al. [11], Chandok et al. [14], Mohanta [18], Mondal et al. [20], Radenovic et al. [21], Omran and Masmali [24] and references therein. In 2020, Rahimi et al. [19] introduces the concept of $\mathcal{R}$ -metric space by introducing a relational contraction on metric space.

In the present manuscript, we give the notion of C^* -algebra valued $b_{\mathcal{R}}$ -metric space, which enlarges the class of some metric space presented in the literature. We also give the generalization of the Banach Contraction Principle in the framework of such space and proved fixed point results for some well-known contractions in the existing literature. As an application, we apply our result to examine the existence and uniqueness of the solution of bounded linear operator equation. Moreover, some examples are presented here to illustrate the usability of obtained results.

2 Preliminaries

We begin this section by discussing a few of the basic concepts of C^* -algebra valued b -metric space (C^* -av- b -MS), followed by some definitions of relational theoretic. Readers should note that, throughout the manuscript $\mathbb{A}$ denotes a unital C^* -algebra with the unity element I , $\theta_{\mathbb{A}}$ represents the zero element of $\mathbb{A}$, $\mathbb{A}^+ = \{\rho \in \mathbb{A} : \rho \succeq \theta_{\mathbb{A}}\}$ with $\|\rho\| = (\rho^* \rho)^{\frac{1}{2}}$, $\mathbb{A}'$ denote the set $\{\xi \in \mathbb{A} : \xi \eta = \eta \xi \text{ for all } \eta \in \mathbb{A}\}$ and $\mathcal{R}$ denotes a non-empty relation on a non-empty set X .

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Definition 1 ([19]) A sequence $\{\rho_n\}$ in an $\mathcal{R}$ -metric space X is called an $\mathcal{R}$ -sequence if $\rho_n \mathcal{R} \rho_{n+k}$ for each $n, k \in \mathbb{N}$.

Definition 2 ([19]) For an $\mathcal{R}$ -metric space $(X, d, \mathcal{R})$, a self map $\Gamma : X \rightarrow X$ is said to be $\mathcal{R}$ -continuous at $\rho \in X$ if for any $\mathcal{R}$ -sequence $\{\rho_n\}_{n \in \mathbb{N}} \subset X$ with $\lim_{n \rightarrow \infty} \rho_n \rightarrow \rho$ we have $\lim_{n \rightarrow \infty} \Gamma \rho_n \rightarrow \Gamma \rho$. Also, Γ is said to be $\mathcal{R}$ -continuous on X if it is $\mathcal{R}$ -continuous at each point of X .

Remark 1 ([19]) Every continuous map is $\mathcal{R}$ -continuous but not conversely.

Definition 3 ([19]) For an $\mathcal{R}$ -metric space $(X, d, \mathcal{R})$, a self map $\Gamma : X \rightarrow X$ is said to be $\mathcal{R}$ -preserving if for every $\rho \mathcal{R} \sigma$ we have $\Gamma \rho \mathcal{R} \Gamma \sigma$.

Definition 4 ([26]) Suppose X is a non-empty set and $\Lambda \in \mathbb{A}'$ such that $\Lambda \succeq I$. A function $d : X \times X \rightarrow \mathbb{A}$ satisfies:

- (i) $d(\rho, \sigma) \succeq \theta_{\mathbb{A}}$ and $d(\rho, \sigma) = \theta_{\mathbb{A}}$ if and only if $\rho = \sigma$;
- (ii) $d(\rho, \sigma) = d(\sigma, \rho)$;
- (iii) $d(\rho, \sigma) \preceq \Lambda(d(\rho, \mu) + d(\mu, \sigma))$ for all $\rho, \sigma, \mu \in X$.

Then, d is called a C^* -algebra valued b -metric and $(X, \mathbb{A}, d)$ is called a C^* -av- b -MS.

Lemma 1 ([24]) Suppose $\mathbb{A}$ is a unital C^* -algebra with unity I . Then, the following hold:

- (i) If $\xi \in \mathbb{A}$ with $\|\xi\| \leq 1/2$, then $(I - \xi)$ is invertible and $\|\xi(I - \xi)^{-1}\| \leq 1$;
- (ii) For any $\rho \in \mathbb{A}$ and $\xi, \eta \in \mathbb{A}^+$ such that $\xi \preceq \eta$, we have $\rho^* \xi \rho$ and $\rho^* \eta \rho$ are positive elements and $\rho^* \xi \rho \preceq \rho^* \eta \rho$;
- (iii) If $\theta_{\mathbb{A}} \preceq \xi \preceq \eta$, then $\|\xi\| \leq \|\eta\|$;
- (iv) If $\xi, \eta \in \mathbb{A}^+$ and $\xi \eta = \eta \xi$, then $\xi \cdot \eta \succeq \theta_{\mathbb{A}}$;
- (v) Let $\xi \in \mathbb{A}'$. If $\eta, \zeta \in \mathbb{A}$ with $\eta \succeq \zeta \succeq \theta_{\mathbb{A}}$ and $(I - \xi) \in (\mathbb{A}')^+$ is an invertible element, then $(I - \xi)^{-1} \eta \preceq (I - \xi)^{-1} \zeta$.

3 Main Results

In this section, we give the notion of C^* -av- $b_{\mathcal{R}}$ -MS and C^* -av- $b_{\mathcal{R}}$ -contractive map and establish some fixed point results.

Definition 5 For a non-empty set X together with a unital C^* -algebra $\mathbb{A}$, and binary relation $\mathcal{R}$, define a function $d : X \times X \rightarrow \mathbb{A}$. Then, $(X, \mathbb{A}, d, \mathcal{R})$ is called C^* -av- $b_{\mathcal{R}}$ -MS if the following are satisfied:

- (i) $(X, \mathbb{A}, d)$ is C^* -av- b -MS.
- (ii) $\mathcal{R}$ is a binary relation on X .

Remark 2 Every C^* -algebra valued $\mathcal{R}$ -metric space is a C^* -av- $b_{\mathcal{R}}$ -MS but not conversely.

Example 1 Let $X = \mathbb{R}$ and $\mathbb{A} = M_n(\mathbb{R})$. Define

$$d(\rho, \sigma) = \text{diag}(c_1 |\rho - \sigma|^p, c_2 |\rho - \sigma|^p, \dots, c_n |\rho - \sigma|^p),$$

where diag denotes the diagonal matrix, $\rho, \sigma \in X$, $c_i > 0$ for all $i = 1, 2, \dots, n$ are constants and $p > 1$. Define $\mathcal{R}$ on X such that $\rho \mathcal{R} \sigma \Leftrightarrow (\rho - \sigma) < \rho$. It is easy to verify that d is C^* -algebra valued $b_{\mathcal{R}}$ -metric and $(X, \mathbb{A}, d, \mathcal{R})$ is a C^* -av- $b_{\mathcal{R}}$ -MS where $\Lambda = 3^p I \in \mathbb{A}$ and $\Lambda \succ I$ with $3^p > 1$. But $|\rho - \sigma|^p \leq |\rho - \mu|^p + |\mu - \sigma|^p$ is impossible for $\rho > \mu > \sigma > 0$. Thus, $(X, \mathbb{A}, d, \mathcal{R})$ is not a C^* -algebra valued $\mathcal{R}$ -metric space.

Example 2 Assume that

$$P = \begin{pmatrix} 10 & 10 \\ 12 & 14 \end{pmatrix}, \quad Q = \begin{pmatrix} 5 & 4 \\ 4 & 4 \end{pmatrix}, \quad R = \begin{pmatrix} -4 & 0 \\ -2 & -2 \end{pmatrix}.$$

Let $X = \{P, Q, R\}$ and $\mathbb{A} = M_2(\mathbb{R})$. Define $d(P, Q) = P + Q$, where P, Q denote the matrix of order 2. Define $\mathcal{R}$ on X such that $PRQ \Leftrightarrow \det(P) > \det(Q)$. Thus, $\det(P) > \det(Q)$, $\det(P) > \det(R)$ and $\det(R) > \det(Q)$. Hence, PRQ , RQ and PRR . It is easy to verify that $(X, \mathbb{A}, d, \mathbb{R})$ is C^* -av- $b_{\mathcal{R}}$ -MS, where $\Lambda = 5^p I$, $p > 1$. But, $d(P, Q) \not\preceq d(P, R) + d(R, Q)$. Thus, $(X, \mathbb{A}, d, \mathcal{R})$ is not a C^* -algebra valued $\mathcal{R}$ -metric space.

Definition 6 For a C^* -av- $b_{\mathcal{R}}$ -MS $(X, \mathbb{A}, d, \mathcal{R})$, a self map $\Gamma : X \rightarrow X$ is said to be a C^* -av- $b_{\mathcal{R}}$ -contractive map if for all $\rho, \sigma \in X$ with $(\rho, \sigma) \in \mathcal{R}$, there exists a $\kappa \in \mathbb{A}$ where $\|\kappa\| < 1$ such that $d(\Gamma\rho, \Gamma\sigma) \preceq \kappa^* d(\rho, \sigma) \kappa$.

Example 3 Let $X = \mathbb{R}$ and $\mathbb{A} = M_2(\mathbb{R})$ with involution on $\mathbb{A}$ define $\kappa^* = \kappa^t$ for all $\kappa \in \mathbb{A}$, where κ^t denotes the transpose of κ . For $\kappa = [\lambda_{ij}]$, let $|\kappa| = \max_{1 \leq i, j \leq 2} |\lambda_{ij}|$. Define

$$d(\rho, \sigma) = \text{diag}(c_1 |\rho - \sigma|^p, c_2 |\rho - \sigma|^p),$$

where diag denotes the diagonal matrix, $\rho, \sigma \in X$, $c_i > 0$ for $i = 1, 2$ are constants and $p > 1$. $\mathcal{R}$ is defined on X such that $\rho \mathcal{R} \sigma \Leftrightarrow (\rho - \sigma) < \rho$. It is easy to verify that d is C^* -algebra valued $b_{\mathcal{R}}$ -metric and $(X, \mathbb{A}, d, \mathcal{R})$ is a C^* -av- $b_{\mathcal{R}}$ -MS. For proving triangle inequality of Definition 5, we need to use following inequality

$$|\rho - \sigma|^p \leq 3^p (|\rho - \mu|^p + |\mu - \sigma|^p),$$

implies that $d(\rho, \sigma) \preceq \Lambda(d(\rho, \mu) + d(\mu, \sigma))$ for all $\rho, \sigma, \mu \in X$, where $\Lambda = 3^p I \in \mathbb{A}$ and $\Lambda \succ I$ with $3^p > 1$. Define a self map $\Gamma : X \rightarrow X$ as

$$\Gamma\rho = \begin{cases} 1/10, & \text{if } \rho > 10, \\ 0, & \text{otherwise.} \end{cases}$$

Following cases arises:

Case (i): If $\rho, \sigma < 10$, then $d(\Gamma\rho, \Gamma\sigma) = d(0, 0) = \theta_{\mathbb{A}}$. For any $\kappa \in \mathbb{A}$ with $|\kappa| < 1$, we have $\kappa^* d(\rho, \sigma) \kappa \succeq \theta_{\mathbb{A}}$. Hence, $d(\Gamma\rho, \Gamma\sigma) \preceq \kappa^* d(\rho, \sigma) \kappa$.

Case (ii): If $\rho > 10$ and $\sigma < 10$, then

$$d(\Gamma\rho, \Gamma\sigma) = \begin{bmatrix} c_1 |1/10|^p & 0 \\ 0 & c_2 |1/10|^p \end{bmatrix}$$

and for $\kappa = \begin{bmatrix} 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{bmatrix}$, we have

$$\kappa^* d(\rho, \sigma) \kappa = \kappa^t d(\rho, \sigma) \kappa = \begin{bmatrix} c_1 |\rho - \sigma|^p / 2 & 0 \\ 0 & c_2 |\rho - \sigma|^p / 2 \end{bmatrix}.$$

Hence, $d(\Gamma\rho, \Gamma\sigma) \preceq \kappa^* d(\rho, \sigma) \kappa$.

Case (iii): If $\rho < 10$ and $\sigma > 10$, then the similar lines as Case (ii), we get $d(\Gamma\rho, \Gamma\sigma) \preceq \kappa^* d(\rho, \sigma) \kappa$.

Case (iv): If $\rho, \sigma > 10$, then $d(\Gamma\rho, \Gamma\sigma) = d(1/10, 1/10) = \theta_{\mathbb{A}}$. For any $\kappa \in \mathbb{A}$ with $\|\kappa\| < 1$, we have $\kappa^* d(\rho, \sigma) \kappa \succeq \theta_{\mathbb{A}}$. Hence, $d(\Gamma\rho, \Gamma\sigma) \preceq \kappa^* d(\rho, \sigma) \kappa$.

Thus, Γ is a C^* -av- $b_{\mathcal{R}}$ -contractive map.

Definition 7 A sequence in $(X, \mathbb{A}, d, \mathcal{R})$ is said to be a $b_{\mathcal{R}}$ -sequence if $\rho_n \mathcal{R} \rho_{n+k}$ for each $n, k \in \mathbb{N}$.

Definition 8 Let $(X, \mathbb{A}, d, \mathcal{R})$ be a C^* -av- $b_{\mathcal{R}}$ -MS. Suppose $\{\rho_n\} \subset X$ and $\rho \in X$. Then,

- (i) a $b_{\mathcal{R}}$ -sequence $\{\rho_n\}$ is said to be convergent with respect to $\mathbb{A}$, if for any $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\|d(\rho_n, \rho)\| \leq \epsilon$ for all $n \geq n_0$. We say ρ is a limit of ρ_n denoted by $\lim_{n \rightarrow \infty} \rho_n \xrightarrow{\mathcal{R}} \rho$.
- (ii) a $b_{\mathcal{R}}$ -sequence $\{\rho_n\}$ is said to be $b_{\mathcal{R}}$ -Cauchy sequence with respect to $\mathbb{A}$, if for any $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\|d(\rho_n, \rho_m)\| \leq \epsilon$ for all $n, m > n_0$. We say $(X, \mathbb{A}, d, \mathcal{R})$ is a complete C^* -av- $b_{\mathcal{R}}$ -MS if every $b_{\mathcal{R}}$ -Cauchy sequence with respect to $\mathbb{A}$ is convergent in X .

Theorem 1 Let $(X, \mathbb{A}, d, \mathcal{R})$ be a $b_{\mathcal{R}}$ -complete C^* -av- $b_{\mathcal{R}}$ -MS (not necessarily complete metric space). Let $\Gamma : X \rightarrow X$ be $\mathcal{R}$ -continuous, $b_{\mathcal{R}}$ -contraction and $\mathcal{R}$ -preserving. Suppose there exists $\rho_0 \in X$ such that $\rho_0 \mathcal{R} \sigma$ for all $\sigma \in \Gamma(X)$. Then, Γ has a unique fixed point ρ^* . Also, Γ is a Picard operator, that is $\lim_{n \rightarrow \infty} \Gamma^n(\rho) = \rho^*$ for all $\rho \in X$.

Proof. Let $\rho_1 = \Gamma(\rho_0)$, $\rho_2 = \Gamma(\rho_1) = \Gamma^2(\rho_0)$, ... , $\rho_n = \Gamma(\rho_{n-1}) = \Gamma^n(\rho_0)$, for all $n \in \mathbb{N}$. Let $n, m \in \mathbb{N}$ and $n < m$. Substituting $k = m - n$, there exists $\rho_0 \in X$ such that $\rho_0 \mathcal{R} \Gamma^k(\rho_0)$. Since, Γ is $\mathcal{R}$ -preserving, $\rho_n = \Gamma^n(\rho_0) \mathcal{R} \Gamma^{n+k}(\rho_0) = \rho_m$. Hence, $\{\rho_n\}$ is a $b_{\mathcal{R}}$ -sequence. Since, Γ satisfies $b_{\mathcal{R}}$ -contraction, we have

$$\begin{aligned} d(\rho_{n+1}, \rho_n) &= d(\Gamma \rho_n, \Gamma \rho_{n-1}) \preceq \kappa^* d(\rho_n, \rho_{n-1}) \kappa \\ &\preceq (\kappa^*)^2 d(\rho_{n-1}, \rho_{n-2}) \kappa^2 \\ &\preceq \dots \\ &\preceq (\kappa^*)^n d(\rho_1, \rho_0) \kappa^n \\ &= (\kappa^*)^n \beta \kappa^n, \end{aligned}$$

where $\beta = d(\rho_0, \rho_1)$ and $\kappa \in \mathbb{A}$ and $\|\kappa\| < 1$. For any $m, p \in \mathbb{N}$, we get

$$\begin{aligned} d(\rho_{m+p}, \rho_m) &\preceq \Lambda(d(\rho_{m+p}, \rho_{m+p-1}) + d(\rho_{m+p-1}, \rho_m)) \\ &= \Lambda d(\rho_{m+p}, \rho_{m+p-1}) + \Lambda d(\rho_{m+p-1}, \rho_m) \\ &\preceq \Lambda d(\rho_{m+p}, \rho_{m+p-1}) + \Lambda^2(d(\rho_{m+p-1}, \rho_{m+p-2}) + d(\rho_{m+p-2}, \rho_m)) \\ &= \Lambda d(\rho_{m+p}, \rho_{m+p-1}) + \Lambda^2 d(\rho_{m+p-1}, \rho_{m+p-2}) + \Lambda^2 d(\rho_{m+p-2}, \rho_m) \\ &\preceq \Lambda d(\rho_{m+p}, \rho_{m+p-1}) + \Lambda^2 d(\rho_{m+p-1}, \rho_{m+p-2}) + \dots + \Lambda^{p-1} d(\rho_{m+2}, \rho_{m+1}) \\ &\quad + \Lambda^{p-1} d(\rho_{m+1}, \rho_m) \\ &\preceq \Lambda(\kappa^*)^{m+p-1} \beta \kappa^{m+p-1} + \Lambda^2(\kappa^*)^{m+p-2} \beta \kappa^{m+p-2} + \dots + \Lambda^{p-1}(\kappa^*)^{m+1} \beta \kappa^{m+1} \\ &\quad + \Lambda^{p-1}(\kappa^*)^m \beta \kappa^m \\ &= \sum_{k=1}^{p-1} \Lambda^k (\kappa^*)^{m+p-k} \beta \kappa^{m+p-k} + \Lambda^{p-1} (\kappa^*)^m \beta \kappa^m \\ &= \sum_{k=1}^{p-1} \left((\kappa^*)^{m+p-k} \Lambda^{\frac{k}{2}} \sqrt{\beta} \right) \left(\sqrt{\beta} \Lambda^{\frac{k}{2}} \kappa^{m+p-k} \right) + \left((\kappa^*)^m \Lambda^{\frac{p-1}{2}} \sqrt{\beta} \right) \left(\sqrt{\beta} \Lambda^{\frac{p-1}{2}} \kappa^m \right) \\ &= \sum_{k=1}^{p-1} \left((\kappa)^{m+p-k} \Lambda^{\frac{k}{2}} \sqrt{\beta} \right)^* \left(\sqrt{\beta} \Lambda^{\frac{k}{2}} \kappa^{m+p-k} \right) + \left((\kappa)^m \Lambda^{\frac{p-1}{2}} \sqrt{\beta} \right)^* \left(\sqrt{\beta} \Lambda^{\frac{p-1}{2}} \kappa^m \right) \\ &= \sum_{k=1}^{p-1} \left| \sqrt{\beta} \Lambda^{\frac{k}{2}} \kappa^{m+p-k} \right|^2 + \left| \sqrt{\beta} \Lambda^{\frac{p-1}{2}} \kappa^m \right|^2 \\ &\leq \sum_{k=1}^{p-1} \left\| \sqrt{\beta} \Lambda^{\frac{k}{2}} \kappa^{m+p-k} \right\|^2 I + \left\| \sqrt{\beta} \Lambda^{\frac{p-1}{2}} \kappa^m \right\|^2 I \\ &\leq \left\| \sqrt{\beta} \right\|^2 \sum_{k=1}^{p-1} \|\kappa\|^{2(m+p-k)} \|\Lambda\|^k I + \left\| \sqrt{\beta} \right\|^2 \|\Lambda\|^{p-1} \|\kappa^m\|^2 I \end{aligned}$$

$$\begin{aligned}
&= \|\beta\| \|\kappa\|^{2(m+p)} \frac{\|\Lambda\|((\|\Lambda\| \|\kappa\|^{-2})^{p-1}) - 1}{\|\Lambda\| - \|\kappa\|^2} I + \|\beta\| \|\Lambda\|^{p-1} \|\kappa^m\|^2 I \\
&\leq \|\beta\| \frac{\|\Lambda\|^p \|\kappa\|^{2(m+1)}}{\|\Lambda\| - \|\kappa\|^2} I + \|\beta\| \|\Lambda\|^{p-1} \|\kappa\|^{2m} I \rightarrow 0 \quad (m \rightarrow \infty).
\end{aligned}$$

Thus, we get $\{\rho_n = \Gamma\rho_{n-1}\}_{n \in \mathbb{N}}$ is a $b_{\mathcal{R}}$ -Cauchy sequence in X . Since, X is a $b_{\mathcal{R}}$ -complete C^* -av- $b_{\mathcal{R}}$ -MS. Therefore, there exists $\rho^* \in X$ such that $\lim_{n \rightarrow \infty} \rho_n = \rho^*$. Since, Γ is $\mathcal{R}$ -continuous. Therefore, $\Gamma(\rho_n) \xrightarrow{\mathcal{R}} \Gamma\rho^*$. Hence,

$$\Gamma\rho^* = \Gamma(\lim_{n \rightarrow \infty} \rho_n) = \lim_{n \rightarrow \infty} \Gamma\rho_n = \lim_{n \rightarrow \infty} \rho_{n+1} = \rho^*.$$

Thus, ρ^* is a fixed point of Γ .

Uniqueness: Let σ^* be another fixed point of Γ . Then, there exists $\rho_0 \in X$ such that $\rho_0 \mathcal{R} \Gamma\sigma^* = \sigma^*$. Hence, $\rho_n = \Gamma^n(\rho_0) \mathcal{R} \sigma^*$ for all $n \in \mathbb{N}$. Using triangle inequality

$$\begin{aligned}
d(\rho^*, \sigma^*) = d(\Gamma^n \rho^*, \Gamma^n \sigma^*) &\preceq \Lambda(d(\Gamma^n \rho^*, \Gamma^n \rho_0) + d(\Gamma^n \rho_0, \Gamma^n \sigma^*)) \\
&\preceq \Lambda((\kappa^*)^n d(\rho^*, \rho_0) \kappa^n + (\kappa^*)^n d(\rho_0, \sigma^*) \kappa^n).
\end{aligned} \tag{1}$$

Taking the limit as $n \rightarrow \infty$ in (1), we get

$$d(\rho^*, \sigma^*) = \theta_{\mathbb{A}}.$$

Thus, $\rho^* = \sigma^*$. This implies uniqueness.

Finally, let ρ be an arbitrary element of X . Then, there exists $\rho_0 \in X$ such that $\rho_0 \mathcal{R} \Gamma\rho$. Thus, $\rho_n = \Gamma^n(\rho_0) \mathcal{R} \Gamma^{n+1}\rho$ for all $n \in \mathbb{N}$. Therefore,

$$\begin{aligned}
d(\rho^*, \Gamma^n(\rho)) = d(\Gamma^n(\rho^*), \Gamma^n(\rho)) &\preceq \Lambda(d(\Gamma^{n-1}(\Gamma\rho^*), \Gamma^{n-1}(\rho_0)) + d(\Gamma^{n-1}(\rho_0), \Gamma^{n-1}(\Gamma\rho))) \\
&= \Lambda(d(\Gamma^{n-1}(\rho^*), \Gamma^{n-1}(\rho_0)) + d(\Gamma^{n-1}(\rho_0), \Gamma^{n-1}(\Gamma\rho))) \\
&\preceq \Lambda((\kappa^*)^{n-1} d(\rho^*, \rho_0) \kappa^{n-1} + (\kappa^*)^{n-1} d(\rho_0, \Gamma\rho) \kappa^{n-1}) \\
&\rightarrow \theta_{\mathbb{A}} \text{ as } n \rightarrow \infty.
\end{aligned}$$

Hence, $\lim_{n \rightarrow \infty} \Gamma^n(\rho) = \rho^*$. ■

Theorem 2 Let $(X, \mathbb{A}, d, \mathcal{R})$ be a $b_{\mathcal{R}}$ -complete C^* -av- $b_{\mathcal{R}}$ -MS (not necessarily complete metric space). Let $\Gamma : X \rightarrow X$ be $\mathcal{R}$ -continuous, $\mathcal{R}$ -preserving and satisfy

$$d(\Gamma\rho, \Gamma\sigma) \preceq \alpha d(\rho, \sigma) + \kappa d(\Gamma\rho, \rho) + \beta d(\Gamma\sigma, \sigma) + \delta d(\rho, \Gamma\sigma) + \mu d(\sigma, \Gamma\rho), \tag{2}$$

for all $\rho, \sigma \in X$, where $\kappa, \alpha, \beta, \delta, \mu \in \mathbb{A}^{+}$ and $\kappa + \alpha + \beta + \delta + \mu < 1$. Suppose there exists $\rho_0 \in X$ such that $\rho_0 \mathcal{R} \sigma$ for all $\sigma \in \Gamma(X)$. Then, Γ has a unique fixed point ρ^* .

Proof. Let $\rho_1 = \Gamma(\rho_0)$, $\rho_2 = \Gamma(\rho_1) = \Gamma^2(\rho_0)$, ..., $\rho_n = \Gamma(\rho_{n-1}) = \Gamma^n(\rho_0)$, for all $n \in \mathbb{N}$. Let $m, n \in \mathbb{N}$ and $n < m$. Substituting $k = m - n$, there exists $\rho_0 \in X$ such that $\rho_0 \mathcal{R} \Gamma^k(\rho_0)$. Since, Γ is $\mathcal{R}$ -preserving. Thus, $\rho_n = \Gamma^n(\rho_0) \mathcal{R} \Gamma^{n+k}(\rho_0) = \rho_m$. Hence, $\{\rho_n\}$ is a $b_{\mathcal{R}}$ -sequence. Using (2), we have

$$\begin{aligned}
d(\rho_{n+1}, \rho_n) = d(\Gamma\rho_n, \Gamma\rho_{n-1}) &\preceq \alpha d(\rho_n, \rho_{n-1}) + \kappa d(\Gamma\rho_n, \rho_n) + \beta d(\Gamma\rho_{n-1}, \rho_{n-1}) \\
&\quad + \delta d(\Gamma\rho_n, \rho_{n-1}) + \mu d(\rho_n, \Gamma\rho_{n-1}) \\
&= \alpha d(\rho_n, \rho_{n-1}) + \kappa d(\rho_{n+1}, \rho_n) + \beta d(\rho_n, \rho_{n-1}) \\
&\quad + \delta d(\rho_{n+1}, \rho_{n-1}) + \mu d(\rho_n, \rho_n) \\
&\preceq \alpha d(\rho_n, \rho_{n-1}) + \kappa d(\rho_{n+1}, \rho_n) + \beta d(\rho_n, \rho_{n-1}) \\
&\quad + \delta d(\rho_{n+1}, \rho_n) + \delta d(\rho_n, \rho_{n-1}),
\end{aligned}$$

$$(I - (\kappa + \delta))d(\rho_{n+1}, \rho_n) \preceq (\alpha + \beta + \delta)d(\rho_n, \rho_{n-1})$$

$$\begin{aligned}
d(\rho_{n+1}, \rho_n) &\preceq (\alpha + \beta + \delta)(I - (\kappa + \delta))^{-1}d(\rho_n, \rho_{n-1}) \\
&= hd(\rho_n, \rho_{n-1}) \\
&\preceq h^2d(\rho_{n-1}, \rho_{n-2}) \\
&\preceq \dots \\
&\preceq h^nd(\rho_1, \rho_0) \\
&= h^n\omega,
\end{aligned}$$

where $h = (\alpha + \beta + \delta)(I - (\kappa + \delta))^{-1}$ with $\|h\| < 1$ and $\omega = d(\rho_1, \rho_0)$.

On the similar lines of Theorem 1, $\{\rho_n = \Gamma\rho_{n-1}\}_{n \in \mathbb{N}}$ is a $b_{\mathcal{R}}$ -Cauchy sequence in X . Since, X is a complete C^* -av- $b_{\mathcal{R}}$ -MS. Therefore, there exists $\rho^* \in X$ such that $\lim_{n \rightarrow \infty} \rho_n = \rho^*$. Since, Γ is $\mathcal{R}$ -continuous. Therefore, $\Gamma(\rho_n) \xrightarrow{\mathcal{R}} \Gamma\rho^*$. Hence,

$$\Gamma\rho^* = \Gamma(\lim_{n \rightarrow \infty} \rho_n) = \lim_{n \rightarrow \infty} \Gamma\rho_n = \lim_{n \rightarrow \infty} \rho_{n+1} = \rho^*.$$

Thus, ρ^* is a fixed point of Γ .

Uniqueness : Let σ^* be another fixed point of Γ in fact $\Gamma^n\sigma^* = \sigma^*$. Then, there exists $\rho_0 \in X$ such that $\rho_0\mathcal{R}\Gamma\sigma^* = \sigma^*$. Hence, $\rho_n = \Gamma^n(\rho_0)\mathcal{R}\Gamma^n\sigma^* = \sigma^*$ for all $n \in \mathbb{N}$. Using (2), we get

$$\begin{aligned}
d(\rho_n, \sigma^*) = d(\Gamma^n\rho_0, \Gamma^n\sigma^*) &\preceq \alpha d(\Gamma^{n-1}\rho_0, \Gamma^{n-1}\sigma^*) + \kappa d(\Gamma^n\rho_0, \Gamma^{n-1}\rho_0) + \beta d(\Gamma^n\sigma^*, \Gamma^{n-1}\sigma^*) \\
&\quad + \delta d(\Gamma^{n-1}\rho_0, \Gamma^n\sigma^*) + \mu d(\Gamma^{n-1}\sigma^*, \Gamma^n\rho_0) \\
&= \alpha d(\rho_n, \sigma^*) + \kappa d(\rho_{n+1}, \rho_n) + \beta d(\sigma^*, \sigma^*) + \delta d(\rho_n, \sigma^*) + \mu d(\sigma^*, \rho_{n+1}),
\end{aligned}$$

$$\begin{aligned}
(I - (\alpha + \delta))d(\rho_n, \sigma^*) &\preceq \kappa d(\rho_{n+1}, \rho_n) + \mu d(\sigma^*, \rho_{n+1}) \\
d(\rho_n, \sigma^*) &\preceq \kappa(I - (\alpha + \delta))^{-1}d(\rho_{n+1}, \rho_n) + \mu(I - (\alpha + \delta))^{-1}d(\sigma^*, \rho_{n+1}) \\
&\preceq \kappa(I - (\alpha + \delta))^{-1}h^{n-1}\omega + \mu(I - (\alpha + \delta))^{-1}d(\sigma^*, \rho_{n+1}).
\end{aligned}$$

Taking limit as $n \rightarrow \infty$ in (3), we get

$$d(\rho^*, \sigma^*) \preceq \mu(I - (\alpha + \delta))^{-1}d(\rho^*, \sigma^*).$$

We would have $\alpha + \delta + \mu \geq 1$, which is a contradiction. Thus, $\rho^* = \sigma^*$. ■

4 Remarks

For different values of $\alpha, \beta, \gamma, \delta$ and μ in (2), we can extend the following version of well known results of literature for self mapping in the framework of C^* -algebra valued $b_{\mathcal{R}}$ -metric space.

- (1) (Kannan type, see [12]) There exists a number $\kappa \in \mathbb{A}'^+$ and $\|\kappa\| < \frac{1}{2}$ such that $\rho, \sigma \in X$.

$$d(\Gamma\rho, \Gamma\sigma) \preceq \kappa(d(\Gamma\rho, \rho) + d(\Gamma\sigma, \sigma)).$$

- (2) (Chatterjea type, see [6]) There exists a number $\delta \in \mathbb{A}'^+$ and $\|\delta\Lambda\| < \frac{1}{2}$ such that $\rho, \sigma \in X$.

$$d(\Gamma\rho, \Gamma\sigma) \preceq \delta(d(\Gamma\rho, \sigma) + d(\Gamma\sigma, \rho)).$$

- (3) (Reich type, see [22]) There exist $\alpha, \beta, \kappa \in \mathbb{A}'^+$ and $\kappa + \alpha + \beta < 1$ such that $\rho, \sigma \in X$.

$$d(\Gamma\rho, \Gamma\sigma) \preceq \alpha d(\rho, \sigma) + \kappa d(\Gamma\rho, \rho) + \beta d(\Gamma\sigma, \sigma).$$

(4) (Ciric type, see [17]) There exist $\kappa, \alpha, \beta, \delta \in \mathbb{A}'^+$ and $\kappa + \alpha + \beta + 2\delta < 1$ such that $\rho, \sigma \in X$.

$$d(\Gamma\rho, \Gamma\sigma) \preceq \alpha d(\rho, \sigma) + \kappa d(\Gamma\rho, \rho) + \beta d(\Gamma\sigma, \sigma) + \delta(d(\rho, \Gamma\sigma) + d(\sigma, \Gamma\rho)).$$

Example 4 Consider $X = [0, \infty)$ and a unital C^* -algebra $\mathbb{A} = M_2(\mathbb{R})$ with $A^* = A^t$ for all $A \in \mathbb{A}$, where A^t denotes the transpose of A . For $A = [a_{ij}]$, $\|A\| = \max_{1 \leq i, j \leq 2} |a_{ij}|$. Let

$$\theta = \hat{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and $d : X \times X \rightarrow M_2(\mathbb{R})$ be defined as follow

$$d(\rho, \sigma) = \begin{cases} \hat{0}, & \text{if } \rho = \sigma, \\ \rho I, & \text{otherwise.} \end{cases}$$

For $A = [a_{ij}]$ and $B = [b_{ij}]$, $A \preceq B$ if and only if $a_{ij} \leq b_{ij}$ for $1 \leq i, j \leq 2$. Define relation $\mathcal{R}$ on X as $\rho \mathcal{R} \sigma$ if and only if either $\rho = \sigma$ or $\rho\sigma = 0$. Then, $(X, \mathbb{A}, d, \mathcal{R})$ is a complete C^* -av- $b_{\mathcal{R}}$ -MS (since every $b_{\mathcal{R}}$ -Cauchy sequence in X is convergent). Let $\Gamma : X \rightarrow X$ be defined as

$$\Gamma\rho = \begin{cases} \rho/300, & \text{if } \rho \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

For $\rho \mathcal{R} \sigma$. Then, following cases arise:

Case (i) For $\rho = \sigma \leq 1$. Clearly, for any $\kappa \in \mathbb{A}$ with $\|\kappa\| < 1$, we get

$$d(\Gamma\rho, \Gamma\sigma) \preceq \kappa^* d(\rho, \sigma) \kappa$$

or

$$d(\Gamma\rho, \Gamma\sigma) \preceq \kappa(d(\rho, \Gamma\rho) + d(\sigma, \Gamma\sigma))$$

or

$$d(\Gamma\rho, \Gamma\sigma) \preceq \kappa(d(\rho, \Gamma\sigma) + d(\sigma, \Gamma\rho)).$$

Case (ii) For $\rho = \sigma \geq 1$. Clearly, for any $\kappa \in \mathbb{A}$ with $\|\kappa\| < 1$, we get

$$d(\Gamma\rho, \Gamma\sigma) \preceq \kappa^* d(\rho, \sigma) \kappa$$

or

$$d(\Gamma\rho, \Gamma\sigma) \preceq \kappa(d(\rho, \Gamma\rho) + d(\sigma, \Gamma\sigma))$$

or

$$d(\Gamma\rho, \Gamma\sigma) \preceq \kappa(d(\rho, \Gamma\sigma) + d(\sigma, \Gamma\rho)).$$

Let $\rho \neq \sigma$. Then $\rho\sigma = 0$ i.e. either $\rho = 0$ or $\sigma = 0$. Without loss of generality, let $\sigma = 0$.

Case (iii) For $\rho \leq 1$, we have

$$d(\Gamma\rho, \Gamma\sigma) = d(\rho/300, 0) = \frac{\rho}{300}I \text{ and } d(\rho, \sigma) = \rho I.$$

Then, for $\kappa \in \mathbb{A}$ with $\|\kappa\| < 1$ such that $\kappa = 1/\sqrt{3}$, we have

$$d(\Gamma\rho, \Gamma\sigma) \preceq \kappa^* d(\rho, \sigma) \kappa$$

or

$$d(\Gamma\rho, \Gamma\sigma) \preceq \kappa(d(\rho, \Gamma\rho) + d(\sigma, \Gamma\sigma))$$

or

$$d(\Gamma\rho, \Gamma\sigma) \preceq \kappa(d(\rho, \Gamma\sigma) + d(\sigma, \Gamma\rho)).$$

Case (iv) For $\rho \geq 1$, we have

$$d(\Gamma\rho, \Gamma\sigma) = d(0, 0) = 0 \text{ and } d(\rho, \sigma) = \rho I.$$

Then, for $\kappa \in \mathbb{A}$ with $\|\kappa\| < 1$ such that $\kappa = 1/\sqrt{3}$, we have

$$d(\Gamma\rho, \Gamma\sigma) \preceq \kappa^* d(\rho, \sigma) \kappa$$

or

$$d(\Gamma\rho, \Gamma\sigma) \preceq \kappa(d(\rho, \Gamma\rho) + d(\sigma, \Gamma\sigma))$$

or

$$d(\Gamma\rho, \Gamma\sigma) \preceq \kappa(d(\rho, \Gamma\sigma) + d(\sigma, \Gamma\rho)).$$

Thus, Γ is a C^* -av- $b_{\mathcal{R}}$ -contractive map. Also, Γ satisfy $\mathcal{R}$ -preserving and $\mathcal{R}$ -continuous. Then, by Theorem 1, Kannan type contraction and Chatterjea type contraction we conclude that Γ possess a unique fixed point which in this case is $\rho = 0$.

Remark 3 The metric space (X, d) considered in above example is an incomplete metric space and thus violating the applicability of Theorems 2.1–2.3 in Ma et al. [26] and Banach contraction principle in [13], Kannan contraction principle [12] and Chatterjea contraction in [17].

Example 5 Consider $X = [0, 1)$ equipped with usual metric and let the unital C^* -algebra $\mathbb{A} = (-\infty, +\infty)$ together with $\|\kappa\| = |\kappa|$, for $\gamma, \kappa \in \mathbb{A}$ $\kappa \preceq \gamma$ if and only if $\kappa \leq \gamma$ and involution given by $\kappa^* = \kappa$. Define relation $\mathcal{R}$ on X as $\rho \mathcal{R} \sigma$ if and only if $(\rho - \sigma) \in \{\rho\}$ and let $\Gamma : X \rightarrow X$ be defined as

$$\Gamma\rho = \begin{cases} 0, & \text{if } \rho \in [0, 1/2], \\ 1/13, & \text{otherwise.} \end{cases}$$

Then, X is a $b_{\mathcal{R}}$ -complete C^* -av- $b_{\mathcal{R}}$ -MS (since every $b_{\mathcal{R}}$ -Cauchy sequence in X is convergent). For $\rho \mathcal{R} \sigma$, then we have either $\sigma = 0$ or both are zero. Let us consider $\sigma = 0$. Following cases arises :

Case (i) : If $\rho \in [0, 1/2]$, then

$$d(\Gamma\rho, \Gamma\sigma) = d(0, 0) = \theta_{\mathbb{A}} \text{ and } d(\rho, \sigma) = d(\rho, 0) = \rho. \quad (3)$$

Then, for any $\kappa \in \mathbb{A}$ with $\|\kappa\| < 1$ and from (3), we get $d(\Gamma\rho, \Gamma\sigma) \preceq \kappa^* d(\rho, \sigma) \kappa$.

Case (ii) : If $\rho \in (1/2, 1)$, then

$$d(\Gamma\rho, \Gamma\sigma) = d(1/13, 0) = 1/13 \text{ and } d(\rho, \sigma) = d(\rho, 0) = \rho. \quad (4)$$

Then, for $\kappa \in \mathbb{A}$ with $\|\kappa\| < 1$ such that $\kappa = 1/\sqrt{3}$ and from (4), we get $d(\Gamma\rho, \Gamma\sigma) \preceq \kappa^* d(\rho, \sigma) \kappa$.

Thus, Γ is a C^* -av- $b_{\mathcal{R}}$ -contractive map. Also, Γ satisfy $\mathcal{R}$ -preserving and $\mathcal{R}$ -continuous. Then, by Theorem 1, we conclude that Γ possess a unique fixed point which in this case is $\rho = 0$.

Remark 4 The metric space (X, d) considered in above example is an incomplete metric space and thus violating the applicability of Theorem 2.1 in Ma et al. [26] and Banach contraction principle in [13].

Example 6 Let $X = [0, 1)$ and $\mathbb{A} = M_2(\mathbb{R})$ with involution on $\mathbb{A}$ define $\kappa^* = \kappa^t$ for all $\kappa \in \mathbb{A}$, where κ^t denotes the transpose of κ . For $\kappa = [\lambda_{ij}]$, let $\|\kappa\| = \max_{1 \leq i, j \leq 2} |\lambda_{ij}|$. Define

$$d(\rho, \sigma) = \text{diag}(c_1 |\rho - \sigma|^p, c_2 |\rho - \sigma|^p),$$

where diag denotes the diagonal matrix, $\rho, \sigma \in X$, $c_i > 0$ for all $(i = 1, 2)$ are constants and $p > 1$. $\mathcal{R}$ is defined on X such that $\rho \mathcal{R} \sigma \Leftrightarrow (\rho - \sigma) \in \{\rho\}$. It is easy to verify that $(X, \mathbb{A}, d, \mathcal{R})$ is a $b_{\mathcal{R}}$ -complete C^* -av- $b_{\mathcal{R}}$ -MS, for proving triangle inequality of Definition 5, we need to use following inequality

$$|\rho - \sigma|^p \leq 3^p(|\rho - \mu|^p + |\mu - \sigma|^p),$$

implies that $d(\rho, \sigma) \preceq \Lambda(d(\rho, \mu) + d(\mu, \sigma))$ for all $\rho, \sigma, \mu \in X$ where $\Lambda = 3^p I \in \mathbb{A}$ and $\Lambda \succ I$ with $3^p > 1$. Define a self map $\Gamma : X \rightarrow X$ as

$$\Gamma\rho = \begin{cases} 0, & \text{if } \rho \in [0, 1/2], \\ 3/25, & \text{if } \rho \in (1/2, 1). \end{cases}$$

For $\rho \mathcal{R} \sigma$ then, we have either $\sigma = 0$ or both are zero. Let us consider $\sigma = 0$. Following cases arises:

Case (i): If $\rho \in [0, 1/2]$, then $d(\Gamma\rho, \Gamma\sigma) = d(0, 0) = \theta_{\mathbb{A}}$. For any $\kappa \in \mathbb{A}$ with $\|\kappa\| < 1/2$, we have $\kappa(d(\Gamma\rho, \rho) + d(\Gamma\sigma, \sigma)) = \kappa\rho \succeq \theta_{\mathbb{A}}$. Thus, $d(\Gamma\rho, \Gamma\sigma) \preceq \kappa(d(\Gamma\rho, \rho) + d(\Gamma\sigma, \sigma))$.

Case (ii): If $\rho \in (1/2, 1)$, then

$$d(\Gamma\rho, \Gamma\sigma) = \begin{bmatrix} c_1 |3/25|^p & 0 \\ 0 & c_2 |3/25|^p \end{bmatrix}$$

and for $\kappa = \begin{bmatrix} 1/\sqrt{3} & 0 \\ 0 & 1/\sqrt{3} \end{bmatrix}$, we have

$$\kappa(d(\rho, \Gamma\rho) + d(\sigma, \Gamma\sigma)) = \begin{bmatrix} c_1 |\rho - 3/25|^p / \sqrt{3} & 0 \\ 0 & c_2^p |\rho - 3/25|^p / \sqrt{3} \end{bmatrix}.$$

Hence, $d(\Gamma\rho, \Gamma\sigma) \preceq \kappa(d(\rho, \Gamma\rho) + d(\sigma, \Gamma\sigma))$.

Also, Γ satisfy $\mathcal{R}$ -preserving and $\mathcal{R}$ -continuous. Hence, by Kannan type contraction, we conclude that Γ possess a unique fixed point which in this case is $\rho = 0$.

Remark 5 The metric space (X, d) considered in above example is an incomplete metric space and thus violating the applicability of Theorem 2.3 in Ma et al. [26] and Kannan contraction principle [12].

Example 7 Consider $X = [0, 1)$ equipped with usual metric and let the unital C^* -algebra valued metric space $\mathbb{A} = (-\infty, +\infty)$ together with $\|\kappa\| = |\kappa|$ and involution given by $\kappa^* = \kappa$. Define relation $\mathcal{R}$ on X as $\rho \mathcal{R} \sigma$ if and only if $\rho\sigma \in \{0\}$ and let $\Gamma : X \rightarrow X$ be defined as

$$\Gamma\rho = \begin{cases} 0, & \text{if } \rho \in [0, 1/2], \\ 2/33, & \text{otherwise.} \end{cases}$$

Then, X is a $b_{\mathcal{R}}$ -complete C^* -av- $b_{\mathcal{R}}$ -MS (since every $b_{\mathcal{R}}$ -Cauchy sequence in X is convergent). For $\rho \mathcal{R} \sigma$ then, we have either $\rho = 0$ or $\sigma = 0$ or both are zero. Let us consider $\sigma = 0$. Following cases arises :

Case (i): If $\rho \in [0, 1/2]$, then

$$d(\Gamma\rho, \Gamma\sigma) = d(0, 0) = \theta_{\mathbb{A}} \quad (5)$$

and

$$d(\rho, \Gamma\sigma) = d(\rho, 0) = \rho \text{ and } d(\sigma, \Gamma\rho) = d(0, 0) = \theta_{\mathbb{A}}. \quad (6)$$

Then, for any $\kappa \in \mathbb{A}$ with $\|\kappa\| < 1/2$ and from (5) and (6), we get

$$d(\Gamma\rho, \Gamma\sigma) \preceq \kappa(d(\rho, \Gamma\sigma) + d(\sigma, \Gamma\rho)).$$

Case (ii): If $\rho \in (1/2, 1)$, then

$$d(\Gamma\rho, \Gamma\sigma) = d(2/33, 0) = 2/33 \quad (7)$$

and

$$d(\rho, \Gamma\sigma) = d(\rho, 0) = \rho \text{ and } d(\sigma, \Gamma\rho) = d(0, 2/33) = 2/33. \quad (8)$$

Then, for $\kappa \in \mathbb{A}$ with $\|\kappa\| < 1/2$ such that $\kappa = 1/\sqrt{3}$ and from (7) and (8), we get

$$d(\Gamma\rho, \Gamma\sigma) \preceq \kappa(d(\rho, \Gamma\sigma) + d(\sigma, \Gamma\rho)).$$

Also, Γ satisfy $\mathcal{R}$ -preserving and $\mathcal{R}$ -continuous. Hence, by Chatterjea type contraction, we conclude that Γ possess a unique fixed point which in this case is $\rho = 0$.

Remark 6 The metric space (X, d) considered in above example is an incomplete metric space and thus violating the applicability of Theorem 2.2 in Ma et al. [26] and Chatterjea contraction in [17].

Example 8 Consider $X = [0, 1)$ equipped with usual metric and let the unital C^* -algebra valued metric space $\mathbb{A} = (-\infty, +\infty)$ together with $\|\kappa\| = |\kappa|$ and involution given by $\kappa^* = \kappa$. Define relation $\mathcal{R}$ on X as $(\rho, \sigma) \in \mathcal{R}$ if and only if $(\rho - \sigma) \in \{\rho\}$ and let $\Gamma : X \rightarrow X$ be defined as

$$\Gamma\rho = \begin{cases} 0, & \text{if } \rho \in [0, 1/2], \\ 1/133, & \text{otherwise.} \end{cases}$$

Then, X is a $b_{\mathcal{R}}$ -complete C^* -algebra valued $b_{\mathcal{R}}$ -metric space (since every $b_{\mathcal{R}}$ -Cauchy sequence in X is convergent). For $(\rho, \sigma) \in \mathcal{R}$ then, we have either $\sigma = 0$ or both are zero. Let us consider $\sigma = 0$.

Following cases arises :

Case (i) : If $\rho \in [0, 1/2]$, then

$$d(\Gamma\rho, \Gamma\sigma) = d(0, 0) = \theta_{\mathbb{A}} \quad (9)$$

and

$$d(\rho, \sigma) = \rho, \quad d(\rho, \Gamma\rho) = \rho \quad \text{and} \quad d(\sigma, \Gamma\sigma) = \theta_{\mathbb{A}}. \quad (10)$$

Then, for any $\kappa, \alpha, \beta \in \mathbb{A}^+$ with $\|\kappa\| + \|\alpha\| + \|\beta\| < 1$ and from (9) and (10), we get

$$d(\Gamma\rho, \Gamma\sigma) \preceq \alpha d(\rho, \sigma) + \kappa d(\rho, \Gamma\rho) + \beta d(\sigma, \Gamma\sigma).$$

Case (ii) : If $\rho \in (1/2, 1)$, then

$$d(\Gamma\rho, \Gamma\sigma) = d(1/133, 0) = 1/133. \quad (11)$$

$$d(\rho, \sigma) = \rho, \quad d(\rho, \Gamma\rho) = |\rho - 1/133| \quad \text{and} \quad d(\sigma, \Gamma\sigma) = \theta_{\mathbb{A}}. \quad (12)$$

Then, for $\kappa, \alpha, \beta \in \mathbb{A}$ with $\|\kappa\| + \|\alpha\| + \|\beta\| < 1$ such that $\kappa = \alpha = 1/3, \beta = 1/4$ and from (11) and (12), we get

$$d(\Gamma\rho, \Gamma\sigma) \preceq \alpha d(\rho, \sigma) + \kappa d(\rho, \Gamma\rho) + \beta d(\sigma, \Gamma\sigma).$$

Also, Γ satisfy $\mathcal{R}$ -preserving and $\mathcal{R}$ -continuous. Hence, by Reich type contraction, we conclude that Γ possess a unique fixed point which in this case is $\rho = 0$.

Remark 7 The metric space (X, d) considered in above example is an incomplete metric space and thus violating the applicability of Theorem [3] of Reich type contraction in [22].

Example 9 Consider $X = [0, 1)$ equipped with usual metric and let the unital C^* -algebra valued metric space $\mathbb{A} = (-\infty, +\infty)$ together with $\|\kappa\| = |\kappa|$ and involution given by $\kappa^* = \kappa$. Define relation $\mathcal{R}$ on X as $(\rho, \sigma) \in \mathcal{R}$ if and only if $(\rho - \sigma) \in \{\rho\}$ and let $\Gamma : X \rightarrow X$ be defined as

$$\Gamma\rho = \begin{cases} 0, & \text{if } \rho \in [0, 1/2], \\ 5/253, & \text{otherwise.} \end{cases}$$

Then, X is a $b_{\mathcal{R}}$ -complete C^* -algebra valued $b_{\mathcal{R}}$ -metric space (since every $b_{\mathcal{R}}$ -Cauchy sequence in X is convergent). For $(\rho, \sigma) \in \mathcal{R}$ then, we have either $\sigma = 0$ or both are zero. Let us consider $\sigma = 0$. Following cases arises:

Case (i) : If $\rho \in [0, 1/2]$, then

$$d(\Gamma\rho, \Gamma\sigma) = d(0, 0) = \theta_{\mathbb{A}}. \quad (13)$$

$$\begin{aligned} d(\rho, \sigma) &= \rho, & d(\rho, \Gamma\rho) &= \rho, & d(\sigma, \Gamma\sigma) &= \theta_{\mathbb{A}}, \\ d(\rho, \Gamma\sigma) &= \rho & \text{and} & & d(\sigma, \Gamma\rho) &= \theta_{\mathbb{A}}. \end{aligned} \quad (14)$$

Then, for any $\kappa, \alpha, \beta, \delta \in \mathbb{A}^+$ with $\|\kappa\| + \|\alpha\| + \|\beta\| + 2\|\delta\| < 1$ and from (13) and (14), we get

$$d(\Gamma\rho, \Gamma\sigma) \preceq \alpha d(\rho, \sigma) + \kappa d(\rho, \Gamma\rho) + \beta d(\sigma, \Gamma\sigma) + \delta(d(\rho, \Gamma\sigma) + d(\sigma, \Gamma\rho)).$$

Case (ii): If $\rho \in (1/2, 1)$, then

$$d(\Gamma\rho, \Gamma\sigma) = d(5/253, 0) = 5/253. \quad (15)$$

$$\begin{aligned} d(\rho, \sigma) &= \rho, & d(\rho, \Gamma\rho) &= |\rho - 5/253|, & d(\sigma, \Gamma\sigma) &= \theta_{\mathbb{A}}, \\ d(\rho, \Gamma\sigma) &= \rho & \text{and} & & d(\sigma, \Gamma\rho) &= 5/253. \end{aligned} \quad (16)$$

Then, for $\kappa, \alpha, \beta, \delta \in \mathbb{A}$ with $\|\kappa\| + \|\alpha\| + \|\beta\| + 2\|\delta\| < 1$ such that $\kappa = \alpha = 1/5 = \beta$, $\delta = 1/8$ and from (15) and (16), we get

$$d(\Gamma\rho, \Gamma\sigma) \preceq \alpha d(\rho, \sigma) + \kappa d(\rho, \Gamma\rho) + \beta d(\sigma, \Gamma\sigma) + \delta(d(\rho, \Gamma\sigma) + d(\sigma, \Gamma\rho)).$$

Also, Γ satisfy $\mathcal{R}$ -preserving and $\mathcal{R}$ -continuous. Hence, by Ciric contraction, we conclude that Γ possess a unique fixed point which in this case is $\rho = 0$.

Remark 8 The metric space (X, d) considered in above example is an incomplete metric space and thus violating the applicability of Theorem 1 of Ciric contraction in [17].

Example 10 Consider $X = [0, 1)$ equipped with usual metric and let the unital C^* -algebra valued metric space $\mathbb{A} = (-\infty, +\infty)$ together with $\|\kappa\| = |\kappa|$ and involution given by $\kappa^* = \kappa$. Define relation $\mathcal{R}$ on X as $(\rho, \sigma) \in \mathcal{R}$ if and only if $(\rho - \sigma) \in \{\rho\}$ and let $\Gamma : X \rightarrow X$ be defined as

$$\Gamma\rho = \begin{cases} 0, & \text{if } \rho \in [0, 1/2], \\ 3/342, & \text{otherwise.} \end{cases}$$

Then, X is a $b_{\mathcal{R}}$ -complete C^* -algebra valued $b_{\mathcal{R}}$ -metric space (since every $b_{\mathcal{R}}$ -Cauchy sequence in X is convergent). For $(\rho, \sigma) \in \mathcal{R}$ then, we have either $\sigma = 0$ or both are zero. Let us consider $\sigma = 0$. Following cases arises :

Case (i): If $\rho \in [0, 1/2]$, then

$$d(\Gamma\rho, \Gamma\sigma) = d(0, 0) = \theta_{\mathbb{A}}. \quad (17)$$

$$\begin{aligned} d(\rho, \sigma) &= \rho, & d(\rho, \Gamma\rho) &= \rho, & d(\sigma, \Gamma\sigma) &= \theta_{\mathbb{A}}, \\ d(\rho, \Gamma\sigma) &= \rho & \text{and} & & d(\sigma, \Gamma\rho) &= \theta_{\mathbb{A}}. \end{aligned} \quad (18)$$

Then, for any $\mu, \kappa, \alpha, \beta, \delta \in \mathbb{A}^+$ with $\|\kappa\| + \|\alpha\| + \|\beta\| + \|\delta\| + \|\mu\| < 1$ and from (17) and (18), we get

$$d(\Gamma\rho, \Gamma\sigma) \preceq \alpha d(\rho, \sigma) + \kappa d(\rho, \Gamma\rho) + \beta d(\sigma, \Gamma\sigma) + \delta d(\rho, \Gamma\sigma) + \mu d(\sigma, \Gamma\rho).$$

Case (ii): If $\rho \in (1/2, 1)$, then

$$d(\Gamma\rho, \Gamma\sigma) = d(3/342, 0) = 3/342. \quad (19)$$

$$\begin{aligned} d(\rho, \sigma) &= \rho, & d(\rho, \Gamma\rho) &= |\rho - 3/342|, & d(\sigma, \Gamma\sigma) &= \theta_{\mathbb{A}}, \\ d(\rho, \Gamma\sigma) &= \rho \text{ and } d(\sigma, \Gamma\rho) &= 3/342. \end{aligned} \quad (20)$$

Then, for $\kappa, \alpha, \beta, \delta, \mu \in \mathbb{A}^+$ with $\|\kappa\| + \|\alpha\| + \|\beta\| + \|\delta\| + \|\mu\| < 1$ such that $\kappa = \alpha = 1/8 = \beta = \delta = \mu$ and from (19) and (20), we get

$$d(\Gamma\rho, \Gamma\sigma) \preceq \alpha d(\rho, \sigma) + \kappa d(\rho, \Gamma\rho) + \beta d(\sigma, \Gamma\sigma) + \delta d(\rho, \Gamma\sigma) + \mu d(\sigma, \Gamma\rho).$$

Also, Γ satisfy $\mathcal{R}$ -preserving and $\mathcal{R}$ -continuous. Then, by Theorem 2, we conclude that Γ possess a unique fixed point which in this case is $\rho = 0$.

Remark 9 The metric space (X, d) considered in above example is an incomplete metric space and thus violating the applicability of Theorem 1 Hardy-Roger type contraction in [4].

5 Application to Operator Equation

As an applications of contractive mapping theorem on complete C^* -av- $b_{\mathcal{R}}$ -MS, existence and uniqueness of solution of the operator equation is presented.

Theorem 3 Suppose that H is a Hilbert space, $L(H)$ is the set of linear bounded operators on H . Let $\Lambda_1, \Lambda_2, \dots, \Lambda_n, \dots$ which satisfy $\sum_{n=1}^{\infty} \|\Lambda_n\| < 1$. Then, the operator equation

$$\rho = \sum_{n=1}^{\infty} \Lambda_n^* \rho \Lambda_n$$

has a unique solution in $L(H)$.

Proof. Set $\kappa = (\sum_{n=1}^{\infty} \|\Lambda_n\|)^p$ with $p \geq 1$, then $\|\kappa\| < 1$. Without loss of generality, one can suppose that $\kappa > \theta_{\mathbb{A}}$. Choose a positive operator $T \in L(H)$. For $\rho, \sigma \in L(H)$ and $p \geq 1$, set $d(\rho, \sigma) = \|\rho - \sigma\|^p T$. $\mathcal{R}$ is defined on $L(H)$ such that $\rho \mathcal{R} \sigma \Leftrightarrow (\rho - \sigma) \in \{\rho\}$. It is easy to show that $d(\rho, \sigma)$ is a C^* -av- $b_{\mathcal{R}}$ -MS and $(L(H), d)$ is complete, since $L(H)$ is a Banach space. Indeed, it suffices to check the condition of Definition 5 as follow:

$$d(\rho, \sigma) \leq \Lambda(d(\rho, \mu) + d(\mu, \sigma)) \quad \text{where} \quad \Lambda = 2^p I.$$

Consider, the map $F : L(H) \rightarrow L(H)$ defined by

$$F(\rho) = \sum_{n=1}^{\infty} \Lambda_n^* \rho \Lambda_n.$$

Following step has to be follow:

Step (I) : F is $\mathcal{R}$ -preserving. Let $\rho \mathcal{R} \sigma$. Consider

$$F(\rho) - F(\sigma) = \sum_{n=1}^{\infty} \Lambda_n^* \rho \Lambda_n - \sum_{n=1}^{\infty} \Lambda_n^* \sigma \Lambda_n = \sum_{n=1}^{\infty} \Lambda_n^* (\rho - \sigma) \Lambda_n.$$

Thus, if $\rho \mathcal{R} \sigma$ then $F(\rho) \mathcal{R} F(\sigma)$.

Step (II) : F is $\mathcal{R}$ -contraction. Let $\rho \mathcal{R} \sigma$. Consider

$$\begin{aligned} d(F(\rho), F(\sigma)) &= \|F(\rho) - F(\sigma)\|^p T \\ &= \left\| \sum_{n=1}^{\infty} \Lambda_n^* (\rho - \sigma) \Lambda_n \right\|^p \end{aligned}$$

$$\begin{aligned}
& \preceq \sum_{n=1}^{\infty} \|\Lambda_n\|^{2p} \|\rho - \sigma\|^p T \\
& = \kappa^2 d(\rho, \sigma) \\
& = (\kappa I)^* d(\rho, \sigma) (\kappa I).
\end{aligned}$$

Hence, F follow $b_{\mathcal{R}}$ -contraction.

Step (III) : F is $\mathcal{R}$ -continuous. Let $\{\Lambda_n\}$ be a $b_{\mathcal{R}}$ -sequence converging to $\Lambda \in L(H)$. Using $\|\Lambda\| < 1$, hence $(\Lambda_n - \Lambda) \in \{\Lambda_n\}$ for all $n \in \mathbb{N}$. Therefore, $\Lambda_n \mathcal{R} \Lambda$. By step (II), we have

$$\|F(\Lambda_n) - F(\Lambda)\|^p T \leq \sum_{n=1}^{\infty} \|\Lambda_n\|^{2p} \|\Lambda_n - \Lambda\|^p T.$$

Thus,

$$\|F(\Lambda_n) - F(\Lambda)\| \leq \kappa^2 d(\Lambda_n, \Lambda) \preceq d(\Lambda_n, \Lambda).$$

Therefore, $F(\Lambda_n) \rightarrow F(\Lambda)$.

By Theorem 1, it follows that there exists a unique fixed point X in $L(H)$ which is the solution of operator equation. ■

6 Conclusion

As discussed in the beginning, the key purpose of this manuscript is to generalize and unify some of the well known results present in the literature. Following points give a visualization of deductions made through the present manuscript.

- (i) On considering binary relation $\mathcal{R}$ as a universal relation (that is, relation $\mathcal{R}$ on X such that $\rho \mathcal{R} \sigma$ for all $\rho, \sigma \in X$) in Theorem 1, we obtain the Theorem 2.1 of Ma et al. [26].
- (ii) If we consider $\mathbb{A} = \mathbb{R}$, $\|\alpha\| = |\alpha|$ and $\alpha^* = \alpha$ with $\Lambda = 1$ in Theorem 1, then we obtain the analogue of Theorem 3.1 of Rahimi et al. [19].
- (iii) On combining the above two conditions in Theorem 1, we obtain the Banach contraction principle [13].
- (iv) On combining conditions (i) and (ii) in Kannan, Chatterjea, Riech, Ćirić type contraction and Theorem 2 in the manuscript, we derive the fixed point result of Chatterjea [17], Ćirić [6], Hardy and Roger [4], Kannan [12] and Reich [22].
- (v) On considering binary relation $\mathcal{R}$ as a universal relation (that is, relation $\mathcal{R}$ on X such that $\rho \mathcal{R} \sigma$ for all $\rho, \sigma \in X$) in Kannan and Chatterjea type contraction in the manuscript, we obtain Theorems 2.2 and 2.3 in Ma et al. [26].

However, readers should note that from either of the results obtained in Chatterjea [17], Kannan [12], Rahimi et al. [19] and Ma et al. [26], we cannot obtain the results proved in this manuscript and to further substantiate the utility Examples 5, 6 and 7 were considered.

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