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Bound Estimates For The Number Of Zeros Of Polynomials*

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Abstract

In this paper, we shall obtain the estimates for the number of zeros of a polynomial whose intermediate coefficients satisfy monotonicity condition. Our results besides generalizing the result due to Mohammad [8] also provide some interesting results as special cases.

1 Introduction

In the vast landscape of mathematics, polynomials stand as pillars of exploration and understanding. They weave through algebra, calculus, and beyond, shaping the foundation of countless mathematical concepts and applications. Among the many fascinating aspects of polynomials lies the investigation of their roots, the values at which they equal zero. This exploration leads us to the intriguing concept of bounds on the number of zeros a polynomial can possess within a given interval. The study of polynomial roots dates back centuries, with roots deeply embedded in the history of mathematics. Notably, mathematicians like Ren Descartes and Isaac Newton laid crucial groundwork in understanding the behavior of polynomial roots. Over time, the exploration of these roots evolved, culminating in fundamental theorems that shed light on the bounds of their distribution. One such theorem is the Fundamental Theorem of Algebra (for reference see [9]), a cornerstone result that asserts every non-constant polynomial with complex coefficients has at least one complex root. Although the theorem guarantees the existence of as many zeros of a polynomial as its degree, there is no method for finding these zeros when the degree exceeds four. So, the problem of locating at least the regions which contain all or some zeros of a polynomial gain importance in the theory of polynomials. In this direction, Cauchy [3] proved an interesting result known as Cauchy's Classical Theorem which states that all the zeros of a polynomial $P(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n$, $a_n \neq 0$ lie in

$$|z| < 1 + \max_{0 \leq j \leq n-1} \left| \frac{a_j}{a_n} \right|.$$

In the literature, there exists several results concerning the bounds for the zeros of polynomials and related topics (for references see [1], [2], [7], [10] and [11]). One such elegant result that give the location of the zeros of a polynomial with restricted coefficients is known as Eneström-Kakeya Theorem [7] which states :

Theorem 1 *If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n such that $a_n \geq a_{n-1} \geq ... \geq a_1 \geq a_0 > 0$, then $P(z)$ has all its zeros in $|z| \leq 1$.*

Later on Joyall et al. [6] extended Theorem 1 by relaxing the condition of non-negativity and proved the following result.

Theorem 2 *If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n such that $a_n \geq a_{n-1} \geq ... \geq a_1 \geq a_0$, then $P(z)$ has all its zeros in*

$$|z| \leq \frac{|a_n| - a_0 + |a_0|}{|a_n|}.$$

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There exists several extensions and generalizations of Theorem 1. However the following result concerning the number of zeros of a polynomial in a closed disk can be found in Tichmarsh's book, see[12].

Theorem 3 *Let $F(z)$ be analytic in $|z| \leq R$ such that $|F(z)| \leq M$ in $|z| \leq R$ and $F(0) \neq 0$. Then for $0 < \delta < 1$, the number of zeros of $F(z)$ in the disk $|z| \leq R\delta$ does not exceed*

$$\frac{1}{\log \frac{1}{\delta}} \log \frac{M}{|F(0)|}.$$

Mohammad [8] proved the special case of Theorem 3 by imposing a restriction on the coefficients of a polynomial similar to that of Eneström-Kakeya Theorem by establishing

Theorem 4 *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ bea polynomial of degree n such that $a_n \geq a_{n-1} \geq ... \geq a_1 \geq a_0 > 0$. Then the number of zeros of $P(z)$ in $|z| \leq \frac{1}{2}$ does not exceed*

$$1 + \frac{1}{\log 2} \log \frac{a_n}{a_0}.$$

Dewan [4] extended Theorem 4 to the polynomials with complex coefficients and proved the following results:

Theorem 5 *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients such that for some real β, $|arga_j - \beta| \leq \alpha \leq \frac{\pi}{2}$ for all $j = 0, 1, 2, ..., n$ and $|a_n| \geq |a_{n-1}| \geq ... \geq |a_1| \geq |a_0| > 0$. Then the number of zeros of $P(z)$ in $|z| \leq \frac{1}{2}$ does not exceed*

$$1 + \frac{1}{\log 2} \log \frac{|a_n|(1 + \cos\alpha + \sin\alpha) + 2\sin\alpha \sum_{j=0}^{n-1} |a_j|}{|a_0|}.$$

Theorem 6 *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with complex coefficients. If $\mathrm{Re}(a_i) = \alpha_i$, $\mathrm{Im}(a_i) = \beta_i$, $i = 0, 1, ..., n$ are such that $\alpha_n \geq \alpha_{n-1} \geq ... \geq \alpha_1 \geq \alpha_0 > 0$, then the number of zeros of $P(z)$ in $|z| \leq \frac{1}{2}$ does not exceed*

$$1 + \frac{1}{\log 2} \log \frac{\alpha_n + \sum_{j=0}^{n} |\beta_j|}{|a_0|}.$$

2 Main Results

In this paper, we present the generalization of Theorems 4, 5 and 6 by relaxing the condition of monotonicity for extreme co-efficients. In fact, we prove

Theorem 7 *Let $P(z) = \sum_{j=0}^{n} a_j z^j$, $a_0 \neq 0$ be a polynomial of degree n such that for some positive integer p and a non-negative integer q with $p \geq q$, $a_p \geq a_{p-1} \geq ... \geq a_{q+1} \geq a_q$. Then for $0 < \delta < 1$, the number of zeros of $P(z)$ in $|z| \leq \delta$ does not exceed*

$$\frac{1}{\log \frac{1}{\delta}} \log \frac{|a_n| + M_p + M_q + a_p - a_q + |a_0|}{|a_0|},$$

where

$$M_p = \sum_{j=p+1}^{n} |a_j - a_{j-1}| \quad \text{and} \quad M_q = \sum_{j=1}^{q} |a_j - a_{j-1}|.$$

Remark 1 *If $a_0 > 0$, then for $p = n$, $q = 0$ and $\delta = \frac{1}{2}$, Theorem 7 reduces to Theorem 4 due to Mohammad [8] and if we take $q = 0$ in Theorem 7, then we get the following result.*

Corollary 1 *Let $P(z) = \sum_{j=0}^{n} a_j z^j, a_0 \neq 0$ be a polynomial of degree n such that for some positive integer $0 < p \leq n$, $a_p \geq a_{p-1} \geq ... \geq a_1 \geq a_0$. Then for $0 < \delta < 1$, the number of zeros of $P(z)$ in $|z| \leq \delta$ does not exceed*

$$\frac{1}{\log \frac{1}{\delta}} \log \frac{|a_n| + M_p + a_p - a_0 + |a_0|}{|a_0|},$$

where

$$M_p = \sum_{j=p+1}^{n} |a_j - a_{j-1}|.$$

If we choose $p = n$ in Theorem 7, we obtain the following result.

Corollary 2 *Let $P(z) = \sum_{j=0}^{n} a_j z^j, a_0 \neq 0$ be a polynomial of degree n such that for a non-negative integer q, $a_n \geq a_{n-1} \geq ... \geq a_{q+1} \geq a_q$. Then for $0 < \delta < 1$, the number of zeros of $P(z)$ in $|z| \leq \delta$ does not exceed*

$$\frac{1}{\log \frac{1}{\delta}} \log \frac{|a_n| + a_n - a_q + |a_0| + M_q}{|a_0|},$$

where

$$M_q = \sum_{j=1}^{q} |a_j - a_{j-1}|.$$

Next we extend Theorem 7 to the polynomials with complex coefficients and present the following results.

Theorem 8 *Let $P(z) = \sum_{j=0}^{n} a_j z^j$, $a_0 \neq 0$ be a polynomial of degree n with complex coefficients. If $\mathrm{Re}(a_i) = \alpha_i$ and $\mathrm{Im}(a_i) = \beta_i$, $i = 0, 1, ..., n$ are such that for some positive integer p and a non-negative integer q with $p \geq q$, $\alpha_p \geq \alpha_{p-1} \geq ... \geq \alpha_{q+1} \geq \alpha_q$. Then for $0 < \delta < 1$, the number of zeros of $P(z)$ in $|z| \leq \delta$ does not exceed*

$$\frac{1}{\log \frac{1}{\delta}} \log \frac{|\alpha_n| + M_p + M_q + \alpha_p - \alpha_q + |\alpha_0| + 2\sum_{j=0}^{n} |\beta_j|}{|a_0|},$$

where

$$M_p = \sum_{j=p+1}^{n} |\alpha_j - \alpha_{j-1}| \quad \text{and} \quad M_q = \sum_{j=1}^{q} |\alpha_j - \alpha_{j-1}|.$$

Remark 2 *Setting $\alpha_0 > 0$, then for $p = n$, $q = 0$ and $\delta = \frac{1}{2}$, Theorem 8 reduces to Theorem 6 due to Dewan [4] and for $q = 0$, it yields the following.*

Corollary 3 *Let $P(z) = \sum_{j=0}^{n} a_j z^j$, $a_0 \neq 0$ be a polynomial of degree n with complex coefficients. If $\mathrm{Re}(a_i) = \alpha_i$, $\mathrm{Im}(a_i) = \beta_i$, $i = 0, 1, ..., n$ are such that for some positive integer p, $\alpha_p \geq \alpha_{p-1} \geq ... \geq \alpha_1 \geq \alpha_0$, then for $0 < \delta < 1$, the number of zeros of $P(z)$ in $|z| \leq \delta$ does not exceed*

$$\frac{1}{\log \frac{1}{\delta}} \log \frac{|\alpha_n| + M_p + \alpha_p - \alpha_0 + |\alpha_0| + 2\sum_{j=0}^{n} |\beta_j|}{|a_0|},$$

where

$$M_p = \sum_{j=p+1}^{n} |\alpha_j - \alpha_{j-1}|.$$

If we put $p = n$ in Theorem 8 , then we acquire the following result.

Corollary 4 *Let $P(z)=\sum_{j=0}^{n}a_jz^j, a_0\neq 0$ be a polynomial of degree n with complex coefficients. If $\mathrm{Re}(a_i)=\alpha_i$, $\mathrm{Im}(a_i)=\beta_i$, $i=0,1,...,n$ are such that for some non-negative integer q, $\alpha_n\geq\alpha_{n-1}\geq...\geq\alpha_{q+1}\geq\alpha_q$, then for $0<\delta<1$, the number of zeros of $P(z)$ in $|z|\leq\delta$ does not exceed*

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{|\alpha_n|+M_q+\alpha_n-\alpha_q+|\alpha_0|+2\sum_{j=0}^{n}|\beta_j|}{|a_0|},$$

where

$$M_q=\sum_{j=1}^{q}|\alpha_j-\alpha_{j-1}|.$$

Theorem 9 *Let $P(z)=\sum_{j=0}^{n}a_jz^j$, $a_0\neq 0$ be a polynomial of degree n with complex coefficients such that for some real β and for some integers p and q with $0\leq q<p\leq n$, $|\arg a_i-\beta|\leq\alpha\leq\frac{\pi}{2}$, $q\leq i\leq p$ we have*

$$|a_p|\geq|a_{p-1}|\geq...\geq|a_{q+1}|\geq|a_q|.$$

Then for $0<\delta<1$, the number of zeros of $P(z)$ in $|z|\leq\delta$ does not exceed

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{|a_n|+M_p+|a_p|(\cos\alpha+\sin\alpha)+2\sin\alpha\sum_{j=q}^{p-1}|a_j|-|a_q|(\cos\alpha+\sin\alpha)+|a_0|+M_q}{|a_0|},$$

where

$$M_p=\sum_{j=p+1}^{n}|a_j-a_{j-1}| \quad \text{and} \quad M_q=\sum_{j=1}^{q}|a_j-a_{j-1}|.$$

Remark 3 *If $a_0>0$, then for $p=n$, $q=0$ and $\delta=\frac{1}{2}$, Theorem 9 reduces to Theorem 5 due to Dewan [4] and on taking $q=0$ in Theorem 9, we produce following result :*

Corollary 5 *Let $P(z)=\sum_{j=0}^{n}a_jz^j$, $a_0\neq 0$ be a polynomial of degree n with complex coefficients such that for some real β, $|\arg a_i-\beta|\leq\alpha\leq\frac{\pi}{2}$, $0\leq i\leq p$ and satisfies $|a_p|\geq|a_{p-1}|\geq...\geq|a_1|\geq|a_0|$, $p\geq 0$. Then for $0<\delta<1$, the number of zeros of $P(z)$ in $|z|\leq\delta$ does not exceed*

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{|a_n|+M_p+|a_p|(\cos\alpha+\sin\alpha)+2\sin\alpha\sum_{j=0}^{p-1}|a_j|-|a_0|(\cos\alpha+\sin\alpha)+|a_0|}{|a_0|},$$

where

$$M_p=\sum_{j=p+1}^{n}|a_j-a_{j-1}|.$$

If $p=n$ in Theorem 9 , then we attain the following result.

Corollary 6 *Let $P(z)=\sum_{j=0}^{n}a_jz^j$, $a_0\neq 0$ be a polynomial of degree n with complex coefficients such that for some real β, $|\arg a_i-\beta|\leq\alpha\leq\frac{\pi}{2}$, $q\leq i\leq n$ and satisfies $|a_n|\geq|a_{n-1}|\geq...\geq|a_{q+1}|\geq|a_q|$, $q\geq 0$.*

Then for $0<\delta<1$, the number of zeros of $P(z)$ in $|z|\leq\delta$ does not exceed

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{|a_n|+|a_n|(\cos\alpha+\sin\alpha)+2\sin\alpha\sum_{j=q}^{n-1}|a_j|-|a_q|(\cos\alpha+\sin\alpha)+|a_0|+M_q}{|a_0|},$$

where

$$M_q=\sum_{j=1}^{q}|a_j-a_{j-1}|.$$

3 Lemmas

For the proofs of the Theorems stated above, we need the following lemma due to Govil and Rehman [5].

Lemma 1 *If for some real* β, $|\arg \alpha_i - \beta| \leq \alpha \leq \frac{\pi}{2}$ *and* $|a_i| \geq |a_{i-1}|$ *for* $i = 0, 1, 2, ..., n$, *then*

$$|a_i - a_{i-1}| \leq (|a_i| - |a_{i-1}|)\cos\alpha + (|a_i| + |a_{i-1}|)\sin\alpha.$$

4 Proofs of Theorems

Proof of Theorem 7. Consider the polynomial

$$\begin{aligned} F(z) &= (1-z)P(z) = (1-z)(a_n z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + ... + a_{p+1}z^{p+1} \\ &\quad + a_p z^p + ... + a_{q+1}z^{q+1} + a_q z^q + a_{q-1}z^{q-1} + ... + a_1 z + a_0) \\ &= -a_n z^{n+1} + (a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + ... + (a_{p+1} - a_p)z^{p+1} \\ &\quad + (a_p - a_{p-1})z^p + ... + (a_{q+1} - a_q)z^{q+1} + (a_q - a_{q-1})z^q \\ &\quad + (a_{q-1} - a_{q-2})z^{q-1} + ... + (a_1 - a_0)z + a_0. \end{aligned}$$

This gives for $|z| \leq 1$

$$\begin{aligned} |F(z)| &\leq |a_n| + |a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + ... + |a_{p+1} - a_p| \\ &\quad + |a_p - a_{p-1}| + ... + |a_{q+1} - a_q| + |a_q - a_{q-1}| \\ &\quad + |a_{q-1} - a_{q-2}| + ... + |a_1 - a_0| + |a_0|. \end{aligned} \tag{1}$$

Now by hypothesis $a_p \geq a_{p-1} \geq ... \geq a_{q+1} \geq a_q, p \geq q$, using in inequality (1), we get,

$$\begin{aligned} |F(z)| \leq & |a_n| + |a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + ... + |a_{p+1} - a_p| \\ & + (a_p - a_{p-1}) + (a_{p-1} - a_{p-2}) + (a_{p-2} - a_{p-3}) ... + (a_{q+1} - a_q) \\ & + |a_q - a_{q-1}| + |a_{q-1} - a_{q-2}| + ... + |a_1 - a_0| + |a_0|. \end{aligned}$$

This implies for $|z| \leq 1$

$$|F(z)| \leq |a_n| + M_p + a_p - a_q + M_q + |a_0| = M.$$

Since $F(z)$ is analytic in $|z| \leq 1$ with $|F(z)| \leq M$ for $|z| \leq 1$ and $F(0) = a_0$. Therefore applying Theorem 3 to the polynomial $F(z)$, it follows that the number of zeros of $F(z)$ in $|z| \leq \delta$ does not exceed

$$\frac{1}{\log \frac{1}{\delta}} \log \frac{M}{|F(0)|}.$$

As the number of zeros of $P(z)$ in $|z| \leq \delta$ is equal to number of zeros of $F(z)$ in $|z| \leq \delta$. Hence the number of $P(z)$ in $|z| \leq \delta$ does not exceed

$$\frac{1}{\log \frac{1}{\delta}} \log \frac{M}{|a_0|}.$$

That proves Theorem 7. ■

Proof of Theorem 8. Define a polynomial $T(z)$ by

$$T(z) = (1-z)P(z) = (1-z)\sum_{j=0}^{n} a_j z^j = -a_n z^{n+1} + \sum_{j=1}^{n}(a_j - a_{j-1})z^j + a_0.$$

Therefore for $|z| \leq 1$, we have

$$\begin{aligned} |T(z)| &\leq |a_n| + \sum_{j=1}^{n} |a_j - a_{j-1}| + |a_0| \\ &\leq |\alpha_n| + |\beta_n| + |\alpha_0| + |\beta_0| + \sum_{j=1}^{n} \left\{ |\alpha_j - \alpha_{j-1}| + |\beta_j - |\beta_{j-1}| \right\} \\ &= |\alpha_n| + |\beta_n| + |\alpha_0| + |\beta_0| + \sum_{j=1}^{n} |\alpha_j - \alpha_{j-1}| + \sum_{j=1}^{n} |\beta_j - \beta_{j-1}|, \end{aligned}$$

or equivalently

$$\begin{aligned} |T(z)| \leq\ & |\alpha_n| + |\beta_n| + |\alpha_0| + |\beta_0| + \sum_{j=p+1}^{n} |\alpha_j - \alpha_{j-1}| + \sum_{j=q+1}^{p} |\alpha_j - \alpha_{j-1}| \\ & + \sum_{j=1}^{q} |\alpha_j - \alpha_{j-1}| + \sum_{j=1}^{n} (|\beta_j| + |\beta_{j-1}|). \end{aligned} \tag{2}$$

By hypothesis, we have for given p and q, $p \geq q$, $\alpha_p \geq \alpha_{p-1} \geq \cdots \geq \alpha_{q+1} \geq \alpha_q$. Therefore inequality (2) becomes

$$\begin{aligned} |T(z)| &\leq |\alpha_n| + |\alpha_0| + M_p + M_q + 2\sum_{j=o}^{n} |\beta_j| \\ &\quad + (\alpha_p - \alpha_{p-1}) + (\alpha_{p-1} - \alpha_{p-2}) + \cdots + (\alpha_{q+1} - \alpha_q) \\ &= |\alpha_n| + M_p + \alpha_p - \alpha_q + M_q + 2\sum_{j=0}^{n} |\beta_j| + |\alpha_0|. \end{aligned}$$

This implies for $|z| \leq 1$

$$|T(z)| \leq |\alpha_n| + M_p + \alpha_p - \alpha_q + M_q + 2\sum_{j=0}^{n} |\beta_j| + |\alpha_0| = N.$$

This shows that $|T(z)| \leq N$ for $|z| \leq 1$. Further $T(z)$ is analytic in $|z| \leq 1$ and $T(0) = a_0 \neq 0$. Therefore by Theorem 3, the number of zeros of $T(z)$ in $|z| \leq \delta$ does not exceed

$$\frac{1}{\log \frac{1}{\delta}} \log \frac{N}{|T(0)|}.$$

Since by definition of $T(z)$, it is clear that $T(z)$ and $P(z)$ have the same number of zeros in $|z| \leq \delta$. Hence the number of $P(z)$ in $|z| \leq \delta$ does not exceed

$$\frac{1}{\log \frac{1}{\delta}} \log \frac{N}{|a_0|}.$$

This completes the proof Theorem 8. ■

Proof of Theorem 9. Contemplate the polynomial

$$H(z) = (1 - z)P(z) = (1 - z)\sum_{j=0}^{n} a_j z^j = -a_n z^{n+1} + \sum_{j=1}^{n} (a_j - a_{j-1}) z^j + a_0.$$

This gives for $|z| \leq 1$

$$\begin{aligned}
|H(z)| &\leq |a_n| + \sum_{j=1}^{n} |a_j - a_{j-1}| + |a_0| \\
&\leq |a_n| + |a_0| + \sum_{j=p+1}^{n} (|a_j - a_{j-1}|) + \sum_{j=q+1}^{p} (|a_j - a_{j-1}|) + \sum_{j=1}^{q} (|a_j - a_{j-1}|) \\
&= |a_n| + |a_0| + \sum_{j=p+1}^{n} (|a_j - a_{j-1}|) + \sum_{j=1}^{q} (|a_j - a_{j-1}|) + |a_p - a_{p-1}| + |a_{p-1} - a_{p-2}| + ... + |a_{q+1} - a_q|.
\end{aligned}$$

Since by hypothesis, we have for $p \geq q$, $|a_p| \geq |a_{p-1}| \geq ... \geq |a_{q+1}| \geq |a_q|$, therefore it follows by using Lemma 1 that

$$\begin{aligned}
|H(z)| &\leq |a_n| + |a_0| + M_p + M_q + [(|a_p| - |a_q|)\cos\alpha] + 2\sin\alpha \sum_{j=q}^{p-1} |a_j| + |a_p|\sin\alpha - |a_q|\sin\alpha \\
&= |a_n| + M_p + M_q + |a_0| + |a_p|(\cos\alpha + \sin\alpha) + 2\sin\alpha \sum_{j=q}^{p-1} |a_j| - |a_q|(\sin\alpha + \cos\alpha).
\end{aligned}$$

This implies for $|z| \leq 1$

$$|H(z)| \leq |a_n| + M_p + |a_p|(\cos\alpha + \sin\alpha) + 2\sin\alpha \sum_{j=q}^{p-1} |a_j| - |a_q|(\sin\alpha + \cos\alpha) + M_q + |a_0| = R.$$

This shows that $H(z)$ is analytic and $|H(z)| \leq R$ for $|z| \leq 1$. Also by hypothesis $H(0) = a_0 \neq 0$. Therefore involving Theorem 3, we get the number of zeros of $H(z)$ in $|z| \leq \delta$ does not exceed

$$\frac{1}{\log \frac{1}{\delta}} \log \frac{R}{|H(0)|}.$$

As the number of zeros of $H(z)$ in $|z| \leq \delta$ is equal to number of zeros of $P(z)$ in $|z| \leq \delta$. Hence the number of $P(z)$ in $|z| \leq \delta$ does not exceed

$$\frac{1}{\log \frac{1}{\delta}} \log \frac{R}{|a_0|}.$$

This proves Theorem 9. ■

5 Conclusion

Studying the location of zeros of a complex polynomial is a classical topic in geometric function theory. In this context, several results can be found in the literature for the class of polynomials whose coefficients satisfy certain monotonicity conditions. In this paper, we extended this classical inquiry by relaxing some conditions related to the monotonicity on the coefficients of underlying polynomial. We specifically examined how these conditions influence the zero distribution, particularly discussing the maximum number of zeros that the polynomial in consideration can have within a particular region of complex plane. These results not only contribute to the broader understanding of polynomial roots but also generalize many known results. In addition to this, they provide open avenues for future research in polynomial theory.

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