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L^r Inequalities For Polar Derivative Of Polynomials With S-Fold Zero*

Raju Laishangbam[†], Mayanglambam Singhajit Singh[‡], Reingachan Ngamchui[§]
Barchand Chanam[¶]

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Abstract

This paper deals with the extension of ordinary inequalities on polynomial to integral inequalities for the polar derivative of the polynomial having s-fold zero at the origin and remaining zeros in $|z| \geq k$, $k \geq 1$.

1 Introduction

Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n. We define

$$\|p\|_r = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |p(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}}, \quad 0 < r < \infty. \tag{1}$$

If we let $r \to \infty$ in the above equality and make use of the well-known fact from analysis [12] that

$$\lim_{r\to\infty} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |p(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}} = \max_{|z|=1} |p(z)|,$$

we can suitably denote

$$\|p\|_\infty = \max_{|z|=1} |p(z)|.$$

A famous result due to Bernstein [13] states that if $p(z)$ is a polynomial of degree n, then

$$\|p'\|_\infty \leq n\|p\|_\infty. \tag{2}$$

Inequality (2) can be obtained by letting $r \to \infty$ in the inequality

$$\|p'\|_r \leq n\|p\|_r, r > 0. \tag{3}$$

Inequality (3) for $r \geq 1$ is due to Zygmund [15]. Arestov [1] proved that (3) remains valid for $0 < r < 1$ as well.

If we restrict ourselves to the class of polynomials having no zero in $|z| < 1$, then inequality (2) and (3) can be respectively improved by

$$\|p'\|_\infty \leq \frac{n}{2}\|p\|_\infty, \tag{4}$$

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†Department of Mathematics, National Institute of Technology Manipur, Manipur, 795004, India
‡Department of Mathematics, National Institute of Technology Manipur, Manipur, 795004, India
§Department of Mathematics, National Institute of Technology Manipur, Manipur, 795004, India
¶Department of Mathematics, National Institute of Technology Manipur, Manipur, 795004, India

$$\|p'\|_r \leq \frac{n}{\|1+z\|_r}\|p\|_r, \quad r>0. \tag{5}$$

Inequality (4) was conjectured by Erdös and later verified by Lax [7] whereas, inequality (5) was proved by de-Bruijn [4] for $r \geq 1$ and by Rahman and Schmeisser [11] for $0<r<1$.

As a generalization of (4), Malik [8] proved that if $p(z)$ does not vanish in $|z|<k$, $k \geq 1$, then

$$\|p'\|_\infty \leq \frac{n}{1+k}\|p\|_\infty. \tag{6}$$

Under the same hypothesis of the polynomial $p(z)$, Govil and Rahman [6] extended inequality (6) to L^r setting by showing that

$$\|p'\|_r \leq \frac{n}{\|z+k\|_r}\|p\|_r, \quad r \geq 1. \tag{7}$$

Chan and Malik [2] considered the lacunary polynomial $p(z)=a_0+\sum_{j=\mu}^{n} a_j z^j$, $1 \leq \mu \leq n$, and proved an extension of inequality (6) as

$$\|p'\|_\infty \leq \frac{n}{1+k^\mu}\|p\|_\infty. \tag{8}$$

With the same assumptions of the polynomial $p(z)$, inequality (8) was improved by Pukhta [10], and proved

$$\|p'\|_\infty \leq \frac{n}{1+k^\mu}\left\{\|p\|_\infty - \min_{|z|=k}|p(z)|\right\}. \tag{9}$$

Singh and Shah [14, Theorem 2.3] proved a generalization and improvement of (9) which Chanam et al. [3] recently pointed out some missing part in it while trying to obtain the integral setting of the corrected form of the above result due to Singh and Shah.

Theorem 1 *Let $p(z)=z^s\left(a_0+\sum_{j=\mu}^{n} a_j z^j\right)$, $1 \leq \mu \leq n-s$, $0 \leq s \leq n-1$, be a polynomial of degree n having s-fold zero at the origin and remaining $n-s$ zeros in $|z| \geq k$, $k \geq 1$, then*

$$\begin{aligned}
\max_{|z|=1}|p'(z)| \leq & \left[\frac{1}{(n-s)\left(|a_0|-\frac{m}{k^s}\right)(1+k^{\mu+1})+\mu|a_\mu|(k^{\mu+1}+k^{2\mu})}\right] \\
& \times\left[(n-s)^2\left(|a_0|-\frac{m}{k^s}\right)+(n-s)\mu|a_\mu|k^{\mu+1}\right. \\
& \left.+s(n-s)\left(|a_0|-\frac{m}{k^s}\right)(1+k^{\mu+1})+s\mu|a_\mu|(k^{\mu+1}+k^{2\mu})\right]\max_{|z|=1}|p(z)| \\
& -\frac{1}{k^s}\frac{(n-s)^2\left(|a_0|-\frac{m}{k^s}\right)+(n-s)\mu|a_\mu|k^{\mu+1}}{(n-s)\left(|a_0|-\frac{m}{k^s}\right)(1+k^{\mu+1})+\mu|a_\mu|(k^{\mu+1}+k^{2\mu})}\min_{|z|=k}|p(z)|.
\end{aligned} \tag{10}$$

For a polynomial $p(z)$ of degree n, we now define the polar derivative of $p(z)$ with respect to a real or complex number α as

$$D_\alpha p(z) = np(z)+(\alpha-z)p'(z).$$

This polynomial $D_\alpha p(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative $p'(z)$ in the sense that

$$\lim_{\alpha\to\infty}\frac{D_\alpha p(z)}{\alpha} = p'(z),$$

uniformly with respect to z for $|z| \leq R$, $R>0$.

Over the past four decades, many different authors produced a large number of different versions and generalizations of the above inequalities by involving the comparison of the polar derivative $D_\alpha p(z)$ with various choices of $p(z)$, α and other parameters.

2 Lemmas

For the proof of the theorem, we require the following lemmas.

Lemma 1 *If $p(z)$ is a polynomial of degree n having no zero in $|z| < k$, $k > 0$, then*

$$|p(z)| \geq m \quad \text{for } |z| \leq k, \tag{11}$$

where $m = \min_{|z|=k} |p(z)|$.

This lemma is due to Gardner et al. [5].

Lemma 2 *The function*

$$f(x) = k^{t+1} \left\{ \frac{\frac{t}{n}\frac{|a_t|}{x}k^{t-1} + 1}{\frac{t}{n}\frac{|a_t|}{x}k^{t+1} + 1} \right\},$$

where $t = 1, 2, 3, \ldots$ and $k \geq 1$, is a non-decreasing function of $x > 0$.

Lemma 2 is due to Gardner et al. [5].

Lemma 3 *If $p(z) = z^s \left(a_0 + \sum_{j=\mu}^{n-s} a_j z^j\right)$, $1 \leq \mu \leq n-s$, $0 \leq s \leq n-1$, is a polynomial of degree n having s-fold zero at the origin and remaining $n-s$ zeros in $|z| \geq k$, $k \geq 1$, then for every real or complex number λ with $|\lambda| < 1$,*

$$A' = k^{\mu+1} \left\{ \frac{\frac{\mu}{n-s}\frac{|a_\mu|}{\left|a_0 - \frac{\lambda m}{k^s}\right|}k^{\mu-1} + 1}{\frac{\mu}{n-s}\frac{|a_\mu|}{\left|a_0 - \frac{\lambda m}{k^s}\right|}k^{\mu+1} + 1} \right\} \geq k^{\mu+1} \left\{ \frac{\frac{\mu}{n-s}\frac{|a_\mu|}{|a_0| - |\lambda|\frac{m}{k^s}}k^{\mu-1} + 1}{\frac{\mu}{n-s}\frac{|a_\mu|}{|a_0| - |\lambda|\frac{m}{k^s}}k^{\mu+1} + 1} \right\} = A, \tag{12}$$

where m is defined in Lemma 1.

Proof. Since $\phi(z) = \frac{p(z)}{z^s} = a_0 + \sum_{j=\mu}^{n-s} a_j z^j$ has no zero in $|z| < k$, $k \geq 1$, by Lemma 1, we have

$$\begin{aligned} |\phi(z)| &\geq \min_{|z|=k} |\phi(z)| \qquad \text{for} \qquad |z| \leq k \\ &= \min_{|z|=k} \left\{ \frac{|p(z)|}{|z|^s} \right\} \\ &\geq \frac{1}{k^s} \min_{|z|=k} |p(z)| \\ &= \frac{m}{k^s}, \end{aligned} \tag{13}$$

where $m = \min_{|z|=k} |p(z)|$. In particular, (13) gives for $z = 0$ that

$$|a_0| \geq \frac{m}{k^s}. \tag{14}$$

Now, for $m > 0$

$$\left| a_0 - \lambda \frac{m}{k^s} \right| \geq \left| |a_0| - |\lambda| \frac{m}{k^s} \right| = |a_0| - |\lambda| \frac{m}{k^s} > 0. \tag{15}$$

Applying Lemma 2 to (15), we have the required conclusion of Lemma 3. ∎

Lemma 4 *For $k \geq 1$, $\mu = 1, 2, 3, \ldots$,*

$$A = k^{\mu+1} \left\{ \frac{\frac{\mu}{n-s}\frac{|a_\mu|}{|a_0| - |\lambda|\frac{m}{k^s}}k^{\mu-1} + 1}{\frac{\mu}{n-s}\frac{|a_\mu|}{|a_0| - |\lambda|\frac{m}{k^s}}k^{\mu+1} + 1} \right\} \geq 1, \tag{16}$$

where m is defined in Lemma 1.

Proof. Since $k \geq 1$ and $\mu = 1, 2, 3, ...,$

$$k^{\mu} \geq 1,$$

which is equivalent to

$$k^{\mu} - k \geq 1 - k^{\mu+1}. \tag{17}$$

Also

$$\frac{\mu}{n-s} \frac{|a_{\mu}|}{|a_0| - |\lambda| \frac{m}{k^s}} k^{\mu} > 0. \tag{18}$$

Since L.H.S of (18) is > 0 and L.H.S of (17) is ≥ 0, whereas its R.H.S is ≤ 0, we have

$$\frac{\mu}{n-s} \frac{|a_{\mu}|}{|a_0| - |\lambda| \frac{m}{k^s}} k^{\mu}(k^{\mu} - k) \geq 1 - k^{\mu+1},$$

from which the claim of Lemma 4 follows. ■

Lemma 5 *If $p(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$, $1 \leq \mu \leq n$ is a polynomial of degree n having no zeros in $|z| < k$, $k \geq 1$, then for any complex number α with $|\alpha| \geq 1$ and for $r > 0$,*

$$\left\{ \int_0^{2\pi} |D_{\alpha} p(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}} \leq n\{|\alpha| + A(\mu)\} C_r\{A(\mu)\} \left\{ \int_0^{2\pi} |p(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}}, \tag{19}$$

where

$$A(\mu) = k^{\mu+1} \left\{ \frac{(\frac{\mu}{n})\frac{|a_{\mu}|}{|a_0|} k^{\mu-1} + 1}{(\frac{\mu}{n})\frac{|a_{\mu}|}{|a_0|} k^{\mu+1} + 1} \right\}$$

and

$$C_r\{A(\mu)\} = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |A(\mu) + e^{i\beta}|^r d\beta \right\}^{\frac{-1}{r}}.$$

The above lemma is due to Mir and Ahmad [9, Corollary 1].

Lemma 6 *If $a \geq 1$, $b \geq c \geq 1$ and $\gamma > 0$, then*

$$\frac{a+b}{\|b+z\|_r} \leq \frac{a+c}{\|c+z\|_r}.$$

The above lemma was proved by Govil and Kumar [18].

3 Main Results

In this paper, we extend Theorem 1 to integral analogue for polar derivative. In fact, we prove

Theorem 2 *Let $p(z) = z^s \left(a_0 + \sum_{j=\mu}^{n} a_j z^j\right)$, $1 \leq \mu \leq n-s$, $0 \leq s \leq n-1$, be a polynomial of degree n having s-fold zero at the origin and remaining $n-s$ zeros in $|z| \geq k$, $k \geq 1$, then for every α and λ with $|\alpha| \geq 1$ and $|\lambda| < 1$ and for all $r > 0$,*

$$\left\| zD_{\alpha}p(z) - s\alpha p(z) - (n-s)\frac{\lambda m z^{s+1}}{k^s} \right\|_r \leq \frac{(n-s)(|\alpha| + A)}{\|A+z\|_r} \left\| \frac{p(z)}{z^s} - \frac{\lambda m}{k^s} \right\|_r, \tag{20}$$

where A is as defined in (16) and $m = \min_{|z|=k} |p(z)|$.

Proof. Let $p(z) = z^s\phi(z)$ where $\phi(z) = a_0 + \sum_{j=\mu}^{n-s} a_j z^j$, $1 \le \mu \le n-s$ and $0 \le s \le n-1$, is a polynomial of degree $n-s$ having all its zeros in $|z| \ge k$, $k \ge 1$.

In case $p(z)$ has no zero on $|z| = k$, then

$$\begin{aligned} 0 < m' &= \min_{|z|=k} |\phi(z)| \\ &= \min_{|z|=k} \left\{ \frac{1}{|z|^s} |p(z)| \right\} \\ &\ge \frac{1}{k^s} \min_{|z|=k} |p(z)| \\ &= \frac{1}{k^s} m, \end{aligned}$$

where

$$m = \min_{|z|=k} |p(z)|,$$

and on $|z| = k$,

$$|\phi(z)| \ge m' \ge \frac{m}{k^s}.$$

For every complex number λ such that $|\lambda| < 1$, it gives that

$$|\phi(z)| > \frac{|\lambda| m}{k^s}.$$

Moreover, if $p(z)$ has a zero on $|z| = k$, then $m' = 0$. Thus either $m' = 0$ or $m' \ne 0$. In any case, it follows by Rouche's theorem that all zeros of the polynomial $\psi(z) = \phi(z) - \lambda m'$ lie in $|z| \ge k, k \ge 1$.

Now,

$$\begin{aligned} D_\alpha \psi(z) &= (n-s)\psi(z) + (\alpha - z)\psi'(z) \\ &= (n-s)\left\{ \phi(z) - \frac{\lambda m}{k^s} \right\} + (\alpha - z)\phi'(z) \\ &= (n-s)\phi(z) + (\alpha - z)\phi'(z) - (n-s)\frac{\lambda m}{k^s} \\ &= D_\alpha \phi(z) - (n-s)\frac{\lambda m}{k^s}. \end{aligned}$$

Applying Lemma 5 to $\psi(z)$, we have for every real or complex number α with $|\alpha| \ge 1$ and for each $r > 0$,

$$\left\| D_\alpha \phi(z) - (n-s)\frac{\lambda m}{k^s} \right\|_r \le \frac{(n-s)(|\alpha| + A')}{\|A' + z\|_r} \left\| \phi(z) - \frac{\lambda m}{k^s} \right\|_r, \tag{21}$$

where

$$A' = k^{\mu+1} \left\{ \frac{\frac{\mu}{n-s} \frac{|a_\mu|}{\left|a_0 - \frac{\lambda m}{k^s}\right|} k^{\mu-1} + 1}{\frac{\mu}{n-s} \frac{|a_\mu|}{\left|a_0 - \frac{\lambda m}{k^s}\right|} k^{\mu+1} + 1} \right\}.$$

By Lemmas 3 and 4,

$$A' \ge A \ge 1, \tag{22}$$

where

$$A = k^{\mu+1} \left\{ \frac{\frac{\mu}{n-s} \frac{|a_\mu|}{|a_0| - |\lambda| \frac{m}{k^s}} k^{\mu-1} + 1}{\frac{\mu}{n-s} \frac{|a_\mu|}{|a_0| - |\lambda| \frac{m}{k^s}} k^{\mu+1} + 1} \right\}.$$

Using Lemma 6 with (22), we have

$$\frac{|\alpha| + A'}{\|z + A'\|_r} \le \frac{|\alpha| + A}{\|z + A\|_r}. \tag{23}$$

Since $p(z) = z^s \phi(z)$, we have

$$D_\alpha p(z) = z^s D_\alpha \phi(z) + s\alpha z^{s-1}\phi(z),$$

i.e.,

$$zD_\alpha p(z) = z^{s+1} D_\alpha \phi(z) + s\alpha p(z),$$

i.e.,

$$D_\alpha \phi(z) = \frac{zD_\alpha p(z) - s\alpha p(z)}{z^{s+1}} = \frac{D_\alpha p(z)}{z^s} - s\alpha \frac{p(z)}{z^{s+1}}. \tag{24}$$

Using (24) in (21), we have

$$\left\| \frac{D_\alpha p(z)}{z^s} - s\alpha \frac{p(z)}{z^{s+1}} - (n-s)\frac{\lambda m}{k^s} \right\|_r \le \frac{(n-s)(|\alpha| + A')}{\|A' + z\|_r} \left\| \frac{p(z)}{z^s} - \frac{\lambda m}{k^s} \right\|_r. \tag{25}$$

Now

$$\frac{D_\alpha p(z)}{z^s} - s\alpha \frac{p(z)}{z^{s+1}} - (n-s)\frac{\lambda m}{k^s} = \frac{1}{z^{s+1}} \left\{ zD_\alpha p(z) - s\alpha p(z) - (n-s)\frac{\lambda m z^{s+1}}{k^s} \right\}. \tag{26}$$

Using (23) and (26) in (25), we get the desire result. ■

Remark 1 *We are interested to examine whether inquality (20) serves as the integral setting of Theorem 1 concerning polar derivative, which we discuss below.*

Now, taking limit as $r \to \infty$ on both sides of (20), we have

$$\left\| zD_\alpha p(z) - s\alpha p(z) - (n-s)\frac{\lambda m z^{s+1}}{k^s} \right\|_\infty \le \frac{(n-s)(|\alpha| + A)}{\|A + z\|_\infty} \left\| \frac{p(z)}{z^s} - \frac{\lambda m}{k^s} \right\|_\infty. \tag{27}$$

Let z_1 be a point on $|z| = 1$ such that

$$\max_{|z|=1} \left| \frac{p(z)}{z^s} - \frac{\lambda m}{k^s} \right| = \left| \frac{p(z_1)}{z_1^s} - \frac{\lambda m}{k^s} \right|. \tag{28}$$

Now, we choose the argument of λ such that

$$\left| \frac{p(z_1)}{z_1^s} - \frac{\lambda m}{k^s} \right| = \left| \frac{p(z_1)}{z_1^s} \right| - \frac{|\lambda| m}{k^s} \le \max_{|z|=1} |p(z)| - \frac{|\lambda| m}{k^s}. \tag{29}$$

Moreover on $|z| = 1$,

$$\left| zD_\alpha p(z) - s\alpha p(z) - (n-s)\frac{\lambda m z^{s+1}}{k^s} \right| \le |D_\alpha p(z)| - s|\alpha||p(z)| - (n-s)\frac{|\lambda| m}{k^s}. \tag{30}$$

Using (28), (29) and (30) in (27), we have

$$\max_{|z|=1} |D_\alpha p(z)| - s|\alpha| \max_{|z|=1} |p(z)| - (n-s)\frac{|\lambda| m}{k^s} \le \frac{(n-s)(|\alpha| + A)}{A+1} \left\{ \max_{|z|=1} |p(z)| - \frac{|\lambda| m}{k^s} \right\}. \tag{31}$$

Dividing both sides of inequality (31) by $|\alpha|$ and letting $|\alpha| \to \infty$, and considering limit as $|\lambda| \to 1$, we get inequality (10).

Again, taking $s = 0$ in (20), we get the following interesting result.

Corollary 1 *Let* $p(z) = \left(a_0 + \sum_{j=\mu}^{n} a_j z^j\right)$, $1 \le \mu \le n$, *be a polynomial of degree* n *having no zero in* $|z| < k$, $k \ge 1$, *then for every* α, λ *with* $|\alpha| \ge 1$ *and* $|\lambda| < 1$, *and for all* $r > 0$

$$\|D_\alpha p(z) - n\lambda m\|_r \le \frac{n(|\alpha| + A_0)}{\|A_0 + z\|_r} \|p(z) - \lambda m\|_r \,, \tag{32}$$

where

$$A_0 = k^{\mu+1} \left\{ \frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-|\lambda|m} k^{\mu-1} + 1}{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-|\lambda|m} k^{\mu+1} + 1} \right\} \quad \text{and} \quad m = \min_{|z|=k} |p(z)|.$$

Remark 2 *Dividing both sides of* (32) *by* $|\alpha|$ *and taking limit as* $|\alpha| \to \infty$, *we get an integral analogue of a result due to Dewan et al. [16, Theorem 1] as shown below.*

Now, letting $r \to \infty$ on both sides of (32), we get

$$\max_{|z|=1} |D_\alpha p(z) - n\lambda m| \le \frac{n(|\alpha| + A_0)}{A_0 + 1} \max_{|z|=1} |p(z) - \lambda m|. \tag{33}$$

Following similar procedures as in (28) and (29) in the right hand side of (33), inequality (33) would give

$$\max_{|z|=1} |D_\alpha p(z)| - n|\lambda| m \le \frac{n(|\alpha| + A_0)}{A_0 + 1} \left\{ \max_{|z|=1} |p(z)| - |\lambda| m \right\}. \tag{34}$$

Remark 3 *When* $|\lambda| \to 1$, (34) *reduces to a result of Dewan et al. [16, Theorem 1]. Further, on dividing both sides of inequality* (34) *by* $|\alpha|$ *and taking limit as* $|\alpha| \to \infty$, *we get a result due to Gardner et al. [17].*

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