Applied Mathematics E-Notes, 25(2025), 525-536 © ISSN 1607-2510
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New Extensions Of The Well-Known Rivlin's Inequality*

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Received 22 October 2024

Abstract

In this paper, we investigate new extensions of Rivlin's inequality regarding generalization in terms of the relative growth of a polynomial $p(z)$ with respect to two circles $|z| = r$ and $|z| = R$ while taking into account the involvement of certain coefficients of the underlying polynomial. Our results improve and generalize certain well-known polynomial inequalities. As a consequence of our results, we obtain other interesting results too. Further, a numerical example is also given in order to illustrate graphically and compare the obtained inequalities with some recently published results.

1 Introduction

Approximating complex functions with simpler polynomial expressions is a fundamental concept in mathematics and applied sciences. This technique involves constructing polynomial functions that closely mimic the behavior of intricate functions, making analysis, computation, and problem-solving easier across various domains. Polynomial approximation is crucial in fields such as numerical analysis, signal processing, computer-aided design, physics, and engineering.

Several approaches have been developed to address polynomial approximation, each tailored to specific contexts and requirements. Least squares approximation, using techniques like linear regression, minimizes the overall error between the polynomial and data points. Chebyshev approximation minimizes the maximum absolute error over an interval, ensuring robustness. Rational function approximation introduces flexibility by representing functions as ratios of polynomials. Each method has strengths and is chosen based on data characteristics, desired accuracy, and computational efficiency. Researchers continue to refine these techniques, ensuring that polynomial approximation remains a powerful and adaptable tool in mathematics and computational sciences.

However, another approach involves applying Bernstein's inequality, specifically, its trigonometric version, which plays a crucial role in the literature for establishing inverse theorems in approximation theory (see Borwein and Erdélyi [15], Ivanov [27], Lorentz [6], Telyakovskii [24]) and have their intrinsic interests. The first result in this area was connected with some investigation of the well-known Russian chemist Mendeleev [4]. In fact, Mendeleev's problem was to determine $\max_{-1\le x\le 1} |p'(x)|$, where $p(x)$ is a quadratic polynomial of real variable x with real coefficients and satisfying $-1 \le p(x) \le 1$ for $-1 \le x \le 1$. He himself was able to prove that if $p(x)$ is a quadratic polynomial and $|p(x)| \le 1$ on $[-1, 1]$, then $|p'(x)| \le 4$ on the same interval. A. A. Markov [1] generalized this result for a polynomial of degree n in the real line. He proved that if $p(x)$ is an algebraic polynomial of degree at most n with real coefficients, then

$$\max_{-1\le x\le 1} |p'(x)| \le n^2 \max_{-1\le x\le 1} |p(x)|.$$

After about twenty years, Bernstein [25] needed the analog of Markov's Theorem for the unit disk in the complex plane instead of the interval $[-1, 1]$ to prove an inverse theorem of approximation (see Borwein and Erdélyi [15, p. 241]) to estimate how well a polynomial of a certain degree approximates a given continuous

*Mathematics Subject Classifications: 30A10, 30C10, 30C15.

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function in terms of its derivatives and Lipschitz constants. This leads to the famous well-known result known as Bernstein's inequality which states that if $t \in \tau_n$ (the set of all real trigonometric polynomials of degree at most n), then for $K := [0, 2\pi)$,

$$\max_{\theta \in K} |t^{(m)}(\theta)| \le n^m \max_{\theta \in K} |t(\theta)|. \tag{1}$$

The above inequality remains true for all $t \in \tau_n^c$ (the set of all complex trigonometric polynomials of degree at most n), which implies, as a particular case, the following algebraic polynomial version of Bernstein's inequality on the unit disk.

Theorem 1 *If $p(z)$ is a polynomial of degree n, then*

$$\max_{|z|=1} |p'(z)| \le n \max_{|z|=1} |p(z)|. \tag{2}$$

Equality holds in (2) if and only if $p(z)$ has all its zeros at the origin.

It is really of interest both in theoretical and practical aspects that continuous functions are approximated by polynomials. In this regard, we have the following interesting result (Theorem 2) [15, p. 241, Part (a) of E.18] which approximates m times differentiable real-valued function on a half-closed interval $[0, 2\pi)$ by trigonometric polynomials. For the sake of convenience of the readers, we state the above result more precisely.

Let Lip_α, $\alpha \in (0, 1]$, denote the family of all real-valued functions g defined on K satisfying

$$\sup\left\{\frac{|g(x) - g(y)|}{|x - y|^\alpha} : x \ne y \in K\right\} < \infty.$$

If $C(K)$ denotes the set of all continuous functions on K, then for $f \in C(K)$, let

$$E_n(f) := \inf\left\{\sup_{\theta \in K} |t - f| : t \in \tau_n\right\}.$$

Theorem 2 (Direct theorem) *Suppose f is m times differentiable on K and $f^{(m)} \in \mathrm{Lip}_\alpha$ for some $\alpha \in (0, 1]$. Then there is a constant C depending only on f so that*

$$E_n(f) \le Cn^{-(m+\alpha)}, n = 1, 2, \ldots.$$

On the other hand, the converse (inverse) of Theorem 2 is essentially of interest and is stated below.

Theorem 3 (Inverse theorem) *Suppose m is a non-negative integer, $\alpha \in (0, 1)$, and $f \in C(K)$. Suppose there is a constant $C > 0$ depending only on f such that*

$$E_n(f) \le Cn^{-(m+\alpha)}, n = 1, 2, \ldots.$$

Then f is m times continuously differentiable on K and $f^{(m)} \in \mathrm{Lip}_\alpha$.

The proof of Theorem 3 is done by the application of the well-known result due to Bernstein (inequality (1)) given in [15].

This discussion shows how important Bernstein and Markov-type inequalities are in approximation theory. For more information on direct and inverse theorems and related topics, you can check out books by Cheney [5], Lorentz [6], and DeVore and Lorentz [19].

Inequality (2) can be sharpened if the zeros of $p(z)$ are restricted. In this direction, Erdös conjectured and later Lax [16] proved that if $p(z)$ has no zero in $|z| < 1$, then

$$\max_{|z|=1} |p'(z)| \le \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{3}$$

As a partial generalization of (3), Malik [9] proved that if $p(z)$ has no zero in $|z| < k$, $k \geq 1$, then

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{1+k} \max_{|z|=1} |p(z)|.$$

Inequality (2) shows how fast a polynomial of degree at most n can change and is of interest both in mathematics, especially in approximation theory, and in application areas such as physical systems. Various analogs of these inequalities are known in which the underlying intervals, the sup-norms, and the families of polynomials are replaced by more general sets, norms, and families of functions, respectively. One such generalization is the relative growth of the polynomial $p(z)$ concerning two circles $|z| = r \leq 1$ and $|z| = 1$, and obtain inequalities about the dependence of sup-norms of $|p(rz)|$ on $|p(z)|$, where $|z| = 1$.

For a polynomial $p(z)$ of degree n, in accordance with the maximum modulus principle, the ensuing result [7] holds for $R \geq 1$ as

$$\max_{|z|=R} |p(z)| \leq R^n \max_{|z|=1} |p(z)|, \tag{4}$$

with equality only for $p(z) = \lambda z^n$.

The reverse analog of the inequality (4) whenever $R \leq 1$ is given by Varga [23], and he proved that if $p(z)$ is a polynomial of degree n, then for $0 < r \leq 1$,

$$\max_{|z|=r} |p(z)| \geq r^n \max_{|z|=1} |p(z)|. \tag{5}$$

Equality in (5) holds whenever $p(z) = az^n$.

For the class of polynomials having no zero inside the unit circle, it was Rivlin [26] who proved that if $p(z)$ is a polynomial of degree n having no zero in $|z| < 1$, then for $0 < r \leq 1$,

$$\max_{|z|=r} |p(z)| \geq \left(\frac{1+r}{2}\right)^n \max_{|z|=1} |p(z)|. \tag{6}$$

Equality holds in (6) if $p(z) = (z + a)^n$ whenever $|a| = 1$.

The following result can be viewed as a consequence of Rivlin's inequality and was proved independently by Jain [28] that if $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$, then for $r \geq 1$,

$$\max_{|z|=r} |p(z)| \geq \left(\frac{r+1}{2}\right)^n \max_{|z|=1} |p(z)|. \tag{7}$$

The inequalities mentioned above serve as the foundation for an extensive body of literature on their extensions, generalizations, and improvements in various directions, see the papers [2, 3, 10, 12, 13, 14, 22]. For a deeper understanding of these types of inequalities and their applications, we refer readers to the monographs [8, 20].

Govil [11] generalized inequality (6) by studying the relative growth of polynomials $p(z)$ having no zero in $|z| < 1$, with respect to two circles $|z| = r$ and $|z| = R$ whenever $0 < r \leq R \leq 1$, and proved that

$$\max_{|z|=r} |p(z)| \geq \left(\frac{1+r}{1+R}\right)^n \max_{|z|=R} |p(z)|. \tag{8}$$

The result is best possible and the equality holds for the polynomial $p(z) = \left(\frac{1+z}{1+R}\right)^n$.

Kumar [17] (see also [18, Corollary 2.2]) obtained a bound that sharpens inequality (6) by proving that if $p(z)$ is a polynomial of degree n having no zero in $|z| < 1$, then for $0 < r \leq 1$,

$$\max_{|z|=r} |p(z)| \geq \left[\left(\frac{1+r}{2}\right)^n + \left[\frac{|a_0| - |a_n|}{|a_0| + |a_n|}\right]\left(\frac{1-r}{2}\right)^n\right] \max_{|z|=1} |p(z)|. \tag{9}$$

Recently, Dhankhar and Kumar [21] further improved the bound of inequality (9) under the same hypothesis and proved that

$$\max_{|z|=r} |p(z)| \geq \left[\left(\frac{1+r}{2}\right)^n + \left[\frac{|a_0| - |a_n|}{|a_0| + |a_n|}\right]\frac{(1-r)}{2^n}\right] \max_{|z|=1} |p(z)|. \tag{10}$$

They [21] further obtained the refinement of inequality (8) by proving that if $p(z)$ is a polynomial of degree n having no zero in $|z|<1$, then for $0<r\le R\le 1$,

$$\max_{|z|=r}|p(z)|\ge\left[\left(\frac{1+r}{1+R}\right)^n+R^{2(n-1)}\left[\frac{|a_0|-|a_n|}{|a_0|+|a_n|R^n}\right]\frac{(R-r)}{(1+R)^n}\right]\max_{|z|=R}|p(z)|. \tag{11}$$

The present paper is mainly motivated by the desire to establish a generalized refinement of inequality (11) and as a consequence, we obtain another result that gives a generalized improvement of inequality (7). The paper is organized as follows. Section 2 introduces the main results, accompanied by remarks and corollaries. Section 3 provides and constructs auxiliary results essential for proving the main findings. The proofs of these main results are detailed in Section 4. Section 5 presents a numerical example to graphically illustrate and compare the new inequalities with previously established ones. Finally, Section 6 offers the conclusion.

2 Main Results

In this paper, we first establish a result for the class of polynomials that have no zero in $|z|<1$, except for a zero of multiplicity s at the origin, where $0\le s<n$. This result not only improves but also generalizes the inequalities (8) of Govil [11], and (11) of Dhankhar and Kumar [21]. More precisely, we prove that:

Theorem 4 *If $p(z)=z^s\sum_{\nu=0}^{n-s}a_\nu z^\nu$, $0\le s<n$, is a polynomial of degree n having no zero in $|z|<1$, except zero of multiplicity s at the origin, then for $0<r\le R\le 1$,*

$$\begin{aligned}\max_{|z|=r}|p(z)| \ \ge\ & \left(\frac{r}{R}\right)^s\left[\left(\frac{1+r}{1+R}\right)^{n-s}+R^{n-s-1}\left[\frac{|a_0|-|a_{n-s}|}{|a_0|+|a_{n-s}|R^{n-s}}\right]\frac{(1+r)^{n-s-1}(R-r)}{(1+R)^{n-s}}\right]\\ &\times\max_{|z|=R}|p(z)|.\end{aligned} \tag{12}$$

When $s=0$, inequality (12) of Theorem 4 reduces to the following interesting result which sharpens inequalities (8) due to Govil [11] as well as (11) of Dhankhar and Kumar [21].

Corollary 1 *If $p(z)=\sum_{\nu=0}^{n}a_\nu z^\nu$ is a polynomial of degree n having no zero in $|z|<1$, then for $0<r\le R\le 1$,*

$$\max_{|z|=r}|p(z)|\ge\left[\left(\frac{1+r}{1+R}\right)^n+R^{n-1}\left[\frac{|a_0|-|a_n|}{|a_0|+|a_n|R^n}\right]\frac{(1+r)^{n-1}(R-r)}{(1+R)^n}\right]\max_{|z|=R}|p(z)|. \tag{13}$$

Remark 1 *Since $p(z)$ has no zero in $|z|<1$, we have $|a_0|-|a_n|\ge 0$, and hence for $0<r\le R\le 1$,*

$$R^{n-1}\left[\frac{|a_0|-|a_n|}{|a_0|+|a_n|R^n}\right]\frac{(1+r)^{n-1}(R-r)}{(1+R)^n}\ge 0.$$

Therefore inequality (13) sharpens inequality (8) whenever $|a_0|\ne|a_n|$ and $0<r<R\le 1$. Moreover,

$$R^{n-1}\left[\frac{|a_0|-|a_n|}{|a_0|+|a_n|R^n}\right]\frac{(1+r)^{n-1}(R-r)}{(1+R)^n}\ge R^{2(n-1)}\left[\frac{|a_0|-|a_n|}{|a_0|+|a_n|R^n}\right]\frac{(R-r)}{(1+R)^n},$$

where $0<r\le R\le 1$. In view of these facts, inequality (13) improves over inequality (11) whenever $|a_0|\ne|a_n|$, $0<r<R\le 1$, and $n>1$.

Putting $R=1$ in inequality (13) of Corollary 1, we get the following interesting result which improves the well-known result due to Rivlin [26], and also the inequality (10) of Dhankhar and Kumar [21].

Corollary 2 *If $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n having no zero in $|z| < 1$, then for $0 < r \leq 1$,*

$$\max_{|z|=r} |p(z)| \geq \left[\left(\frac{1+r}{2} \right)^n + \left[\frac{|a_0| - |a_n|}{|a_0| + |a_n|} \right] \frac{(1+r)^{n-1}(1-r)}{2^n} \right] \max_{|z|=1} |p(z)|. \tag{14}$$

As an application of Theorem 4, we prove the following result, which deals with a class of polynomials having all its zeros in $|z| \leq 1$, More precisely, we prove

Theorem 5 *If $p(z) = z^s \sum_{\nu=0}^{n-s} a_\nu z^\nu$, $0 \leq s < n$, is a polynomial of degree n having all its zeros in $|z| \leq 1$ with zero of multiplicity s at the origin, then for $r \geq R \geq 1$,*

$$\max_{|z|=r} |p(z)| \geq \left(\frac{r}{R} \right)^s \left[\left(\frac{r+1}{R+1} \right)^{n-s} + \left[\frac{|a_{n-s}| - |a_0|}{|a_{n-s}| R^{n-s} + |a_0|} \right] \frac{(r+1)^{n-s-1}(r-R)}{(R+1)^{n-s}} \right] \max_{|z|=R} |p(z)|. \tag{15}$$

When $s = 0$ in inequality (15) of Theorem 5, we get the following result which gives an improvement of a result due to Dhankhar and Kumar [21, Corollary 1.4].

Corollary 3 *If $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n having all its zeros in $|z| \leq 1$ with zero of multiplicity s at the origin $0 \leq s < n$, then for $r \geq R \geq 1$,*

$$\max_{|z|=r} |p(z)| \geq \left[\left(\frac{r+1}{R+1} \right)^n + \left[\frac{|a_n| - |a_0|}{|a_n| R^n + |a_0|} \right] \frac{(r+1)^{n-1}(r-R)}{(R+1)^n} \right] \max_{|z|=R} |p(z)|. \tag{16}$$

Putting $R = 1$ in inequality (16) of Corollary 3, we have the following interesting result which yields an improvement of inequality (7) due to Jain [28], as well as of a result due to Dhankhar and Kumar [21, Theorem 1.2].

Corollary 4 *If $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n having all its zeros in $|z| \leq 1$, then for $r \geq 1$,*

$$\max_{|z|=r} |p(z)| \geq \left[\left(\frac{r+1}{2} \right)^n + \left[\frac{|a_n| - |a_0|}{|a_n| + |a_0|} \right] \frac{(r+1)^{n-1}(r-1)}{2^n} \right] \max_{|z|=1} |p(z)|. \tag{17}$$

Remark 2 *Since $p(z)$ has all its zeros in $|z| \leq 1$, we have $|a_n| - |a_0| \geq 0$, and we have for $r \geq 1$,*

$$\left[\frac{|a_n| - |a_0|}{|a_n| + |a_0|} \right] \frac{(r+1)^{n-1}(r-1)}{2^n} \geq 0.$$

This conditions shows that inequality (17) sharpens inequality (7) significantly, whenever $|a_n| \neq |a_0|$ and $r > 1$.

3 Lemmas

We shall need the following lemmas to prove the above theorems and verify the claims.

Lemma 1 *For any $e \geq 1$, $f \geq 1$, where $1 \geq R > 0$ and m is any positive integer, then*

$$\left(\frac{1}{R} \right) \frac{e-1}{e+R^m} + \left(\frac{1}{R} \right)^m \frac{f-1}{f+R} \geq \frac{ef-1}{ef+R^{m+1}}.$$

Proof. We need to show

$$\left(\frac{1}{R}\right)\frac{e-1}{e+R^m}+\left(\frac{1}{R}\right)^m\frac{f-1}{f+R}-\frac{ef-1}{ef+R^{m+1}}\geq 0.$$

Equivalently,

$$\begin{aligned}&R^m(e-1)(f+R)(ef+R^{m+1})+R(f-1)(e+R^m)(ef+R^{m+1})\\&-R^{m+1}(ef-1)(e+R^m)(f+R)\geq 0.\end{aligned}$$

Since

$$(e+R^m)(f+R)=ef+eR+fR^m+R^{m+1}\leq 2(ef+R^{m+1})$$

for $ef+R^{m+1}\geq eR+fR^m$ as $e\geq 1$, $f\geq 1$, it suffices to demonstrate that

$$\begin{aligned}&R^m(e-1)(f+R)(ef+R^{m+1})+R(f-1)(e+R^m)(ef+R^{m+1})\\&-2R^{m+1}(ef-1)(ef+R^{m+1})\geq 0.\end{aligned}$$

But

$$\begin{aligned}&R^m(e-1)(f+R)(ef+R^{m+1})+R(f-1)(e+R^m)(ef+R^{m+1})-2R^{m+1}(ef-1)(ef+R^{m+1})\\=\ &(ef+R^{m+1})[fR^m(1-R)(e-1)+eR(f-1)(1-R^m)]\geq 0.\end{aligned}$$

This completes the proof. ■

Lemma 2 *For any $0<r\leq R\leq 1$ and $R_j\geq 1$, where $1\leq j\leq n$, we have*

$$\prod_{j=1}^{n}\left(\frac{R_j+r}{R_j+R}\right)\geq\left(\frac{1+r}{1+R}\right)^n+R^{n-1}\left[\frac{R_1R_2...R_n-1}{R_1R_2...R_n+R^n}\right]\frac{(1+r)^{n-1}(R-r)}{(1+R)^n}. \tag{18}$$

Proof. We prove the result by induction on n. The identity

$$\frac{R_1+r}{R_1+R}=\left(\frac{1+r}{1+R}\right)+\left[\frac{R_1-1}{R_1+R}\right]\left(\frac{R-r}{1+R}\right), \tag{19}$$

justifies the validity of (18) for $n=1$. Let us assume that (18) is true for $n=m$. Then using the result for m and with the help of (19), we have

$$\begin{aligned}\prod_{j=1}^{m+1}\left(\frac{R_j+r}{R_j+R}\right) &= \left(\frac{R_{m+1}+r}{R_{m+1}+R}\right)\prod_{j=1}^{m}\left(\frac{R_j+r}{R_j+R}\right)\\
&\geq \left[\left(\frac{1+r}{1+R}\right)+\left[\frac{R_{m+1}-1}{R_{m+1}+R}\right]\left(\frac{R-r}{1+R}\right)\right]\left[\left(\frac{1+r}{1+R}\right)^m\right.\\
&\quad\left.+R^{m-1}\left[\frac{R_1R_2...R_m-1}{R_1R_2...R_m+R^m}\right]\frac{(1+r)^{m-1}(R-r)}{(1+R)^m}\right]\\
&= \left(\frac{1+r}{1+R}\right)^{m+1}+R^m\left[\left(\frac{1}{R}\right)\frac{R_1R_2...R_m-1}{R_1R_2...R_m+R^m}+\left(\frac{1}{R}\right)^m\frac{R_{m+1}-1}{R_{m+1}+R}\right.\\
&\quad\left.+\left(\frac{1}{R}\right)\left[\frac{R_1R_2...R_m-1}{R_1R_2...R_m+R^m}\right]\left[\frac{R_{m+1}-1}{R_{m+1}+R}\right]\left(\frac{R-r}{1+r}\right)\right]\frac{(1+r)^m(R-r)}{(1+R)^{m+1}}\\
&\geq \left(\frac{1+r}{1+R}\right)^{m+1}+R^m\left[\left(\frac{1}{R}\right)\frac{R_1R_2...R_m-1}{R_1R_2...R_m+R^m}+\left(\frac{1}{R}\right)^m\frac{R_{m+1}-1}{R_{m+1}+R}\right]\\
&\quad\times\frac{(1+r)^m(R-r)}{(1+R)^{m+1}}.\end{aligned}$$

Applying Lemma 1 to the second term on the right-hand side of the above inequality, we obtain

$$\prod_{j=1}^{m+1}\left(\frac{R_j+r}{R_j+R}\right) \geq \left(\frac{1+r}{1+R}\right)^{m+1} + R^m\left[\frac{R_1R_2...R_{m+1}-1}{R_1R_2...R_{m+1}+R^{m+1}}\right]\frac{(1+r)^m(R-r)}{(1+R)^{m+1}},$$

which by induction, the proof completes. ■

4 Proofs of the Main Results

Proof of Theorem 4. We consider $p(z) = z^s G(z)$, where $G(z)$ is a polynomial of degree $n-s$ having no zero in $|z| < 1$. Let $z_j = R_j e^{i\theta_j}$, where $1 \leq j \leq n-s$, be the zeros of $G(z)$. Since $G(z)$ has no zero in $|z| < 1$, $R_j \geq 1$. Then for any $0 < r \leq R \leq 1$ and $0 \leq \theta < 2\pi$, we have

$$\begin{aligned}
\frac{|G(re^{i\theta})|}{|G(Re^{i\theta})|} &= \prod_{j=1}^{n-s}\frac{|re^{i\theta}-R_je^{i\theta_j}|}{|Re^{i\theta}-R_je^{i\theta_j}|} \\
&= \prod_{j=1}^{n-s}\frac{|re^{i(\theta-\theta_j)}-R_j|}{|Re^{i(\theta-\theta_j)}-R_j|} \\
&= \prod_{j=1}^{n-s}\left(\frac{r^2+R_j^2-2rR_j\cos(\theta-\theta_j)}{R^2+R_j^2-2RR_j\cos(\theta-\theta_j)}\right)^{\frac{1}{2}} \\
&\geq \prod_{j=1}^{n-s}\left(\frac{r+R_j}{R+R_j}\right).
\end{aligned}$$

Therefore, we have

$$|G(re^{i\theta})| \geq \left[\prod_{j=1}^{n-s}\left(\frac{r+R_j}{R+R_j}\right)\right]|G(Re^{i\theta})|. \tag{20}$$

Now, applying Lemma 2 to the right-hand side of the inequality (20) and using the fact that

$$R_1R_2...R_{n-s} = \frac{|a_0|}{|a_{n-s}|},$$

since $R_j \geq 1$, $j = 1, 2, ...n-s$, $|a_0| - |a_{n-s}| \geq 0$. We have the following inequality

$$\left|G(re^{i\theta})\right| \geq \left[\left(\frac{1+r}{1+R}\right)^{n-s} + R^{n-s-1}\left[\frac{|a_0|-|a_{n-s}|}{|a_0|+|a_{n-s}|R^{n-s}}\right]\frac{(1+r)^{n-s-1}(R-r)}{(1+R)^{n-s}}\right]\left|G(Re^{i\theta})\right|. \tag{21}$$

Also, we have

$$\frac{|G(re^{i\theta})|}{|G(Re^{i\theta})|} = \frac{\left|\frac{p(re^{i\theta})}{(re^{i\theta})^s}\right|}{\left|\frac{p(Re^{i\theta})}{(Re^{i\theta})^s}\right|}. \tag{22}$$

From inequalities (21) and (22), we have

$$\max_{|z|=r}|p(z)| \geq \left(\frac{r}{R}\right)^s\left[\left(\frac{1+r}{1+R}\right)^{n-s} + R^{n-s-1}\left[\frac{|a_0|-|a_{n-s}|}{|a_0|+|a_{n-s}|R^{n-s}}\right]\frac{(1+r)^{n-s-1}(R-r)}{(1+R)^{n-s}}\right]\max_{|z|=R}|p(z)|.$$

This completes the proof of Theorem 4. ■

Proof of Theorem 5. Since $p(z)$ has all its zeros in $|z| \leq 1$, with zero of multiplicity s at the origin $0 \leq s < n$, its conjugate reciprocal polynomial $q(z) = z^n \overline{p\left(\frac{1}{\bar{z}}\right)}$ of degree $n-s$ has no zero in $|z| < 1$. Now, if $r \geq R \geq 1$, then $\frac{1}{r} \leq \frac{1}{R} \leq 1$. Applying corollary 1 to $q(z)$, we get

$$\max_{|z|=\frac{1}{r}} |q(z)| \geq \left[\left(\frac{1+\frac{1}{r}}{1+\frac{1}{R}} \right)^{n-s} + \left(\frac{1}{R} \right)^{n-s-1} \left[\frac{|a_{n-s}| - |a_0|}{|a_{n-s}| + |a_0| (\frac{1}{R})^{n-s}} \right] \frac{(1+\frac{1}{r})^{n-s-1}(\frac{1}{R} - \frac{1}{r})}{(1+\frac{1}{R})^{n-s}} \right] \max_{|z|=\frac{1}{R}} |q(z)|.$$

Using the facts that

$$\max_{|z|=\frac{1}{R}} |q(z)| = \frac{1}{R^n} \max_{|z|=R} |p(z)| \quad \text{and} \quad \max_{|z|=\frac{1}{r}} |q(z)| = \frac{1}{r^n} \max_{|z|=r} |p(z)|,$$

we have

$$\max_{|z|=r} |p(z)| \geq \left(\frac{r}{R} \right)^s \left[\left(\frac{r+1}{R+1} \right)^{n-s} + \left[\frac{|a_{n-s}| - |a_0|}{|a_{n-s}| R^{n-s} + |a_0|} \right] \frac{(r+1)^{n-s-1}(r-R)}{(R+1)^{n-s}} \right] \max_{|z|=R} |p(z)|.$$

This ends the proof of Theorem 5. ■

5 Numerical Example and Graphical Representations

As an illustration of the obtained results, we consider the following example and compare the bounds obtained from our results with previously known results.

Example 1 *Let $p(z) = (z+2)^2(z+3)^2$ with no zero in $|z| < 1$. Then on the circle $|z| = r$, we have*

$$\begin{aligned} |p(re^{i\theta})| &= [(r\cos\theta + 2)^2 + (r\sin\theta)^2][(r\cos\theta + 3)^2 + (r\sin\theta)^2)] \\ &= (r^2 + 4 + 4r\cos\theta)(r^2 + 9 + 6r\cos\theta). \end{aligned}$$

For convenience, we denote the quantity on the right-hand side by

$$A(\theta) = (r^2 + 4 + 4r\cos\theta)(r^2 + 9 + 6r\cos\theta), \quad \theta \in [0, 2\pi).$$

We must choose θ such that $A(\theta)$ is the maximum. For this we require to solve $A'(\theta) = 0$, i.e.,

$$-(10r^3 \sin\theta + 60r\sin\theta + 48r^2 \sin\theta\cos\theta) = 0,$$

i.e.,

$$\theta = 0, \pi, \quad \text{and} \quad \cos^{-1}[\frac{-10(r^2+6)}{48r}].$$

Further, for $\theta = 0$,

$$A''^3 - 60r - 48r^2 < 0, \quad \text{for} \ \ 0 < r \leq 1,$$

i.e., $A(\theta)$ attains its maximum value as $(r+2)^2(r+3)^2$. Thus

$$\max_{|z|=r} |p(z)| = \max_{0 \leq \theta < 2\pi} |p(re^{i\theta})| = (r+2)^2(r+3)^2,$$

which we can also observe in Figure 1.

We can consider the differences between the values of the left and the right-hand sides respectively in the inequalities (8) due to Govil [11], (11) due to Dhankhar and Kumar [21], and (13) of Corollary 1 as

$$\delta_1(r, R) = \begin{cases} M(p,r) - \left(\dfrac{1+r}{1+R} \right)^4 M(p,R), & \textit{of inequality } (8), \\ M(p,r) - \left[\left(\dfrac{1+r}{1+R} \right)^4 + \dfrac{35R^6(R-r)}{(36+R^4)(1+R)^4} \right] M(p,R), & \textit{of inequality } (11), \\ M(p,r) - \left[\left(\dfrac{1+r}{1+R} \right)^4 + \dfrac{35R^3(1+r)^3(R-r)}{(36+R^4)(1+R)^4} \right] M(p,R), & \textit{of inequality } (13). \end{cases}$$

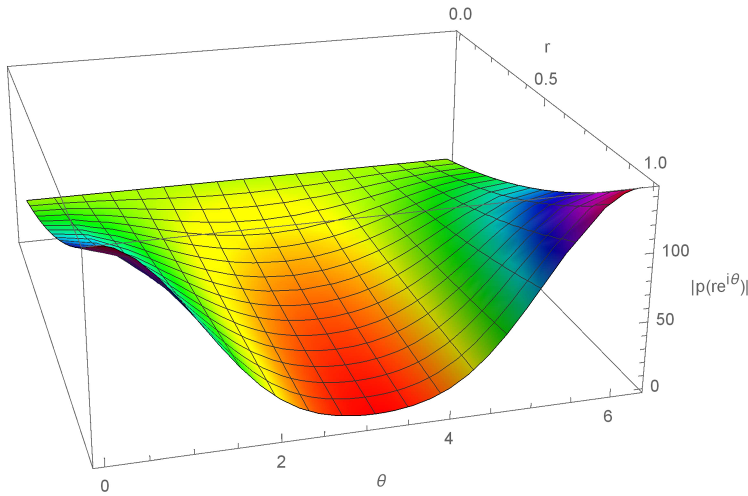

Figure 1: Surface graph of the function $(r,\theta) \mapsto |p(re^{i\theta})|$ for $0 < r \leq 1$ and $0 \leq \theta < 2\pi$, clearly showing the extremals.

In Figure 2, we present the comparison of the surface graphs for the differences $(r,R) \mapsto \delta_1(r,R)$ *of the inequalities* (8), (11), *and* (13) *for* $0 < r \leq 0.5$ *and* $0.5 \leq R \leq 1$ *indicated respectively by blue, grey, and orange colors. It is clear that the lesser the value of* $\delta_1(r,R)$ *for a surface, the more improved the bound is. It can be seen that inequality* (13) *of our Corollary 1 gives a higher improvement than inequalities* (8) *due to Govil [11], and* (11) *due to Dhankhar and Kumar [21] as the values of* r *and* R *vary.*

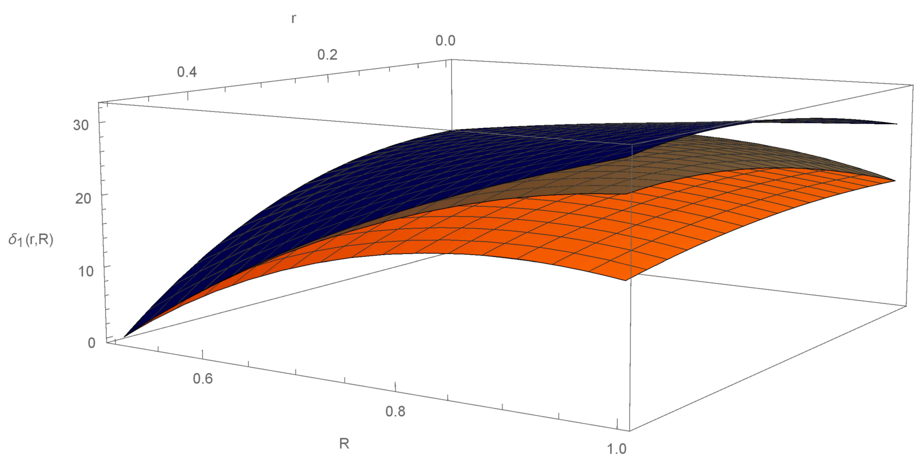

Figure 2: Surface comparison of the differences $(r,R) \mapsto \delta_1(r,R)$ in the inequalities (8), (11), and (13).

For $r = 0.5$, *we have* $\max_{|z|=0.5} |p(z)| = 76.56$, *and for the values of* R *as* 0.6, 0.8, *and* 0.9, *we have* $\max_{|z|=0.6} |p(z)| = 87.61$, $\max_{|z|=0.8} |p(z)| = 113.21$, *and* $\max_{|z|=0.9} |p(z)| = 127.92$, *respectively. Then it is evident that the differences between the values of the left and the right-hand sides in the inequalities* (8) *due to Govil [11],* (11) *due to Dhankhar and Kumar [21], and* (13) *of Corollary 1 are as in the following Table 1.*

When $R = 1$, *inequalities* (8) *due to Govil [11],* (11) *of Dhankhar and Kumar [21], and* (13) *of Corollary 1 reduces respectively to inequalities* (6) *due to Rivlin [26],* (10) *due to Dhankhar and Kumar [21], and* (14) *of Corollary 2.*

Further, we can examine the differences between the left and right-hand sides in the inequalities (9) *due*

	$R=0.6$	$R=0.8$	$R=0.9$
Inequality (8)	8.89	21.97	26.87
Inequality (11)	8.83	21.15	24.88
Inequality (13)	7.74	16.59	17.65

Table 1: Differences between the values of the left and the right-hand sides given by inequalities (8), (11), and (13).

to Kumar [17], (10) of Dhankhar and Kumar [21], and (14) of Corollary 2 to compare their sharpness. For this, we consider

$$\Delta(r)=\begin{cases} M(p,r)-\left[\left(\dfrac{1+r}{2}\right)^4+\left(\dfrac{35}{37}\right)\left(\dfrac{1-r}{2}\right)^4\right]144, & \text{of inequality } (9),\\ M(p,r)-\left[\left(\dfrac{1+r}{2}\right)^4+\left(\dfrac{35}{37}\right)\left(\dfrac{1-r}{2^4}\right)\right]144, & \text{of inequality } (10),\\ M(p,r)-\left[\left(\dfrac{1+r}{2}\right)^4+\left(\dfrac{35}{37}\right)\left(\dfrac{(1+r)^3(1-r)}{2^4}\right)\right]144, & \text{of inequality } (14).\end{cases}$$

In Figure 3, we present the graphics for the difference $r\mapsto\Delta(r)$ between the left and the right-hand sides in the inequalities (9) due to Kumar [17], (10) due to Dhankhar and Kumar [21], and (14) of Corollary 2.

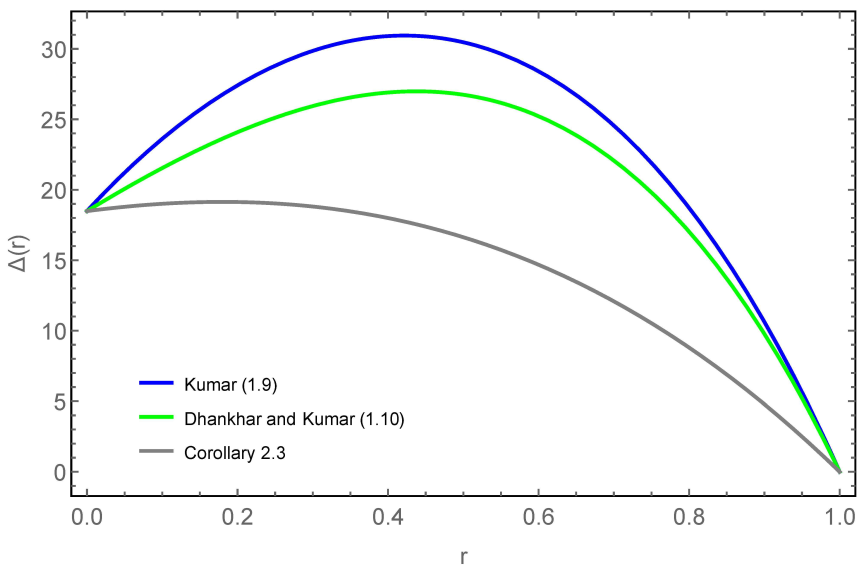

Figure 3: Comparison of the differences $r\mapsto\Delta(r)$ in the inequalities (9), (10), and (14).

For $r=0.3$, 0.5, and 0.7, we have respectively $\max_{|z|=0.3}|p(z)|=57.61$, $\max_{|z|=0.5}|p(z)|=76.56$, *and* $\max_{|z|=0.7}|p(z)|=99.80$. *Then it is evident that the differences between the values of the left and the right-hand sides given by inequalities (9) due to Kumar [17], (10) of Dhankhar and Kumar [21], and (14) of Corollary 2 are as in the following Table 2.*

Remark 3 *At any given point along the r-axis, the inequalities whose $\Delta(r)$ graph is positioned closer to the r-axis offer a more refined and improved bound in comparison to the other inequalities. From Figure 3,*

	$r = 0.3$	$r = 0.5$	$r = 0.7$
Inequality (9)	29.86	30.47	24.56
Inequality (10)	25.95	26.74	22.08
Inequality (14)	18.81	16.63	12.08

Table 2: Differences between the values of the left and the right-hand sides given by inequalities (9), (10), and (14).

it is clear that inequality (14) *of Corollary 2 gives the most improved bound for all* r*,* $0 < r \leq 1$*, for this particular example. It is clear that as the degree of the polynomial rises, our result enhances in precision and accuracy, surpassing significantly the results obtained from inequalities* (9) *due to Kumar [17], and* (10) *of Dhankhar and Kumar [21].*

6 Conclusion

In the past few years, a series of papers on Rivlin-type inequalities has been published, resulting in significant advancements in various areas. This paper continues the exploration of these inequalities for polynomials, establishing new results that consider the placement of specific polynomial coefficients. Our findings not only generalize but also refine several well-known polynomial inequalities. Finally, we provide a numerical example to graphically illustrate and compare our newly obtained inequalities with recent results.

Acknowledgment. We are very grateful to the referee for the valuable and useful suggestions or comments in upgrading the paper to the present form. We are also thankful to the National Institute of Technology Manipur for financial support.

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