Applied Mathematics E-Notes, 25(2025), 183-187 ©
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/ ISSN 1607-2510

A Remark On A Nonlinear System Arising In A Theory Of Thermal Explosion*

Huda Ibrahim Alsayed[†], Elaf Ali Alsisi[‡], Divya Adalja[§], Ramappa Manjunatha[¶]
Udandarao Sarat[‖], Damanjeet Aulakh[**], Kamaljit Kaur[††], Shivakrishna Dasi[‡‡]
Hessan Mohammad[§§¶¶]

Received 21 October 2024

Abstract

The purpose of this paper is to study the existence and multiplicity of positive solutions for a mathematical model of thermal explosion which is described by the system

$$\begin{cases} -\Delta u = \lambda f(v), & x \in \Omega, \\ -\Delta v = \lambda g(u), & x \in \Omega, \\ \mathbf{n}.\nabla u + a(u)u = 0, & x \in \partial\Omega, \\ \mathbf{n}.\nabla v + b(v)v = 0, & x \in \partial\Omega, \end{cases}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$, Δ is the Laplacian operator, $\lambda > 0$ is a parameter, f, g are C^1 strictly increasing functions that have a combined sublinear effect at ∞, and $a, b : [0, \infty) \to (0, \infty)$ are strictly increasing C^1 functions. We establish our existence and multiplicity results by the method of sub- and supersolutions.

1 Introduction

A classical problem in combustion theory is a model of thermal explosion which occurs due to a spontaneous ignition in a rapid combustion process. In this paper, we consider a model involving a nonlinear boundary heat loss which is not a very typical one in classical combustion theory, but is relevant to some more applications (see [4, 10, 12, 5] for details). The model reads as:

$$\begin{cases} \theta(t) - \Delta\theta = \lambda f(\eta), & (t, x) \in (0, \infty) \times \Omega, \\ \eta(t) - \Delta\eta = \lambda g(\theta), & (t, x) \in (0, \infty) \times \Omega, \\ \mathbf{n}.\nabla\theta + a(\theta)\theta = 0, & (t, x) \in (0, \infty) \times \partial\Omega, \\ \mathbf{n}.\nabla\eta + b(\eta)\eta = 0, & (t, x) \in (0, \infty) \times \partial\Omega, \\ \theta(0, x) = 0 = \eta(0, x). \end{cases} \tag{1}$$

*Mathematics Subject Classifications: Primary 35J25, 35J55, 35J65.

†Accounting Department, Faculty of Business School, King Khalid University, P.O. Box 9004, Abha 61413, Saudi Arabia
‡Business Informatics Department, Business College, King Khalid University, Abha, Saudi Arabia
§Department of Sciences, Vivekananda Global University, Jaipur, Rajasthan-303012, India
¶Department of Data analytics and Mathematical Sciences, School of Sciences, JAIN (Deemed to be University), Bangalore, Karnataka
‖Department of Mathematics Raghu Engineering College,Dakamarri-531162, Andhra Pradesh, India
**Centre for Research Impact & Outcome, Chitkara University Institute of Engineering and Technology Chitkara University, Rajpura, 140401, Punjab, India
††Department of Applied Sciences, Chandigarh Engineering College, Chandigarh Group of Colleges-Jhanjeri, Mohali, Pin 140307, Punjab, India
‡‡NIMS School of Computing Science and Artificial Intelligence, NIMS University Rajasthan, Jaipur, India
§§Department of computers Techniques engineering, College of technical engineering, The Islamic University, Najaf, Iraq
¶¶Department of computers Techniques engineering, College of technical engineering, The Islamic University of Al Diwaniyah, Al Diwaniyah, Iraq

Here θ, η are the appropriately scaled temperature in a bounded smooth domain $\Omega \subset \mathbb{R}^{N}, N \geq 1$, and f, g are the normalized reaction rate. We assume that f, g satisfy the following assumptions:

$(H1)$ $f, g \in C([0,\infty))$ are strictly increasing functions,

$(H2)$ $\lim_{s\to\infty} \frac{f(Ag(s))}{s} = 0$, for all $A > 0$.

On the C^2 boundary $\partial\Omega$, with the outward unit normal denoted by $\mathbf{n}$, the heat-loss parameters $a(\theta), b(\eta)$ are assumed to satisfy the following hypothesis:

$(H3)$ $a, b : [0,\infty) \to (0,\infty)$ are strictly increasing bounded C^1 functions.

Physically this assumption means that a heat loss through the boundary always exists and increases linearly with the temperature even in the small temperature regime.

A bifurcation (or scaling) parameter $\lambda > 0$ can be associated with the size of domain Ω in (1) which grows linearly as the measure of Ω increases. It is well known that, after normalizing for the size of Ω, the long term behavior of solution of the system (1) is close to the solution of the time-independent system:

$$\begin{cases} -\Delta u = \lambda f(v), & x \in \Omega, \\ -\Delta v = \lambda g(u), & x \in \Omega, \\ \mathbf{n}.\nabla u + a(u)u = 0, & x \in \partial\Omega, \\ \mathbf{n}.\nabla v + b(v)v = 0, & x \in \partial\Omega. \end{cases} \tag{2}$$

The motivation for this study cames from the work in [7] where the authors established the existence, uniqueness and multiplicity of positive solutions for certain range of λ for the single equation of the form

$$\begin{cases} -\Delta u = \lambda f(u), & x \in \Omega, \\ \mathbf{n}.\nabla u + a(u)u = 0, & x \in \partial\Omega. \end{cases}$$

Here we extend this study to Laplacian system of the form (2). In [1], Ali-Shivaji-Ramaswamy discussed the existence of multiple positive solutions to such systems with Dirichlet boundary conditions. One can refer to [3, 8] for some recent existence and uniqueness results of elliptic problems with nonlinear boundary conditions.

2 Existence Results

In this section, we shall establish our existence results via the method of sub - supersolution. A pair of nonnegative functions $(\psi_1, \psi_2) \in W^{1,2} \cap C(\overline{\Omega}) \times W^{1,2} \cap C(\overline{\Omega})$ and a pair $(z_1, z_2) \in W^{1,2} \cap C(\overline{\Omega}) \times W^{1,2} \cap C(\overline{\Omega})$ are called a subsolution and supersolution of the system (2) if they satisfy

$$\begin{cases} -\Delta \psi_1 \leq \lambda f(\psi_2), & x \in \Omega, \\ -\Delta \psi_2 \leq \lambda g(\psi_1), & x \in \Omega, \\ \mathbf{n}.\nabla \psi_1 + a(\psi_1)\psi_1 \leq 0, & x \in \partial\Omega, \\ \mathbf{n}.\nabla \psi_2 + b(\psi_2)\psi_2 \leq 0, & x \in \partial\Omega, \end{cases} \tag{3}$$

and

$$\begin{cases} -\Delta z_1 \geq \lambda f(z_2), & x \in \Omega, \\ -\Delta z_2 \geq \lambda g(z_1), & x \in \Omega, \\ \mathbf{n}.\nabla z_1 + a(z_1)z_1 \geq 0, & x \in \partial\Omega, \\ \mathbf{n}.\nabla z_2 + b(z_2)z_2 \geq 0, & x \in \partial\Omega, \end{cases} \tag{4}$$

respectively. It is well known that if there exist sub and supersolutions (ψ_1, ψ_2) and (z_1, z_2) respectively of the system (2) such that $(\psi_1, \psi_2) \leq (z_1, z_2)$. Then the system (2) has a solution (u, v) such that $(u, v) \in [(\psi_1, \psi_2), (z_1, z_2)]$ (see [2, 6]).

By strict sub and super-solutions we understand functions (ψ_1, ψ_2) and (z_1, z_2) for which strict inequalities (3) and (4) hold.

Our multiplicity results are obtained by constructing sub and super-solution pairs that satisfy the following lemma.

Lemma 1 ([6, 9, 11]) *Suppose that the system (2) has a sub-solution (ψ_1, ψ_2), a strict super-solution (ζ_1, ζ_2), a strict sub-solution (w_1, w_2), and a super-solution (z_1, z_2) for the system (2) such that*

$$(\psi_1, \psi_2) \leq (\zeta_1, \zeta_2) \leq (z_1, z_2),$$

$$(\psi_1, \psi_2) \leq (w_1, w_2) \leq (z_1, z_2),$$

and $(w_1, w_2) \not\leq (\zeta_1, \zeta_2)$. Then the system (2) has at least three distinct solutions (u_i, v_i), $i = 1, 2, 3$ such that

$$(u_1, v_1) \in [(\psi_1, \psi_2), (\zeta_1, \zeta_2)], \ \ (u_2, v_2) \in [(w_1, w_2), (z_1, z_2)]$$

and

$$(u_3, v_3) \in \Big[(\psi_1, \psi_2), (z_1, z_2)\Big] \backslash \Big(\Big[(\psi_1, \psi_2), (\zeta_1, \zeta_2)\Big] \cup \Big[(w_1, w_2), (z_1, z_2)\Big]\Big).$$

To precisely state our existence result we consider the unique classical solution e_r of the following linear elliptic problem

$$\begin{cases} -\Delta e_r = 1, & x \in \Omega, \\ \mathbf{n}.\nabla e_r + r_0 e_r = 0, & x \in \partial\Omega, \end{cases}$$

for $r = a, b$, where $r_0 = r(0)$. Then we establish the following theorem.

Theorem 1 *Let (H1)–(H3) hold and $f(0)$ or $g(0)$ be strictly positive. Then the system (2) has a positive solution (u, v) for all $\lambda > 0$.*

Proof. It is easy to see that $(\psi_1, \psi_2) = (0, 0)$ is a subsolution of the system (2). We now construct the supersolution (z_1, z_2). Let $(z_1, z_2) = \Big(C_\lambda e_a, \lambda g(C_\lambda \|e_b\|_\infty) e_b\Big)$, where C_λ is a large number to be chosen later. We shall verify that (z_1, z_2) is a supersolution of the system (2) for all $\lambda > 0$. By $(H2)$ we can choose C_λ large enough so that

$$C_\lambda \geq \lambda f\Big(\lambda g(C_\lambda \|e_b\|_\infty) \|e_b\|_\infty\Big),$$

and therefore

$$\begin{aligned} -\Delta z_1 &= C_\lambda \geq \lambda f\Big(\lambda g(C_\lambda \|e_b\|_\infty) \|e_b\|_\infty\Big) \\ &\geq \lambda f\Big(\lambda g(C_\lambda \|e_b\|_\infty) e_b\Big) \\ &= \lambda f(z_2) \ \ \text{in}\, \Omega, \end{aligned}$$

and

$$\begin{aligned} \mathbf{n}.\nabla z_1 + a(z_1) z_1 &\geq C_\lambda \mathbf{n}.\nabla e_a + C_\lambda e_a a_0 \\ &= C_\lambda (\mathbf{n}.\nabla e_a + e_a a_0) \\ &= 0 \ \ \text{on}\, \partial\Omega. \end{aligned}$$

Next,

$$-\Delta z_2 = \lambda g\Big(C_\lambda \|e_b\|_\infty\Big) \geq \lambda g\Big(C_\lambda e_b\Big) = \lambda g(z_1), \ \ \text{in}\, \Omega,$$

and

$$\begin{aligned} \mathbf{n}.\nabla z_2 + b(z_2) z_2 &\geq \lambda g\Big(C_\lambda \|e_b\|_\infty\Big) \mathbf{n}.\nabla e_b + \lambda g\Big(C_\lambda \|e_b\|_\infty\Big) e_b b_0 \\ &= \lambda g\Big(C_\lambda \|e_b\|_\infty\Big)(\mathbf{n}.\nabla e_b + b_0 e_b) = 0, \ \ \text{on}\, \partial\Omega, \end{aligned}$$

which implies that (z_1, z_2) is indeed a positive supersolution of the system (2). Therefore the system (2) has a positive solution for all $\lambda > 0$. ■

Our second result concerns with multiplicity of solution for the system (2) and gives an estimate on the parameter λ when such a situation occurs. For positive constants a_i, b_i; $i = 1, 2$, define

$$Q_1(a_1, b_1) = \min\{\frac{a_1}{f(b_1)}, \frac{b_1}{g(a_1)}\}$$

and

$$Q_2(a_2, b_2) = \max\{\frac{a_2}{f(b_2)}, \frac{b_2}{g(a_2)}\}.$$

Then we establish:

Theorem 2 *Assume $f(0)$ or $g(0)$ be positive. Let B_R be the largest ball of radius R inscribed in Ω, for $0 < \epsilon < R$, we define*

$$C_1(\Omega) = \inf_{\epsilon} \frac{N}{\epsilon^N} \frac{R^{N-1}}{R - \epsilon},$$

and $C(\Omega) = C_1(\Omega)\|e_r\|_\infty$, for $r = a, b$. Let (H1)–(H3) hold and $\frac{Q_1}{Q_2} > C(\Omega)$ for some a_i, b_i, $i = 1, 2$. Then the system (2) has at least three positive solutions for $\lambda \in (\lambda_*, \lambda^*)$, where $\lambda_* = CQ_2$ and $\lambda^* = \frac{Q_1}{\|e_r\|_\infty}$, for $r = a, b$.

Proof. We will establish a pair of subsolutions (ψ_1, ψ_2), (w_1, w_2) and a pair of supersolutions (ζ_1, ζ_2), (z_1, z_2), satisfying Lemma 1. Clearly $(\psi_1, \psi_2) = (0, 0)$ is a subsolution of the system (2).

We next construct a positive supersolution (ζ_1, ζ_2), of the system (2) when $\lambda < \frac{Q_1}{\|e_r\|_\infty}$, for $r = a, b$. Since $\lambda < \frac{a_1}{f(b_1)\|e_a\|_\infty}$, we can choose $\epsilon > 0$ so small that $\lambda f(b_1) < \frac{a_1}{\epsilon + \|e_a\|_\infty}$. Let

$$(\zeta_1, \zeta_2) = (a_1 \frac{e_a + \epsilon}{\|e_a\|_\infty + \epsilon}, b_1 \frac{e_b + \epsilon}{\|e_b\|_\infty + \epsilon}).$$

Then, we have

$$\begin{aligned} -\Delta\zeta_1 &= \frac{a_1}{\epsilon + \|e_a\|_\infty} > \lambda f(b_1) \\ &\geq \lambda f\Big(b_1 \frac{e_b + \epsilon}{\|e_b\|_\infty + \epsilon}\Big) \\ &= \lambda f(\zeta_2) \ \text{ in}\, \Omega, \end{aligned}$$

and

$$\begin{aligned} \mathbf{n}.\nabla\zeta_1 + a(\zeta_1)\zeta_1 &\geq \frac{a_1}{\epsilon + \|e_a\|_\infty}\Big(\mathbf{n}.\nabla e_a + (e_a + \epsilon)a_0\Big) \\ &= \frac{a_1}{\epsilon + \|e_a\|_\infty}(\mathbf{n}.\nabla e_a + a_0 e_a + a_0\epsilon) \\ &= \frac{a_1 a_0 \epsilon}{\epsilon + \|e_a\|_\infty} \\ &> 0 \ \text{ on}\, \partial\Omega. \end{aligned}$$

Similar argument shows that ζ_2 satisfies $-\Delta\zeta_2 > \lambda g(\zeta_1)$ in Ω, and $\mathbf{n}.\nabla\zeta_2 + b(\zeta_2)\zeta_2 > 0$.

Next let us construct a strict sub-solution (w_1, w_2) of the system (2). First note that a system

$$\begin{cases} -\Delta u_D = \lambda f(v_D), & x \in \Omega, \\ -\Delta v_D = \lambda g(u_D), & x \in \Omega, \\ u_D = 0 = v_D, & x \in \partial\Omega, \end{cases}$$

admits a strict sub-solution (w_{1D}, w_{2D}) with $\|w_{1D}\|_\infty \geq a_2$ and $\|w_{2D}\|_\infty \geq b_2$ provided $\lambda < \lambda^*$ (see [1]). Then we have $(w_1, w_2) \nleq (\zeta_1, \zeta_2)$. From Hopf's lemma, $\mathbf{n}.\nabla w_{iD} < 0$ for $i = 1, 2$. Therefore, setting $w_1 = w_{1D}$ and $w_2 = w_{2D}$ we obtain a strict sub-solution for the system (2). for $\lambda > \lambda_*$.

Let (z_1, z_2) be the super solution as in the proof of Theorem 2 Further, $w_i, \zeta_i \leq z_i$, $i = 1, 2$ for C_λ large. Hence there exist positive solutions (u_i, v_i), $i = 1, 2, 3$ such that

$$(u_1, v_1) \in [(\psi_1, \psi_2), (\zeta_1, \zeta_2)], \quad (u_2, v_2) \in [(w_1, w_2), (z_1, z_2)]$$

and

$$(u_3, v_3) \in \Big[(\psi_1, \psi_2), (z_1, z_2)\Big] \setminus \Big(\Big[(\psi_1, \psi_2), (\zeta_1, \zeta_2)\Big] \cup \Big[(w_1, w_2), (z_1, z_2)\Big]\Big).$$

■

References

[1] J. Ali, R. Shivaji and M. Ramaswamy, Multiple positive solutions for classes of elliptic systems with combined nonlinear effects, Diff. and Int. Eqns., 19(2006), 669–680.

[2] H. Amann, Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, SIAM Rev. 18(1976), 620–709.

[3] D. Butler, E. Ko, E.K. Lee and R. Shivaji, Positive radial solutions for elliptic equations on exterior domains with nonlinear boundary conditions, Commu. Pure. Appl. Anal, 13(2014), 2713–2731.

[4] D. A. Frank-Kamenetskii, Diffusion and Heat Transfer in Chemical Kinetics, Plenum Press, New York, 1969.

[5] P. Gordon, E. Ko and R. Shivaji, Multiplicity and uniqueness of positive solutions for elliptic equations with nonlinear boundary conditions arising in a theory of thermal explosion, Nonlinear Anal. Real World Appl., 15(2014), 51–57.

[6] F. Inkmann, Existence and multiplicity theorems for semilinear elliptic equations with nonlinear boundary conditions, Indiana Univ. Math. J., 31(1982), 213–221.

[7] E. Ko and S. Prashanth, Positive solutions for elliptic equations in two dimensionals arising in a theory of thermal explosion, Taiwanese. J. Math, 19(2015), 1759–1775.

[8] E. Ko, M. Ramaswamy and R. Shivaji, Uniqueness of positive radial solutions for a class of semipositone problems on the exterior domain of a ball, J. Math. Anal, Appl, 423(2015), 399–409.

[9] C. Maya, S. Oruganti and R. Shivaji, Positive solutions for classes of p-Laplacian equations, Dif. and Int. Eqns., 16(2003), 757–768.

[10] N. N. Semenov, Chemical Kinetics and Chain Reactions, Oxford University Press, London, 1935.

[11] R. Shivaji, A remark on the existence of three solutions via sub-super solutions. Nonlinear Analysis and Application, Lecture Notes in Pure and Applied Mathematics, 109(1987), 561–566.

[12] Y. B. Zeldovich, G. I. Barenblatt, V. B. Librovich and G. M. Makhviladze, The Mathematical Theory of Combustion and Explosions, Consultants Bureau, New York, 1985.