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On A Fourth Order Boundary Value Problem Arising In Elastic Beam Analysis*

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Abstract

For a specific class of two-point boundary value problems involving fourth-order ordinary differential equations, we investigate the existence and uniqueness of solutions. Applications of such problems for beam deflection modeling are quite interesting. The principal instruments utilized in this investigation comprise the application of the fixed point theorems of Banach and Rus. When a beam experiences a loading force and is immersed at its left end and free at its right, our theoretical findings are applied to elastic beam deflections. For several classes of linear and nonlinear loading forces, the existence and uniqueness of solutions to the models are guaranteed.

1 Introduction

We consider the nonlinear fourth-order differential equation

$$y'''' + \beta^2 y'' = f(x, y), \quad x \in [0, L], \tag{1}$$

together with the boundary conditions

$$y(0) = 0, \quad y'(0) = 0, \quad y''(L) = 0, \quad y'''(L) = 0, \tag{2}$$

in this article. Here $L, \beta \in \mathbb{R}$, $\beta > 0$, $f : [0, L] \times \mathbb{R} \to \mathbb{R}$ is continuous and $f(x, 0) \neq 0$ for $x \in [0, L]$. The assumption $f(x, 0) \neq 0$ excludes the possibility of the trivial solution. By a solution to (1)–(2) we mean a function $y : [0, L] \to \mathbb{R}$ such that y is four times differentiable, with a continuous fourth-order derivative on $[0, L]$, which we denote by $y \in C^4([0, 1])$, and our y satisfies both (1) and (2).

Using fixed point theorems, the purpose of this work is to establish and compare results on the existence of a unique solution to (1)–(2). Our major findings indicate that there exists a unique nontrivial solution to the problem if and only if L is small and the function f meets the Lipschitz condition. We first construct the related Green's function and rephrase our problem (1)–(2) as an equivalent integral equation in order to get these results. Next, we utilize an infinite strip to apply the Banach fixed point theorem. The Banach fixed point theorem is then used inside a closed and bounded set in order to extend the conclusion to a larger class of functions. Lastly, we extend the length of the interval where the result is valid by Rus's fixed point theorem. We consider examples to compare the obtained results.

An examination of fourth-order boundary value problems is naturally motivated by the analysis of deflections in elastic beams. Imagine a thin beam which is embedded at the end $x = 0$ and free at the other end $x = L$ on the x-axis. A transverse load $h(x)$ and a compressive force P, which vary throughout the beam's length, are two of the forces acting on it. If $y = y(x)$ represents the resultant deflection of the beam at position x, with $\beta = \sqrt{\frac{P}{EI}}$, the differential equation

$$y'''' + \beta^2 y'' = h(x), \quad x \in [0, L], \tag{3}$$

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depicts the displacement of the beam in the transverse direction caused by buckling, where E is the slender member's Young's modulus and I is the beam's moment of inertia along its length. Assume that the compressive load P and $E \cdot I$ are constants for the sake of simplicity. It is obvious that β needs to be bigger than zero; otherwise, $P = 0$ would result from $\beta = 0$. Given that the beam is embedded at the end $x = 0$ and free at the other end $x = L$, the problem in this case is subject to the boundary conditions (2). The fourth-order differential equation (1) is obtained if we take into account the transverse load on the beam, which is provided by $f(x, y)$. This load may not be linear.

Examining the construction of Green functions unique to boundary value problems is a common step in the study of their solutions. As a result, Green functions are important in boundary value problem theory. In general, the expressions of Green functions are more complicated. This statement is true even for a simple-looking second-order differential equation $y'' + 2ay' + by = g(x)$, $x \in [0, L]$, and the boundary conditions $Ay(0) - By'(0) = 0$, $Cy(L) + Dy'(L) = 0$, or $Ay(0) + By(L) = 0$, $Cy'(0) + Dy'(L) = 0$. (See Appendices $\mathcal{A}$ and $\mathcal{B}$). Consequently, it is tough to derive their properties, which play an important role in the qualitative analysis of the corresponding boundary value problems (See Appendix $\mathcal{C}$). Here A, B, C, D, L, a, $b \in \mathbb{R}$ with $A^2 + B^2 > 0$, $C^2 + D^2 > 0$, and $g : [0, L] \to \mathbb{R}$ is continuous.

Fourth-order boundary value problems and their application to elastic beam deflections have been extensively researched. The solvability of fourth-order boundary value problems and the existence and uniqueness of solutions have been the focus of numerous well-known studies. One powerful and efficient method for proving the existence or uniqueness of solutions to nonlinear boundary value problems is to use fixed point theorems. The presence of solutions to fourth-order boundary value problems using different fixed point theorems has been investigated by numerous writers. Among the several articles discussing the subject of the solvability of fourth-order nonlinear differential equations with respect to a variety of boundary conditions using fixed point theory, we refer to [1, 2, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 17, 19] and the references therein for a few recent publications in this area.

The problem at hand is not the same as the works described above. We also note that our method of ensuring the existence and uniqueness of solutions to fourth-order boundary value problems appears to hold a unique place in the literature: it applies Rus's fixed point theorem. The findings presented here represent a step forward from more conventional methods, including using Banach's fixed point theorem. Rus's fixed point theorem and two metrics are used to accomplish this. As we shall see, this makes it possible to comprehend the existence and uniqueness of solutions to a larger class of problems. This involves honing the Lipschitz constants in closed and limited domains as well as in a global (unbounded) setting.

Since our main tools in this paper are fixed point theorems, let us state Banach and Rus's fixed point theorems for the reader's convenience.

Theorem 1 ([18]) *Let X be a nonempty set, and d be a metric on X such that (X, d) forms a complete metric space. If the mapping $T : X \to X$ satisfies $d(Ty, Tz) \leq \alpha d(y, z)$ for some $\alpha \in (0, 1)$ and all y, $z \in X$, then there is a unique $y_0 \in X$ such that $Ty_0 = y_0$.*

Theorem 2 ([16]) *Let X be a nonempty set, and d and ρ be two metrics on X such that (X, d) forms a complete metric space. If the mapping $T : X \to X$ is continuous with respect to d on X and*

1. *there exists $c > 0$ such that $d(Ty, Tz) \leq c\rho(y, z)$ for all y, $z \in X$, and*

2. *there exists $\alpha \in (0, 1)$ such that $\rho(Ty, Tz) \leq \alpha\rho(y, z)$ for all y, $z \in X$,*

then there is a unique $y_0 \in X$ such that $Ty_0 = y_0$.

This is how the remainder of the paper is structured. Using the variation of parameters formula and a few more assumptions, we construct the Green's function in Section 2 that corresponds to the boundary value problem (1)–(2). The estimation of an integral involving the Green's function is the focus of Section 3. Our primary theorems regarding the uniqueness of the solution to the boundary value problem (1)–(2) are demonstrated in Section 4. We also give some instances to show how well-established results can be applied.

2 Construction of the Green's Function

Rewriting the boundary value problem (1)–(2) as an equivalent integral equation is the aim of this section. Thus, we will examine the linear equation (3) in conjunction with the boundary conditions (2).

Proposition 1 *If $h : [0, L] \to \mathbb{R}$ is a continuous function, then the boundary value problem (3)–(2) has a unique solution which we can write as*

$$y(x) = \int_0^L G(x, \xi)h(\xi)d\xi, \quad 0 \le x \le L, \tag{4}$$

where the Green's function is given by

$$G(x, \xi) = \begin{cases} G_1(x, \xi), & 0 \le \xi \le x \le L, \\ G_2(x, \xi), & 0 \le x \le \xi \le L. \end{cases} \tag{5}$$

Here

$$\mathcal{K}(x, \xi) = \frac{1}{\beta^3} \left[\beta(x - \xi) - \sin \beta(x - \xi)\right],$$

$$G_1(x, \xi) = \frac{(1 - \cos \beta x) \sin \beta\xi}{\beta^3} + \frac{(\sin \beta x - \beta x) \cos \beta\xi}{\beta^3} + \mathcal{K}(x, \xi),$$

and

$$G_2(x, \xi) = \frac{(1 - \cos \beta x) \sin \beta\xi}{\beta^3} + \frac{(\sin \beta x - \beta x) \cos \beta\xi}{\beta^3}.$$

Proof. The general solution of (3) is given by

$$y(x) = c_1 + c_2x + c_3 \cos \beta x + c_4 \sin \beta x + \int_0^x \mathcal{K}(x, \xi)h(\xi)d\xi, \quad 0 \le x \le L, \tag{6}$$

where c_1, c_2, c_3 and c_4 are arbitrary constants. From (6), we have

$$y'(x) = c_2 - \beta c_3 \sin \beta x + \beta c_4 \cos \beta x + \int_0^x \mathcal{K}_x(x, \xi)h(\xi)d\xi, \quad 0 \le x \le L, \tag{7}$$

$$y''(x) = -\beta^2 c_3 \cos \beta x - \beta^2 c_4 \sin \beta x + \int_0^x \mathcal{K}_{xx}(x, \xi)h(\xi)d\xi, \quad 0 \le x \le L, \tag{8}$$

$$y'''(x) = \beta^3 c_3 \sin \beta x - \beta^3 c_4 \cos \beta x + \int_0^x \mathcal{K}_{xxx}(x, \xi)h(\xi)d\xi, \quad 0 \le x \le L, \tag{9}$$

where

$$\mathcal{K}_x(x, \xi) = \frac{1}{\beta^2} \left[1 - \cos \beta(x - \xi)\right], \quad \mathcal{K}_{xx}(x, \xi) = \frac{\sin \beta(x - \xi)}{\beta}, \quad \mathcal{K}_{xxx}(x, \xi) = \cos \beta(x - \xi).$$

Using boundary conditions (2) in (6)–(9) and rearranging the terms, we get

$$c_1 = \int_0^L \frac{\sin \beta\xi}{\beta^3} h(\xi)d\xi, \quad c_2 = -\int_0^L \frac{\cos \beta\xi}{\beta^2} h(\xi)d\xi, \quad c_3 = -c_1, \quad c_4 = -\frac{1}{\beta} c_2.$$

Substituting the constants c_1, c_2, c_3 and c_4 in (6) and rearranging the terms, we obtain (4). Hence, the boundary value problem (3)–(2) has a unique solution (4). To verify that $y \in C^4[0, L]$, one can differentiate (4) four times and check its continuity. ∎

3 Estimation of the Green's Function

In this section, we prove a useful inequality for an integral that involves the Green's function.

Proposition 2 *The Green's function in (5) satisfies*

$$\int_0^L |G(x,\xi)|\, d\xi \leq Lk_1 + Lk_2 + \frac{L^4}{24},$$

where

$$k_1 = \sup_{x\in[0,L]} \left| \frac{(1-\cos\beta x)}{\beta^3} \right| \quad \textit{and} \quad k_2 = \sup_{x\in[0,L]} \left| \frac{(\sin\beta x - \beta x)}{\beta^3} \right|.$$

Proof. For all $x \in [0,L]$, we have

$$\begin{aligned}
\int_0^L |G(x,\xi)|\, d\xi &= \int_0^x |G(x,\xi)|\, d\xi + \int_x^L |G(x,\xi)|\, d\xi \\
&\leq \int_0^x \left| \frac{(1-\cos\beta x)\sin\beta\xi}{\beta^3} \right| d\xi + \int_0^x \left| \frac{(\sin\beta x - \beta x)\cos\beta\xi}{\beta^3} \right| d\xi \\
&\quad + \int_0^x |\mathcal{K}(x,\xi)|\, d\xi \\
&\quad + \int_x^L \left| \frac{(1-\cos\beta x)\sin\beta\xi}{\beta^3} \right| d\xi + \int_x^L \left| \frac{(\sin\beta x - \beta x)\cos\beta\xi}{\beta^3} \right| d\xi \\
&\leq k_1 \left[\int_0^x |\sin\beta\xi|\, d\xi + \int_x^L |\sin\beta\xi|\, d\xi \right] \\
&\quad + k_2 \left[\int_0^x |\cos\beta\xi|\, d\xi + \int_x^L |\cos\beta\xi|\, d\xi \right] \\
&\quad + \int_0^x \mathcal{K}(x,\xi) d\xi \\
&= k_1 \int_0^L |\sin\beta\xi|\, d\xi + k_2 \int_0^L |\cos\beta\xi|\, d\xi + \int_0^x \mathcal{K}(x,\xi) d\xi \\
&\leq Lk_1 + Lk_2 + \frac{x^4}{24} \\
&\leq Lk_1 + Lk_2 + \frac{L^4}{24}.
\end{aligned}$$

The proof is complete. ■

4 Existence of a Unique Solution

In this section, we will apply fixed point theorems to prove our results on the existence of a unique solution to the boundary value problem (1)–(2) and compare them. For this, let us define two metrics on the set X of continuous functions defined on $[0,L]$ such that

$$d(y,z) = \sup_{x\in[0,L]} |y(x) - z(x)|,$$

and

$$\rho(y,z) = \left(\int_0^L |y(x) - z(x)|^2\, dx \right)^{\frac{1}{2}},$$

for all y, $z \in X$. It is easy to show that (X,ρ) is a metric space and (X,d) forms a complete metric space.

4.1 Application of Theorem 1 on an Infinite Strip

Theorem 3 *Let $f : [0, L] \times \mathbb{R} \to \mathbb{R}$ be a continuous function and $f(x, 0) \neq 0$ for $x \in [0, L]$. Assume f satisfies the Lipschitz condition with respect to its second argument with a Lipschitz constant K. If*

$$Lk_1 + Lk_2 + \frac{L^4}{24} < \frac{1}{K}, \tag{10}$$

then there exists a unique non-trivial solution to the boundary value problem (1)–(2).

Proof. It follows from Proposition 1 that the boundary value problem (1)–(2) is equivalent to the integral equation

$$y(x) = \int_0^L G(x, \xi) f(\xi, y(\xi)) d\xi, \quad 0 \leq x \leq L.$$

Define the mapping $T : X \to X$ by

$$(Ty)(x) = \int_0^L G(x, \xi) f(\xi, y(\xi)) d\xi, \quad 0 \leq x \leq L.$$

Clearly, y is a solution of (1)–(2) iff y is a fixed point of T. To establish the existence of a unique fixed point of T, we show that the conditions of Theorem 1 hold. To see this, let y, $z \in X$, $x \in [0, L]$ and consider

$$\begin{aligned}
|(Ty)(x) - (Tz)(x)| &= \left| \int_0^L G(x, \xi) f(\xi, y(\xi)) d\xi - \int_0^L G(x, \xi) f(\xi, z(\xi)) d\xi \right| \\
&\leq \int_0^L |G(x, \xi)| \, |f(\xi, y(\xi)) - f(\xi, z(\xi))| \, d\xi \\
&\leq K \int_0^L |G(x, \xi)| \, |y(\xi) - z(\xi)| \, d\xi \\
&\leq K d(y, z) \int_0^L |G(x, \xi)| \, d\xi \\
&\leq K \left(Lk_1 + Lk_2 + \frac{L^4}{24} \right) d(y, z),
\end{aligned}$$

implying that

$$d(Ty, Tz) \leq K \left(Lk_1 + Lk_2 + \frac{L^4}{24} \right) d(y, z),$$

for all y, $z \in X$. Since

$$K \left(Lk_1 + Lk_2 + \frac{L^4}{24} \right) < 1,$$

the mapping T is a contraction. Hence, by Theorem 1, T has a unique fixed point in X. Therefore, the boundary value problem (1)–(2) has a unique non-trivial solution $y \in X$. The proof is complete. ■

4.2 Application of Theorem 1 within a Closed and Bounded Set

Consider a closed ball B_N with radius N in X as follows:

$$B_N = \{y \in X : d(y, 0) \leq N\}.$$

Since B_N is a closed subspace of X, the pair (B_N, d) forms a complete metric space. Clearly, $T : B_N \to X$.

Theorem 4 *Let $f : [0, L] \times [-N, N] \to \mathbb{R}$ be a continuous function and $f(x, 0) \neq 0$ for $x \in [0, L]$. Assume f satisfies the Lipschitz condition with respect to its second argument with a Lipschitz constant K. If L satisfies the inequalities (10) and*

$$Lk_1 + Lk_2 + \frac{L^4}{24} \leq \frac{N}{M}, \tag{11}$$

where

$$M = \sup_{(x,y)\in[0,L]\times[-N,N]} |f(x, y)|,$$

then there exists a unique non-trivial solution y to the boundary value problem (1)–(2) such that

$$|y(x)| \leq N, \quad x \in [0, L].$$

Proof. First, we show that $T : B_N \to B_N$. To see this, let $y \in B_N$, $x \in [0, L]$ and consider

$$\begin{aligned} |(Ty)(x)| &\leq \int_0^L |G(x, \xi)|\, |f(\xi, y(\xi))|\, d\xi \\ &\leq M \int_0^L |G(x, \xi)|\, d\xi \\ &\leq M \left(Lk_1 + Lk_2 + \frac{L^4}{24} \right) \end{aligned}$$

implying that

$$d(Ty, 0) \leq M \left(Lk_1 + Lk_2 + \frac{L^4}{24} \right) \leq N.$$

Thus, $Ty \in B_N$. Therefore, $T : B_N \to B_N$. It follows from the proof of Theorem 3 that $T : B_N \to B_N$ is a contraction. Hence, by Theorem 1, T has a unique fixed point in B_N. Therefore, the boundary value problem (1)–(2) has a unique non-trivial solution $y \in B_N$. The proof is complete. ■

4.3 Application of Theorem 2 on an Infinite Strip

Theorem 5 *Let $f : [0, L] \times \mathbb{R} \to \mathbb{R}$ be a continuous function and $f(x, 0) \neq 0$ for $x \in [0, L]$. Assume f satisfies the Lipschitz condition with respect to its second argument with a Lipschitz constant K. If*

$$\left(k_1^2 L + k_2^2 L + 2k_1k_2L + \frac{L^7}{252} + \frac{k_1L^4}{12} + \frac{k_2L^4}{12} \right)^{\frac{1}{2}} < \frac{1}{K}, \tag{12}$$

then there exists a unique non-trivial solution to the boundary value problem (1)–(2).

Proof. To establish the existence of a unique fixed point of T using Theorem 2, we have to show that the conditions of Theorem 2 hold. For this purpose, let $y,\, z \in X$, $x \in [0, L]$ and consider

$$\begin{aligned}
|(Ty)(x) - (Tz)(x)| &= \left| \int_0^L G(x,\xi) f(\xi, y(\xi)) d\xi - \int_0^L G(x,\xi) f(\xi, z(\xi)) d\xi \right| \\
&\le \int_0^L |G(x,\xi)| \, |f(\xi, y(\xi)) - f(\xi, z(\xi))| \, d\xi \\
&\le K \int_0^L |G(x,\xi)| \, |y(\xi) - z(\xi)| \, d\xi \\
&\le K \left(\int_0^L |G(x,\xi)|^2 \, d\xi \right)^{\frac{1}{2}} \left(\int_0^L |y(\xi) - z(\xi)|^2 \, d\xi \right)^{\frac{1}{2}} \\
&\le K \sup_{0 \le x \le L} \left(\int_0^L |G(x,\xi)|^2 \, d\xi \right)^{\frac{1}{2}} \rho(y,z) \\
&\le c\rho(y,z),
\end{aligned}$$

implying that

$$d(Ty, Tz) \le c\rho(y,z),$$

for all $y,\, z \in X$. Here

$$c = K \sup_{0 \le x \le L} \left(\int_0^L |G(x,\xi)|^2 \, d\xi \right)^{\frac{1}{2}} > 0.$$

Also,

$$\begin{aligned}
\rho(y,z) &= \left(\int_0^L |y(x) - z(x)|^2 \, dx \right)^{\frac{1}{2}} \\
&\le \left(\int_0^L \sup_{0 \le x \le L} |y(x) - z(x)|^2 \, dx \right)^{\frac{1}{2}} \\
&\le \sup_{0 \le x \le L} |y(x) - z(x)| \left(\int_0^L dx \right)^{\frac{1}{2}} \\
&= L^{\frac{1}{2}} d(y,z).
\end{aligned}$$

Thus, we obtain that

$$d(Ty, Tz) \le c\rho(y,z) \le cL^{\frac{1}{2}} d(y,z),$$

for all $y,\, z \in X$. Then, for any $\epsilon > 0$, choose $\delta = \frac{\epsilon}{cL^{\frac{1}{2}}}$ such that $d(Ty, Tz) < \epsilon$ whenever $d(y,z) < \delta$. Therefore, T is continuous with respect to d on X. Consider

$$\begin{aligned}
\left(\int_0^L |(Ty)(x) - (Tz)(x)|^2 \, dx \right)^{\frac{1}{2}} &\le \left(\int_0^L \left[K \left(\int_0^L |G(x,\xi)|^2 \, d\xi \right)^{\frac{1}{2}} \rho(y,z) \right]^2 dx \right)^{\frac{1}{2}} \\
&\le K\rho(y,z) \left(\int_0^L \left(\int_0^L |G(x,\xi)|^2 \, d\xi \right) dx \right)^{\frac{1}{2}}.
\end{aligned}$$

Now, consider

$$\begin{aligned}
\int_0^L |G(x,\xi)|^2\, d\xi &= \int_0^x |G(x,\xi)|^2\, d\xi + \int_x^L |G(x,\xi)|^2\, d\xi \\
&\leq \int_0^x \Bigg(k_1^2 \sin^2 \beta\xi + k_2^2 \cos^2 \beta\xi + \frac{(x-\xi)^6}{36} + 2k_1k_2 \sin\beta\xi \cos\beta\xi \\
&\quad + \frac{k_1(x-\xi)^3 \sin\beta\xi}{3} + \frac{k_2(x-\xi)^3\cos\beta\xi}{3} \Bigg) d\xi \\
&\quad + \int_x^L \left(k_1^2 \sin^2 \beta\xi + k_2^2 \cos^2 \beta\xi + 2k_1k_2 \sin\beta\xi\cos\beta\xi \right) d\xi \\
&= \int_0^L \Bigg(k_1^2 \sin^2 \beta\xi + k_2^2 \cos^2 \beta\xi + 2k_1k_2 \sin\beta\xi \cos\beta\xi \Bigg) d\xi \\
&\quad + \int_0^x \Bigg(\frac{(x-\xi)^6}{36} + \frac{k_1(x-\xi)^3 \sin\beta\xi}{3} + \frac{k_2(x-\xi)^3\cos\beta\xi}{3} \Bigg) d\xi \\
&\leq k_1^2 L + k_2^2 L + 2k_1k_2 L + \frac{L^7}{252} + \frac{k_1 L^4}{12} + \frac{k_2 L^4}{12}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\left(\int_0^L |(Ty)(x) - (Tz)(x)|^2 dx \right)^{1/2} &\leq K\rho(y,z) \left(\int_0^L \left(k_1^2 L + k_2^2 L + 2k_1k_2 L + \frac{L^7}{252} + \frac{k_1L^4}{12} + \frac{k_2L^4}{12} \right) dx \right)^{1/2} \\
&= K \left(k_1^2 L + k_2^2 L + 2k_1k_2 L + \frac{L^7}{252} + \frac{k_1L^4}{12} + \frac{k_2L^4}{12} \right)^{\frac{1}{2}} \rho(y,z),
\end{aligned}$$

implying that

$$\rho(Ty, Tz) \leq \alpha\rho(y,z),$$

for all y, $z \in X$. Here

$$\alpha = K \left(k_1^2 L + k_2^2 L + 2k_1k_2 L + \frac{L^7}{252} + \frac{k_1L^4}{12} + \frac{k_2L^4}{12} \right)^{\frac{1}{2}} < 1.$$

Hence, by Theorem 2, T has a unique fixed point in X. Therefore, the boundary value problem (1)–(2) has a unique non-trivial solution $y \in X$. The proof is complete. ■

5 Examples

In this section, we provide a few examples to illustrate the applicability of results established in the previous section.

Example 1 *Consider (1)–(2) with $\beta = L = 1$ and*

$$f(x,y) = \frac{y^2}{y^2+1} + 10x + 1.$$

Clearly, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $f(x,0) \neq 0$ for $x \in [0,1]$. Also, f satisfies the Lipschitz condition with respect to its second argument with a Lipschitz constant $K = 1$. Further, we obtain

$$k_1 = \sup_{x\in[0,1]} |1 - \cos x| \approx 0.4597,$$

and

$$k_2 = \sup_{x\in[0,1]} |\sin x - x| \approx 0.1585.$$

Clearly,

$$Lk_1 + Lk_2 + \frac{L^4}{24} \approx 0.6599 < 1,$$

implying the inequality (10) holds. Therefore, by Theorem 3, (1)–(2) has a unique non-trivial solution $y \in X$.

Example 2 *Consider (1)–(2) with $\beta = 1$, $L = 2$ and*

$$f(x,y) = \frac{y^2}{y^2+1} + 10x + 1.$$

Clearly, $f : [0,2] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $f(x,0) \neq 0$ for $x \in [0,2]$. Also, f satisfies the Lipschitz condition with respect to its second argument with a Lipschitz constant $K = 1$. Further, we obtain

$$k_1 = \sup_{x\in[0,2]} |1-\cos x| \approx 1.4162,$$

and

$$k_2 = \sup_{x\in[0,2]} |\sin x - x| \approx 1.0907.$$

Since

$$Lk_1 + Lk_2 + \frac{L^4}{24} \approx 5.6805 > 1,$$

the inequality (10) does not hold. Hence, Theorem 3 is not applicable in this case.

Example 3 *Consider (1)–(2) with $\beta = L = 1$ and $f(x,y) = x^2y^2 + 1$. Clearly, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $f(x,0) \neq 0$ for $x \in [0,1]$. But, f doesn't satisfies the Lipschitz condition with respect to its second argument. Hence, Theorem 3 is not applicable in this case.*

Example 4 *Consider (1)–(2) with $\beta = 1$, $L = 0.5$ and $f(x,y) = x^2y^2 + 1$. Choose $N = 1$. Clearly, $f : [0,0.5] \times [-1,1] \to \mathbb{R}$ is a continuous function and $f(x,0) \neq 0$ for $x \in [0,0.5]$. Also, f satisfies the Lipschitz condition with respect to its second argument with a Lipschitz constant $K = 0.5$. Further, we obtain*

$$k_1 = \sup_{x\in[0,0.5]} |1-\cos x| \approx 0.1224,$$

$$k_2 = \sup_{x\in[0,0.5]} |\sin x - x| \approx 0.0206,$$

and

$$M = \sup_{(x,y)\in[0,0.5]\times[-1,1]} |f(x,y)| = 1.25.$$

Since

$$Lk_1 + Lk_2 + \frac{L^4}{24} \approx 0.0741 < 0.8 < 2,$$

where $\frac{1}{K} = 2$ and $\frac{N}{M} = 0.8$, the inequalities (10) and (11) hold. Hence, by Theorem 4, (1)–(2) has a unique non-trivial solution $y \in B_N$.

Example 5 *Consider Example 4. We obtain that*

$$\left(k_1^2L + k_2^2L + 2k_1k_2L + \frac{L^7}{252} + \frac{k_1L^4}{12} + \frac{k_2L^4}{12}\right)^{\frac{1}{2}} \approx 0.1044 < \frac{1}{K}.$$

Then, by Theorem 5, (1)–(2) has a unique non-trivial solution $y \in X$.

6 Conclusion

In this paper, we applied the fixed point theorems of Banach and Rus to investigate the existence and uniqueness of solutions to a specific class of two-point boundary value problems involving fourth-order ordinary differential equations. These problems have fascinating implications in beam deflection modeling. Future work on this project will examine whether there are any solutions for either a functionally graded elastic beam subjected to a loading force or a variable cross-section elastic beam subjected to a loading force [3, 13] such that the beam is embedded at the left end and free at the right end.

Appendix $\mathcal{A}$

Consider the second-order differential equation

$$y'' + 2ay' + by = g(x), \quad x \in [0, L], \tag{13}$$

together with the boundary conditions

$$Ay(0) - By'(0) = 0, \quad Cy(L) + Dy'(L) = 0. \tag{14}$$

Proposition 3 *Assume $a^2 = b$ and $\Lambda_1 = (A + aB)(cL - aDL + D) + B(C - aD) \neq 0$. If $g : [0, L] \to \mathbb{R}$ is a continuous function, then the boundary value problem (13)–(14) has a unique solution which we can write as*

$$y(x) = e^{-ax} \int_0^L P(x, \xi) g(\xi) d\xi, \quad 0 \leq x \leq L,$$

where the Green's function is given by

$$P(x, \xi) = \frac{1}{\Lambda_1} \begin{cases} P_1(x, \xi), & 0 \leq \xi \leq x \leq L, \\ P_2(x, \xi), & 0 \leq x \leq \xi \leq L. \end{cases} \tag{15}$$

Here

$$P_2(x, \xi) = -\left[(A + aB)x + B\right]\left[(C - aD)(L - \xi) + D\right],$$

and

$$P_1(x, \xi) = P_2(x, \xi) + \Lambda_1(x - \xi).$$

Proposition 4 *Assume $a^2 < b$ and take $c = \sqrt{b - a^2}$. If*

$$\Lambda_2 = (A + aB)\left[(C - aD) \sin cL + cD \cos cL\right] + cB\left[(C - aD) \cos cL - cD \sin cL\right] \neq 0$$

and $g : [0, L] \to \mathbb{R}$ is a continuous function, then the boundary value problem (13)–(14) has a unique solution which we can write as

$$y(x) = e^{-ax} \int_0^L Q(x, \xi) g(\xi) d\xi, \quad 0 \leq x \leq L,$$

where the Green's function is given by

$$Q(x, \xi) = \frac{1}{\Lambda_2} \begin{cases} Q_1(x, \xi), & 0 \leq \xi \leq x \leq L, \\ Q_2(x, \xi), & 0 \leq x \leq \xi \leq L. \end{cases} \tag{16}$$

Here

$$Q_2(x, \xi) = -\left[B \cos cx + \frac{(A + aB)}{c} \sin cx\right]\left[(C - aD) \sin c(L - \xi) + cD \cos c(L - \xi)\right],$$

and

$$Q_1(x, \xi) = Q_2(x, \xi) + \frac{\Lambda_2}{c} \sin c(x - \xi).$$

Proposition 5 *Assume $a^2 > b$ and take $d = \sqrt{a^2 - b}$. If*

$$\Lambda_3 = (A + aB - dB)\left[(C - aD)e^{-dL} - dDe^{-dL}\right] - (A + aB + dB)\left[(C - aD)e^{dL} + dDe^{dL}\right] \neq 0$$

and $g : [0, L] \to \mathbb{R}$ is a continuous function, then the boundary value problem (13)–(14) has a unique solution which we can write as

$$y(x) = e^{-ax}\int_0^L R(x,\xi)g(\xi)d\xi, \quad 0 \leq x \leq L,$$

where the Green's function is given by

$$R(x,\xi) = \frac{1}{\Lambda_3}\begin{cases} R_1(x,\xi), & 0 \leq \xi \leq x \leq L, \\ R_2(x,\xi), & 0 \leq x \leq \xi \leq L. \end{cases} \tag{17}$$

Here

$$\begin{aligned} R_1(x,\xi) &= \left[(A + aB + dB)e^{dx} - (A + aB - dB)e^{-dx}\right] \\ &\times \left[\left(\frac{C - aD}{2d} + \frac{D}{2}\right)e^{d(L-\xi)} + \left(\frac{D}{2} - \frac{C - aD}{2d}\right)e^{-d(L-\xi)}\right], \end{aligned}$$

and

$$R_2(x,\xi) = R_1(x,\xi) + \frac{\Lambda_3}{2d}\left[e^{d(x-\xi)} - e^{-d(x-\xi)}\right].$$

Appendix $\mathcal{B}$

Consider the second-order differential equation (13) together with the boundary conditions

$$Ay(0) + By(L) = 0, \quad Cy'(0) + Dy'(L) = 0. \tag{18}$$

Proposition 6 *Assume $a^2 = b$ and $\lambda_4 = \left(Ae^{aL} + B\right)\left(Ce^{aL} - aDL + D\right) + BL\left(aCe^{aL} + aD\right) \neq 0$. If $g : [0, L] \to \mathbb{R}$ is a continuous function, then the boundary value problem (13)–(18) has a unique solution which we can write as*

$$y(x) = e^{-ax}\int_0^L U(x,\xi)g(\xi)d\xi, \quad 0 \leq x \leq L,$$

where the Green's function is given by

$$U(x,\xi) = \frac{1}{\Lambda_4}\begin{cases} \Theta_1(\xi) + x\Theta_2(\xi) + \Lambda_4(x - \xi), & 0 \leq \xi \leq x \leq L, \\ \Theta_1(\xi) + x\Theta_2(\xi), & 0 \leq x \leq \xi \leq L. \end{cases} \tag{19}$$

Here

$$\Theta_1(\xi) = -BL\left[aD(L - \xi) - D\right] - B(L - \xi)\left[Ce^{aL} - aDL + D\right], \quad 0 \leq \xi \leq L,$$

and

$$\Theta_2(\xi) = \left[Ae^{aL} + B\right]\left[aD(L - \xi) - D\right] - B(L - \xi)\left[aCe^{aL} + aD\right], \quad 0 \leq \xi \leq L.$$

Proposition 7 *Assume $a^2 < b$ and take $c = \sqrt{b - a^2}$. If*

$$\begin{aligned} \Lambda_5 &= \left[Ae^{aL} + B\cos cL\right]\left[cCe^{aL} - aD\sin cL + cD\cos cL\right] \\ &+ B\left[aCe^{aL} + aD\cos cL + cD\sin cL\right]\sin cL \neq 0, \end{aligned}$$

and $g : [0, L] \to \mathbb{R}$ is a continuous function, then the boundary value problem (13)–(18) has a unique solution which we can write as

$$y(x) = e^{-ax}\int_0^L V(x,\xi)g(\xi)d\xi, \quad 0 \leq x \leq L,$$

where the Green's function is given by

$$V(x,\xi)=\frac{1}{\Lambda_5}\begin{cases}\Theta_3(\xi)\cos cx+\Theta_4(\xi)\sin cx+\frac{\Lambda_5}{c}\sin c(x-\xi), & 0\le\xi\le x\le L,\\ \Theta_3(\xi)\cos cx+\Theta_4(\xi)\sin cx, & 0\le x\le\xi\le L.\end{cases}\tag{20}$$

Here

$$\Theta_3(\xi)=-\frac{B}{c}\left[cCe^{aL}-aD\sin cL+cD\cos cL\right]\sin c(L-\xi)-B\left[\frac{aD}{c}\sin c(L-\xi)-D\cos c(L-\xi)\right]\sin cL,$$

and

$$\begin{aligned}\Theta_4(\xi) &= -\frac{B}{c}\left[aCe^{aL}+aD\cos cL+cD\sin cL\right]\sin c(L-\xi)\\ &\quad+\left[Ae^{aL}+B\cos cL\right]\left[\frac{aD}{c}\sin c(L-\xi)-D\cos c(L-\xi)\right],\quad 0\le\xi\le L.\end{aligned}$$

Proposition 8 *Assume $a^2>b$ and take $d=\sqrt{a^2-b}$. If*

$$\begin{aligned}\Lambda_6 &= -\left[Ae^{aL}+Be^{dL}\right]\left[aCe^{aL}+dCe^{aL}+aDe^{-dL}+dDe^{-dL}\right]\\ &\quad-\left[Ae^{aL}+Be^{-dL}\right]\left[-aCe^{aL}+dCe^{aL}-aDe^{dL}+dDe^{dL}\right]\neq 0,\end{aligned}$$

and $g:[0,L]\to\mathbb{R}$ is a continuous function, then the boundary value problem (13)*–*(18) *has a unique solution which we can write as*

$$y(x)=e^{-ax}\int_0^L W(x,\xi)g(\xi)d\xi,\quad 0\le x\le L,$$

where the Green's function is given by

$$W(x,\xi)=\frac{1}{\Lambda_6}\begin{cases}\Theta_5(\xi)e^{dx}+\Theta_6(\xi)e^{-dx}+\frac{\Lambda_6}{2d}\left[e^{d(x-\xi)}-e^{-d(x-\xi)}\right], & 0\le\xi\le x\le L,\\ \Theta_5(\xi)e^{dx}+\Theta_6(\xi)e^{-dx}, & 0\le x\le\xi\le L.\end{cases}\tag{21}$$

Here

$$\begin{aligned}\Theta_5(\xi) &= \frac{B}{2d}\left[aCe^{aL}+dCe^{aL}+aDe^{-dL}+dDe^{-dL}\right]\left[e^{d(L-\xi)}-e^{-d(L-\xi)}\right]\\ &\quad-\left[Ae^{aL}+Be^{-dL}\right]\left[\left(\frac{aD}{2d}-\frac{D}{2}\right)e^{d(L-\xi)}-\left(\frac{aD}{2d}+\frac{D}{2}\right)e^{-d(L-\xi)}\right],\quad 0\le\xi\le L,\end{aligned}$$

and

$$\begin{aligned}\Theta_6(\xi) &= \frac{B}{2d}\left[-aCe^{aL}+dCe^{aL}-aDe^{dL}+dDe^{dL}\right]\left[e^{d(L-\xi)}-e^{-d(L-\xi)}\right]\\ &\quad+\left[Ae^{aL}+Be^{dL}\right]\left[\left(\frac{aD}{2d}-\frac{D}{2}\right)e^{d(L-\xi)}-\left(\frac{aD}{2d}+\frac{D}{2}\right)e^{-d(L-\xi)}\right],\quad 0\le\xi\le L.\end{aligned}$$

Appendix $\mathcal{C}$

Proposition 9 *The Green's function in* (15) *satisfies*

$$\int_0^L|P(x,\xi)|\,d\xi\le\left[\frac{|A+aB|\,L+|B|}{|\Lambda_1|}\right]\left[|C-aD|\,\frac{L^2}{2}+|D|\,L\right]+\frac{L^2}{2}.$$

Proposition 10 *The Green's function in* (16) *satisfies*

$$\int_0^L |Q(x,\xi)|\,d\xi \le \frac{k_3 L}{|\Lambda_2|}\left[|C-aD| + |C|\,|D|\right] + \frac{L}{c},$$

where

$$k_3 = \sup_{x\in[0,L]} \Bigg| B\cos cx + (A+aB)\sin cx \Bigg|.$$

Proposition 11 *The Green's function in* (17) *satisfies*

$$\int_0^L |R(x,\xi)|\,d\xi \le \frac{k_4}{d\,|\Lambda_3|}\left[\left|\frac{C-aD}{2d} + \frac{D}{2}\right|\left(e^{dL}-1\right) + \left|\frac{D}{2} - \frac{C-aD}{2d}\right|\left(1-e^{-dL}\right)\right] + \frac{1}{2d^2}\left(e^{dL}+e^{-dL}-2\right),$$

where

$$k_4 = \sup_{x\in[0,L]} \Bigg| (A+aB+dB)e^{dx} - (A+aB-dB)e^{-dx} \Bigg|.$$

Proposition 12 *The Green's function in* (19) *satisfies*

$$\begin{aligned}\int_0^L |U(x,\xi)|\,d\xi \quad &\le \quad \frac{1}{|\Lambda_4|}\Bigg[|D|\,L^2\left(|B| + \left|Ae^{aL}+B\right|\right)\left(\frac{|a|\,L}{2}+1\right) \\ &\qquad + \frac{|B|\,L^2}{2}\left(\left|Ce^{aL} - aDL + D\right| + L\left|aCe^{aL}+aD\right|\right)\Bigg] + \frac{L^2}{2}.\end{aligned}$$

Proposition 13 *The Green's function in* (20) *satisfies*

$$\begin{aligned}\int_0^L |V(x,\xi)|\,d\xi \quad &\le \quad \frac{1}{|\Lambda_5|}\Bigg[\frac{|B|\,k_5 L}{c}\left|cCe^{aL} - aD\sin cL + cD\cos cL\right| + |B|\,|D|\,k_5 L\left(\frac{|a|}{c}+1\right)|\sin cL| \\ &\qquad + \frac{|B|\,k_6 L}{c}\left|aCe^{aL} + aD\cos cL + cD\sin cL\right| + \left|Ae^{aL} + B\cos cL\right|\,|D|\,k_6 L\left(\frac{|a|}{c}+1\right)\Bigg] \\ &\qquad + \frac{L}{c},\end{aligned}$$

where

$$k_5 = \sup_{x\in[0,L]} |\cos cx| \quad \textit{and} \quad k_6 = \sup_{x\in[0,L]} |\sin cx|.$$

Proposition 14 *The Green's function in* (21) *satisfies*

$$\begin{aligned}\int_0^L |W(x,\xi)|\,d\xi \quad &\le \quad \frac{1}{|\Lambda_6|}\Bigg[\frac{|B|\,k_7}{2d^2}\left(e^{dL}+e^{-dL}-2\right) + k_8\left(\left|\frac{aD}{2d} - \frac{D}{2}\right|\left(e^{dL}-1\right) + \left|\frac{aD}{2d}+\frac{D}{2}\right|\left(1-e^{-dL}\right)\right)\Bigg] \\ &\qquad + \frac{1}{2d^2}\left(e^{dL}+e^{-dL}-2\right),\end{aligned}$$

where

$$k_7 = \sup_{x\in[0,L]} \left[\left|aCe^{aL} + dCe^{aL} + aDe^{-dL} + dDe^{-dL}\right| e^{dx} + \left|-aCe^{aL} + dCe^{aL} - aDe^{dL} + dDe^{dL}\right| e^{-dx}\right],$$

and

$$k_8 = \sup_{x\in[0,L]} \left[\left|Ae^{aL} + Be^{-dL}\right| e^{dx} + \left|Ae^{aL} + Be^{dL}\right| e^{-dx}\right].$$

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