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Convergence Analysis Of A New Iterative Algorithm For Generalized (α, β)-Nonexpansive Mapping In Banach Space*

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Abstract

In this paper, we consider the class of generalized (α, β)-nonexpansive mappings in the setting of Banach spaces. We prove the existence of fixed point and convergence results for these mappings under the PJ-iterative process. The weak convergence is obtained with the help of Opial's property while the strong convergence results are obtained under various assumptions. Finally, we give a numerical example and connect our PJ-iterative process with them. Our results obtained in this paper improve, extend and unify some related results in the literature.

1 Introduction

In recent years, iterative processes lead us to solving fixed point problems. In this way a sequence is generated by the algorithm. The intended fixed point value of the equation of fixed point and the given equation's solution is the limit of the series. In case of contraction mappings, Banach fixed point theorem [3] signals the fundamental Picard iteration $x_{n+1} = \mathfrak{T}x_n$. However, when the Picard iterative process for a given mapping does not converge, we employ alternative iterative procedures with different steps. One of the other iterative processes that have been studied by authors are the Mann iteration [7], Noor iteration [8], Ishikawa iteration [6], Abbas et al. [1], Thakur et al. [17] and Hussain et al. [5].

On the other hand, we present a new four step iterative process for generalized nonexpansive mappings and call it as a PJ iterative process. The PJ-iterative process reads as follows: Let $\mathcal{C}$ be a nonempty, closed and convex subset of a uniformly convex Banach space X and $\mathfrak{T}\colon \mathcal{C} \to \mathcal{C}$ be a mapping and the sequence $\{x_n\}$ generated iteratively by

$$\begin{cases} x_1 \in \mathcal{C}, \\ w_n = \mathfrak{T}((1-\alpha_n)x_n + \alpha_n \mathfrak{T}x_n), \\ z_n = \mathfrak{T}((1-\beta_n)w_n + \beta_n \mathfrak{T}w_n), \\ y_n = \mathfrak{T}((1-\gamma_n)z_n + \gamma_n \mathfrak{T}z_n), \\ x_{n+1} = \mathfrak{T}(y_n),\ n \geq 1 \end{cases} \tag{1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$. They demonstrated that among the many other iterative processes, the PJ-iteration (1) gives very high accurate results in less steps of iterations in the setting of Suzuki mappings. We improve here their results to the larger class of generalized α-nonexpansive mappings.

Suppose $\mathfrak{T}$ is a self-map $\mathfrak{T}\colon \mathcal{C} \to \mathcal{C}$ and $x, y \in \mathcal{C}$, where $\mathcal{C}$ is any subset of a Banach space X. Then $\mathfrak{T}$ is called as follows:

(a) Suzuki generalized nonexpansive [15] if

$$\frac{1}{2}\|x - \mathfrak{T}x\| \leq ||x-y|| \implies \|\mathfrak{T}x - \mathfrak{T}y\| \leq ||x-y||.$$

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(b) Generalized α-nonexpansive [11] if

$$\frac{1}{2}\|x-\mathfrak{T}x\|\leq\|x-y\| \implies \|\mathfrak{T}x-\mathfrak{T}y\|\leq\|x-y\|\leq\alpha\|x-\mathfrak{T}x\|+\alpha\|x-\mathfrak{T}y\|+(1-2\alpha)||x-y||.$$

(c) β-Reich-Suzuki type nonexpansive [10] if

$$\frac{1}{2}\|x-\mathfrak{T}x\|\leq||x-y|| \implies \|\mathfrak{T}x-\mathfrak{T}y\|\leq\beta\|x-\mathfrak{T}x\|+\beta\|y-\mathfrak{T}y\|+(1-2\beta)||x-y||.$$

The classes of generalized α-nonexpansive and β-Reich-Suzuki type nonexpansive self-maps properly include the class of Suzuki nonexpansive self-maps. It is very natural to ask whether there exists any class of mappings that includes all the classes of mappings mentioned above.

To answer the above question, Ullah et al. [18] presented a new class of nonlinear mappings that includes all the above mappings.

Definition 1 *Let $\mathfrak{T}$ be a self-map on a subset $\mathcal{C}$ of a Banach space X is said to be Generalized (α,β)-nonexpansive when, for all $x,y\in\mathcal{C}$, one can find two real constants $\alpha,\beta\in[0,1)$ such that $\alpha+\beta<1$ such that*

$$\frac{1}{2}\|x-\mathfrak{T}x\|\leq||x-y||$$

yields

$$\begin{aligned}||\mathfrak{T}x-\mathfrak{T}y|| \quad\leq\quad & ||x-y||\leq\alpha||x-\mathfrak{T}y||+\alpha||y-\mathfrak{T}x|| \\ & +\beta||x-\mathfrak{T}x||+\beta||y-\mathfrak{T}y||+(1-2\alpha-2\beta)||x-y||.\end{aligned}$$

Te class of generalized (α,β)-nonexpansive mappings includes all these mappings, and thus, the concept of generalized (α,β)-nonexpansive mappings is more difficult but more important than the other mappings mentioned above. The purpose of this work is to carry out some new fixed point convergence results under an effective iterative scheme.

The following proposition gives many examples of generalized (α,β)-nonexpansive mpas.

Remark 1 *Let $\mathfrak{T}$ be a self-map on a subset $\mathcal{C}$ of a Banach space X. Then, the following hold:*

1. *If $\mathfrak{T}$ is Suzuki nonexpansive, then $\mathfrak{T}$ is generalized $(0,0)$-nonexpansive.*
2. *If $\mathfrak{T}$ is generalized α-nonexpansive, then $\mathfrak{T}$ is generalized $(\alpha,0)$- nonexpansive.*
3. *If $\mathfrak{T}$ is β-Reich-Suzuki type nonexpansive, then $\mathfrak{T}$ is generalized $(0,\beta)$-nonexpansive.*

2 Preliminaries

We need some of the known results. Suppose a Banach space X is equipped with the norm $||\cdot||$. The space X will be called a uniformly convex Banach space [4] provided that for each choice of $0\leq\epsilon<1$, a real number $0<\delta<\infty$ can be found satisfying $\frac{||a+b||}{2}\leq1-\delta$, for all two elements $a,b\in X$ with $\|a\|\leq1$, $\|b\|\leq1$ and $||a+b||\geq\epsilon$. On the other side, if X satisfies the property that $||a+b||<2$ for all two different $a,b\in X$ with $\|a\|=\|b\|=1$, then X is called strictly convex.

The space X is said to be equipped with the Opial's property [9], if and only if for any given weakly convergent sequence, namely, $\{a_n\}$ in X having limit $a_0\in X$, then for all $b_0\in X-a_0$, one has

$$\limsup_{n\to\infty}\|a_n-a_0\|<\limsup_{n\to\infty}||a_n-b_0||.$$

Definition 2 (see [2, 16]) *Suppose $\{a_n\}$ denotes any bounded sequence in a closed convex subset $\mathcal{C}$ of a uniformly convex Banach space X. In this case, one denotes and defines the asymptotic radius of $\{a_n\}$ on the set $\mathcal{C}$ by $u(\mathcal{C},\{a_n\}) = \inf\{\limsup_{n\to\infty} ||a_n - a||\colon a \in \mathcal{C}\}$, while the asymptotic center of $\{a_n\}$ on the set $\mathcal{C}$ is denoted and defined as*

$$\mathcal{A}(\mathcal{C},\{a_n\}) = \left\{a \in \mathcal{C}\colon \limsup_{n\to\infty} ||a_n - a|| = u(\mathcal{C},\{a_n\})\right\}.$$

Moreover, the set $\mathcal{A}(\mathcal{C},\{a_n\})$ contains only one element.

Remark 2 *The set $\mathcal{A}(\mathcal{C},\{a_n\})$ contains only one point provided that X is a uniformly convex Banach space. The property that $\mathcal{A}(\mathcal{C},\{a_n\})$ is convex is also known in the setting of weakly compact convex sets (see, e.g., [12, 16] and others).*

Definition 3 ([14]) *Let $\mathfrak{T}$ be a self-map on a subset $\mathcal{C}$ of a Banach space X and f be a selfmap of $[0,\infty)$. We say that $\mathfrak{T}$ has condition (I) if the following holds:*

1. *$f(g) = 0$ if and only if $g = 0$.*
2. *$f(g) > 0$ for every $g > 0$.*
3. *$||x - \mathfrak{T}x|| \geq f(d(x,\mathcal{F}(\mathfrak{T})))$.*

Every uniformly convex Banach space has the following important property [13].

Lemma 1 ([13]) *Consider two sequences $\{a_n\}$ and $\{b_n\}$ in a uniformly convex Banach space X with*

$$\limsup_{n\to\infty} ||a_n|| \leq \kappa \quad \text{and} \quad \limsup_{n\to\infty} ||b_n|| \leq \kappa.$$

In addition, if $0 < \mu \leq \mu_n \leq \nu < 1$ and $\lim_{n\to\infty} ||\mu_n a_n + (1-\mu_n) b_n|| = \kappa$ for some $\kappa \geq 0$, then

$$\lim_{n\to\infty} ||a_n - b_n|| = 0.$$

Lemma 2 ([13]) *Suppose $\mathfrak{T}$ is generalized (α,β)-nonexpansive self-map whose domain of definition is possibly a subset $\mathcal{C}$ of X with a fixed point, namely, ρ. In such a case, the estimate $||\mathfrak{T}a - \mathfrak{T}\rho|| \leq ||a - \rho||$ holds for all $a \in \mathcal{C}$ and $\rho \in \mathcal{F}(\mathfrak{T})$.*

Now Lemma 2 suggests the following result.

Lemma 3 ([13]) *Suppose $\mathfrak{T}$ is generalized (α,β)-nonexpansive self-map whose domain of definition is possibly a subset $\mathcal{C}$ of a Banach space X. Consequently, the set $\mathcal{F}(\mathfrak{T})$ is closed. Also, the set $\mathcal{F}(\mathfrak{T})$ is convex provided that $\mathcal{C}$ is convex and the space X is strictly convex.*

The next Lemma shows a very basic property of the generalized (α,β)-nonexpansive mappings.

Lemma 4 ([18]) *Let $\mathfrak{T}$ be a self-map on a subset $\mathcal{C}$ of a Banach space X. If $\mathfrak{T}$ is generalized (α,β)-nonexpansive, then for each $x, y \in \mathcal{C}$:*

1. *$||\mathfrak{T}x - \mathfrak{T}^2x|| \leq ||x - \mathfrak{T}x||$.*
2. *Either $\frac{1}{2}||x - \mathfrak{T}x|| \leq ||x - y||$ or $\frac{1}{2}||\mathfrak{T}x - \mathfrak{T}^2x|| \leq ||\mathfrak{T}x - y||$.*
3. *Either*

$$||\mathfrak{T}x - \mathfrak{T}y|| \leq \alpha||x - \mathfrak{T}y|| + \alpha||y - \mathfrak{T}x|| + \beta||x - \mathfrak{T}x|| + \beta||y - \mathfrak{T}y|| + (1 - 2\alpha - 2\beta)||x - y||$$

or

$$||\mathfrak{T}^2x - \mathfrak{T}y|| \leq \alpha||\mathfrak{T}x - \mathfrak{T}y|| + \alpha||y - \mathfrak{T}^2x|| + \beta||\mathfrak{T}x - \mathfrak{T}^2x|| + \beta||y - \mathfrak{T}y|| + (1 - 2\alpha - 2\beta)||\mathfrak{T}x - y||.$$

Lemma 5 (**[18]**) *Let $\mathfrak{T}$ be a self-map on a subset $\mathcal{C}$ of a Banach space X. If $\mathfrak{T}$ is generalized (α, β)-nonexpansive, then for each $x, y \in \mathcal{C}$, we have*

$$||x_n - \mathfrak{T}p|| \leq \left(\frac{3+\alpha+\beta}{1-\alpha-\beta}\right) ||x_n - \mathfrak{T}p|| + ||x_n - p||.$$

Lemma 6 (**[18]**) *Let $\mathfrak{T}$ be a self-map on a subset $\mathcal{C}$ of a Banach space X having Opial's property. If $\mathfrak{T}$ is generalized (α, β)-nonexpansive, then the following holds:*

$$\{x_n\} \subset \mathcal{C},\ x_n \to p,\ ||x_n - \mathfrak{T}x_n|| \to 0 \implies \mathfrak{T}x_n = p.$$

3 Main Results

This section establishes some significant strong and Δ-convergence results for operators with generalized (α, β)-nonexpansive mapping. Our results will generalize the results of Ullah et al. [18].

Theorem 1 *Suppose that $\mathfrak{T}$ is a generalized (α, β)-nonexpansive self-map on a convex closed subset $\mathcal{C}$ of a uniformly convex Banach space. If the fixed point set $\mathcal{F}(\mathfrak{T})$ is nonempty and $\{x_n\}$ is produced from PJ iteration (1), then, subsequently, $\lim_{n\to\infty} ||x_n - p||$ exists for each choice of q in $\mathcal{F}(\mathfrak{T})$.*

Proof. Let $p \in \mathcal{F}(\mathfrak{T})$. By Lemma 2, we have

$$\begin{aligned} ||w_n - p|| &= ||\mathfrak{T}((1-\alpha_n)x_n + \alpha_n \mathfrak{T}x_n) - p|| \\ &\leq ||(1-\alpha_n)x_n + \alpha_n \mathfrak{T}x_n - p|| \\ &\leq ((1-\alpha_n))||x_n - p|| + \alpha_n||\mathfrak{T}x_n - p|| \\ &\leq (1-\alpha_n)||x_n - p|| + \alpha_n||x_n - p|| \\ &\leq ||x_n - p||. \end{aligned} \tag{2}$$

Using Lemma 2 and (2), we get

$$\begin{aligned} ||z_n - p|| &= ||\mathfrak{T}((1-\beta_n)w_n + \beta_n \mathfrak{T}w_n) - p|| \\ &\leq ||(1-\beta_n)w_n + \beta_n \mathfrak{T}w_n - p|| \\ &\leq (1-\beta_n)||w_n - p|| + \beta_n||\mathfrak{T}w_n - p|| \\ &\leq (1-\beta_n)||w_n - p|| + \beta_n||w_n - p|| \\ &\leq (1-\beta_n)||x_n - p|| + \beta_n||x_n - p|| \\ &\leq ||x_n - p||. \end{aligned} \tag{3}$$

Using Lemma 2, (2) and (3), we get

$$\begin{aligned} |x_{n+1} - p|| &= ||\mathfrak{T}(y_n) - p|| \\ &\leq ||(1-\gamma_n)z_n + \gamma_n \mathfrak{T}z_n - p|| \\ &\leq (1-\gamma_n)||z_n - p|| + \gamma_n||\mathfrak{T}z_n - p|| \\ &\leq (1-\gamma_n)||z_n - p|| + \gamma_n||z_n - p|| \\ &\leq (1-\gamma_n)||x_n - p|| + \gamma_n||x_n - p|| \\ &\leq ||x_n - p||. \end{aligned} \tag{4}$$

Using Lemma 2 and (2)–(4), we get

$$\begin{aligned} ||x_{n+1} - p|| &= ||\mathfrak{T}(y_n) - p|| \\ &\leq ||(1-\gamma_n)z_n + \gamma_n \mathfrak{T}z_n - p|| \end{aligned}$$

$$\begin{aligned} &\leq (1-\gamma_n)||z_n - p|| + \gamma_n||\mathfrak{T}z_n - p|| \\ &\leq (1-\gamma_n)||z_n - p|| + \gamma_n||z_n - p|| \\ &\leq (1-\gamma_n)||x_n - p|| + \gamma_n||x_n - p|| \\ &\leq ||x_n - p||. \end{aligned} \tag{5}$$

Thus, $\{||x_n - p||\}$ is a non-increasing sequence of reals which is bounded below by zero and hence convergent. Therefore, $\lim_{n\to\infty} ||x_n - p||$ exists $\forall\ p \in \mathcal{F}(\mathfrak{T})$. ■

Theorem 2 *Suppose that $\mathfrak{T}$ is a generalized (α, β)-nonexpansive self-map on a convex closed subset $\mathcal{C}$ of a uniformly convex Banach space. If the fixed point set $\mathcal{F}(\mathfrak{T})$ is nonempty and $\{x_n\}$ is produced from PJ iteration (1), then $\mathcal{F}(\mathfrak{T}) \neq \emptyset$ If and only if $\{x_n\}$ is bounded and $\lim_{n\to\infty} ||\mathfrak{T}x_n - x_n|| = 0$.*

Proof. Suppose that $\mathcal{F}(\mathfrak{T}) \neq \emptyset$ and $p \in \mathcal{F}(\mathfrak{T})$. Then by Theorem 1, it follows that $\lim_{n\to\infty} ||x_n - p||$ exists and $\{x_n\}$ is bounded. Put

$$\lim_{n\to\infty} ||x_n - p|| = c. \tag{6}$$

By the proof of Theorem 1 and (6), we have

$$\lim_{n\to\infty} \sup ||w_n - p|| \leq \lim_{n\to\infty} \sup ||x_n - p|| = c. \tag{7}$$

By using Lemma 2, we have

$$\lim_{n\to\infty} \sup ||\mathfrak{T}x_n - p|| \leq \lim_{n\to\infty} \sup ||x_n - p|| = c. \tag{8}$$

Again by the proof of Theorem 1, we have $||z_n - p|| \leq ||x_n - p||$. Therefore,

$$\begin{aligned} ||x_{n+1} - p|| &= ||\mathfrak{T}(y_n) - p|| \\ &= ||\mathfrak{T}((1-\gamma_n)z_n + \gamma_n\mathfrak{T}z_n) - p|| \\ &\leq ||(1-\gamma_n)z_n + \gamma_n\mathfrak{T}z_n - p|| \\ &\leq (1-\gamma_n)||z_n - p|| + \gamma_n d(\mathfrak{T}z_n, p) \\ &\leq (1-\gamma_n)||x_n - p|| + \gamma_n||z_n - p||. \end{aligned}$$

It follows that

$$\begin{aligned} ||x_{n+1} - p|| - ||x_n - p|| &\leq \frac{||x_{n+1} - p|| - ||x_n - p||}{\gamma_n} \\ &\leq ||z_n - p|| - ||x_n - p|| \\ &\leq (1-\beta_n)||w_n - p|| + \beta_n||w_n - p|| - ||x_n - p|| \\ &\leq ||w_n - p|| - ||x_n - p||. \end{aligned}$$

So, we can get $||x_{n+1} - p|| \leq ||w_n - p||$ and from (6), we have

$$c \leq \lim_{n\to\infty} \inf ||w_n - p||. \tag{9}$$

Hence, from (7) and (9), we obtain

$$c = \lim_{n\to\infty} ||w_n - p||. \tag{10}$$

Therefore, from (10), we have

$$\begin{aligned} c &= \lim_{n\to\infty} ||w_n - p|| = \lim_{n\to\infty} ||\mathfrak{T}((1-\alpha_n)x_n + \alpha_n\mathfrak{T}x_n) - p|| \\ &\leq \lim_{n\to\infty} ||(1-\alpha_n)x_n + \alpha_n\mathfrak{T}x_n - p|| \end{aligned}$$

$$
\begin{aligned}
&\leq \lim_{n\to\infty}(1-\alpha_n)||x_n-p||+\alpha_n||\mathfrak{T}x_n-p|| \\
&\leq \lim_{n\to\infty}(1-\alpha_n)||x_n-p||+\lim_{n\to\infty}\alpha_n||\mathfrak{T}x_n-p|| \\
&\leq c. \qquad (11)
\end{aligned}
$$

Hence,

$$\lim_{n\to\infty}(1-\alpha_n)||x_n-p||+\alpha_n||\mathfrak{T}x_n-p|| = c. \qquad (12)$$

Now, from (7), (9), (12) and Lemma 1, we conclude that,

$$\lim_{n\to\infty}||\mathfrak{T}x_n-x_n|| = 0.$$

Conversely, let $p \in A(\{x_n\})$. By Lemma 5, we have

$$||x_n-\mathfrak{T}p|| \leq \left(\frac{3+\alpha+\beta}{1-\alpha-\beta}\right)||x_n-\mathfrak{T}p||+||x_n-p||.$$

This implies that

$$
\begin{aligned}
u(x_n,\mathfrak{T}p) &= \lim_{n\to\infty}\sup\|x_n-\mathfrak{T}p\| \\
&\leq \left(\frac{3+\alpha+\beta}{1-\alpha-\beta}\right)\lim_{n\to\infty}\sup||x_n-p|| \\
&\leq \lim_{n\to\infty}\sup||x_n-p|| = u(x_n,p).
\end{aligned}
$$

So $\mathfrak{T}p \in A\{x_n\}$. By the uniqueness of asymptotic centers, one can conclude that $\mathfrak{T}p = p$. This completes the proof. ■

Theorem 3 *Suppose that $\mathfrak{T}$ is a generalized (α,β)-nonexpansive self-map on a convex closed subset $\mathcal{C}$ of a uniformly convex Banach space. If the fixed point set $\mathcal{F}(\mathfrak{T})$ is nonempty and $\{x_n\}$ is produced from PJ iteration (1), then, sub-sequentially, $\{x_{n_k}\}$ converges (strongly) to some fixed point of $\mathfrak{T}$ if the set $\mathcal{C}$ is compact.*

Proof. Since the subset $\mathcal{C}$ is convex and compact, we have a subsequence of the iterative sequence $\{x_n\}$ that we may denote by $\{x_{n_k}\}$ such that $x_{n_k} \to p$, where p is a point in $\mathcal{C}$. We prove that p is a fixed point of $\mathfrak{T}$ and strong limit of the iterative sequence $\{x_n\}$ and that $x_{n_k} \to Tp$. For this, we used Theorem 2 and, hence, we get $\lim_{n\to\infty}||\mathfrak{T}x_{n_k}-x_{n_k}|| = 0$. Hence, applying Lemma 5, one has the following equation:

$$||x_{n_k}-\mathfrak{T}p|| \leq \left(\frac{3+\alpha+\beta}{1-\alpha-\beta}\right)||x_{n_k}-\mathfrak{T}x_{n_k}||+||x_{n_k}-p|| \to 0.$$

Thus, $p = \mathfrak{T}p$, so q become a fixed point of $\mathfrak{T}$. Using Lemma 1, $\lim_{n\to\infty}||\mathfrak{T}x_n-p||$ exists. It now clear that p is the strong limit of the iterative sequence $\{x_n\}$. Hence, the proof is finished. ■

The next result does not require the compactness condition. This result is valid in general Banach space setting.

Theorem 4 *Suppose that $\mathfrak{T}$ is a generalized (α,β)-nonexpansive self-map on a convex closed subset $\mathcal{C}$ of a uniformly convex Banach space. If the fixed point set $\mathcal{F}(\mathfrak{T})$ is nonempty and $\{x_n\}$ is produced from PJ iteration (1), then, sub-sequentially, $\{x_{n_k}\}$ converges (strongly) to some fixed point of $\mathfrak{T}$ if*

$$\lim_{n\to\infty}\inf dist(x_n,\mathcal{F}(\mathfrak{T})) = 0.$$

Proof. If the sequence $\{x_n\}$ converges to a point $p \in \mathcal{F}(\mathfrak{T})$, then

$$\lim_{n\to\infty} \inf\ d(x_n, p) = 0,$$

so

$$\lim_{n\to\infty} d(x_n, \mathcal{F}(\mathfrak{T})) = 0.$$

For converse part, assume that $\lim_{n\to\infty} \inf d(x_n, \mathcal{F}(\mathfrak{T})) = 0$. From Theorem 1, we have

$$||x_{n+1} - p|| \leq ||x_n - p|| \text{ for any } p \in \mathcal{F}(\mathfrak{T}),$$

so we have,

$$||x_{n+1} - \mathcal{F}(\mathfrak{T})|| \leq ||x_n - \mathcal{F}(\mathfrak{T})||. \tag{13}$$

Thus, $||x_n - \mathcal{F}(\mathfrak{T})||$ forms a decreasing sequence which is bounded below by zero as well, thus $\lim_{n\to\infty} ||x_n - \mathcal{F}(\mathfrak{T})||$ exists. Since, $\lim_{n\to\infty} \inf ||x_n - \mathcal{F}(\mathfrak{T})|| = 0$ so $\lim_{n\to\infty} dist(x_n, \mathcal{F}(\mathfrak{T})) = 0$.

Now, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and a sequence $\{x_j\}$ in $\mathcal{F}(\mathfrak{T})$ such that $d(x_{n_j}, x_j) \leq \frac{1}{2^j}$ for all $j \in \mathbb{N}$. From the proof of Theorem 1, we have

$$||x_{n_{j+1}} - x_j|| \leq ||x_{n_j} - x_j|| \leq \frac{1}{2^j}.$$

Using triangle inequality, we get

$$||x_{n_{j+1}} - x_j|| \leq ||x_{j+1} - x_{n_{j+1}}|| + ||x_{n_{j+1}} - x_j|| \leq \frac{1}{2^{j+1}} + \frac{1}{2^j} \leq \frac{1}{2^{j-1}} \to 0 \text{ as } j \to \infty.$$

So, $\{x_j\}$ is a Cauchy sequence in $\mathcal{F}(\mathfrak{T})$. From Lemma 3 $\mathcal{F}(\mathfrak{T})$ is closed, so $\{x_j\}$ converges to some $x \in \mathcal{F}(\mathfrak{T})$.

Again, owing to triangle inequality, we have

$$||x_{n_j} - x|| \leq ||x_{n_j} - x_j|| + ||x_j - x||.$$

Letting $j \to \infty$, we have $\{x_{n_j}\}$ converges strongly to $x \in \mathcal{F}(\mathfrak{T})$.

Since $\lim_{n\to\infty} \inf ||x_n - x||$ exists by Theorem 1, therefore $\{x_n\}$ converges to $x \in \mathcal{F}(\mathfrak{T})$. ∎

Eventually, we discuss the strong convergence for our scheme (1) by using the condition (I) given by Definition 3.

Theorem 5 *Suppose that $\mathfrak{T}$ is a generalized (α, β)-nonexpansive self-map on a convex closed subset $\mathcal{C}$ of a uniformly convex Banach space. If the fixed point set $\mathcal{F}(\mathfrak{T})$ is nonempty and $\{x_n\}$ is produced from PJ iteration (1), then, sub-sequentially, $\{x_{n_k}\}$ converges (strongly) to some fixed point of $\mathfrak{T}$ if $\mathfrak{T}$ is with condition (I).*

Proof. From (13), $\lim_{n\to\infty} dist(x_n, \mathcal{F}(\mathfrak{T}))$ exists. Also, by Theorem 2 we have $\lim_{n\to\infty} ||x_n - \mathfrak{T}x_n|| = 0$. It follows from the Condition (I) that

$$\lim_{n\to\infty} f(dist(x_n, \mathcal{F}(\mathfrak{T}))) \leq \lim_{n\to\infty} ||x_n - \mathfrak{T}x_n|| = 0.$$

So $\lim_{n\to\infty} f(dist(x_n, \mathcal{F}(\mathfrak{T}))) = 0$. Since f is a non decreasing function satisfying $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$, we see that $\lim_{n\to\infty} ||x_n - \mathfrak{T}x_n|| = 0$.

By Theorem 4, the sequence $\{x_n\}$ converges strongly to a point of $\mathcal{F}(\mathfrak{T})$. ∎

Theorem 6 *Suppose that $\mathfrak{T}$ is a generalized (α, β)-nonexpansive self-map on a convex closed subset $\mathcal{C}$ of a uniformly convex Banach space. If the fixed point set $\mathcal{F}(\mathfrak{T})$ is nonempty and $\{x_n\}$ is produced from PJ iteration (1), then, sub-sequentially, $\{x_{n_k}\}$ converges (weakly) to some fixed point of $\mathfrak{T}$ if X is with Opial's condition.*

Proof. We notice that X is reflexive because uniformly convex Banach space is always reflexive. Now, since the sequence of iterates $\{x_n\}$ is bounded in $\mathcal{C}$ and so it has a convergent (weakly) subsequence, we denote it here by $\{x_{n_j}\}$ and its weak limit is denoted here by x_1. However, using Theorem 1, one has $\lim_{j\to\infty} ||\mathfrak{T}x_{n_j} - x_{n_j}|| = 0$. Thus, applying Lemma 6, the point x_1 becomes the fixed point for $\mathfrak{T}$. We now show that x_1 is a weak limit for $\{x_k\}$ and this will finish the proof. For this purpose, we assume on the contrary that x_1 is not a weak limit for $\{x_k\}$ and so we can find a new subsequence, namely, $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ is a weak convergent to some x_2 that is different from x_1. As shown earlier, we can show that x_2 is a fixed point of $\mathfrak{T}$. Now, using Opial's condition of $\mathfrak{T}$, it follows that

$$\begin{aligned}\lim_{n\to\infty} ||x_n - x_1|| &= \lim_{j\to\infty} ||x_{n_j} - x_1|| < \lim_{j\to\infty} ||x_{n_j} - x_2|| \\ &= \lim_{n\to\infty} ||x_n - x_2|| = \lim_{k\to\infty} ||x_{n_k} - x_2|| \\ &< \lim_{k\to\infty} ||x_{n_k} - x_1|| = \lim_{n\to\infty} ||x_n - x_1||.\end{aligned}$$

The above strict inequality suggests a contradiction because $x_1 \neq x_2$. Thus, we must accept that the sequence of iterates $\{x_n\}$ converges weakly to x_1. ■

4 Numerical Example

The following example shows that there exist maps which are generalized (α, β)-nonexpansive but neither generalized α-nonexpansive nor β-Reich-Suzuki type.

Example 1 *Define a mapping* $\mathfrak{T}\colon \mathbb{R}^+ \to \mathbb{R}^+$ *by*

$$\mathfrak{T}x = \begin{cases} 0 & \textit{if } x \in [0.\frac{1}{2}], \\ \frac{x}{2} & \textit{if } x \in (\frac{1}{2}, \infty). \end{cases}$$

We shall prove that $\mathfrak{T}$ *is generalized* $(\frac{1}{4}, \frac{1}{4})$*-nonexpansive, and divide the proof into three cases.*

(i) If $0 \leq x, y \leq \frac{1}{2}$*, then we have*

$$\frac{1}{4}|x - \mathfrak{T}y| + \frac{1}{4}|y - \mathfrak{T}x| + \frac{1}{4}|x - \mathfrak{T}x| + \frac{1}{4}|y - \mathfrak{T}y| \geq 0 = |\mathfrak{T}x - \mathfrak{T}y|.$$

(ii) If $\frac{1}{2} < x, y < \infty$*, then we have*

$$\begin{aligned}\frac{1}{4}|x - \mathfrak{T}y| + \frac{1}{4}|y - \mathfrak{T}x| + \frac{1}{4}|x - \mathfrak{T}x| + \frac{1}{4}|y - \mathfrak{T}y| \quad &= \quad \frac{1}{4}|x - \frac{y}{2}| + \frac{1}{4}|y - \frac{x}{2}| \\ &\quad +\frac{1}{4}|x - \frac{x}{2}| + \frac{1}{4}|y - \frac{y}{2}| \\ &\quad +\frac{1}{4}|x - \frac{x}{2}| + \frac{1}{4}|y - \frac{y}{2}| \\ &\geq \quad \frac{1}{4}|\frac{4x}{2} - \frac{4y}{2}| \\ &= \quad \frac{1}{2}|x - y| \\ &= \quad |\mathfrak{T}x - \mathfrak{T}y|.\end{aligned}$$

(iii) If $\frac{1}{2} < x < \infty$ and $0 \leq y \leq \frac{1}{2}$, then we have

$$\begin{aligned}
\frac{1}{4}|x-\mathfrak{T}y|+\frac{1}{4}|y-\mathfrak{T}x|+\frac{1}{4}|x-\mathfrak{T}x|+\frac{1}{4}|y-\mathfrak{T}y| &= \frac{1}{4}|x|+\frac{1}{4}|y-\frac{x}{2}| \\
&+\frac{1}{4}|x-\frac{x}{2}|+\frac{1}{4}|y| \\
&= \frac{1}{4}|x|+\frac{1}{4}|y-\frac{x}{2}|+\frac{1}{4}|\frac{x}{2}|+\frac{1}{4}|y| \\
&\geq \frac{1}{4}|\frac{4x}{2}| = \frac{1}{2}|x| \\
&= |\mathfrak{T}x-\mathfrak{T}y|.
\end{aligned}$$

Hence, $\mathfrak{T}$ is generalized $(\frac{1}{4},\frac{1}{4})$-nonexpansive. However, for $x=\frac{1}{2}$ and $y=\frac{4}{5}$, we have $\frac{1}{2}|\mathfrak{T}x-\mathfrak{T}y| < |x-y|$. However,

(i) $|\mathfrak{T}x-\mathfrak{T}y| > |x-y|$.

(ii) $|\mathfrak{T}x-\mathfrak{T}y| > \frac{1}{4}|x-\mathfrak{T}y|+\frac{1}{4}|y-\mathfrak{T}x|+(1-2(\frac{1}{4}))|x-y|$.

(iii) $|\mathfrak{T}x-\mathfrak{T}y| > \frac{1}{4}|x-\mathfrak{T}x|+\frac{1}{4}|y-\mathfrak{T}y|+(1-2(\frac{1}{4}))|x-y|$.

Hence, $\mathfrak{T}$ is neither generalized $\frac{1}{4}$-nonexpansive nor $\frac{1}{4}$-Reich-Suzuki type. We obtained the influence of initial point for the New iteration algorithm (1) by $\alpha_n = 0.90$, $\beta_n = 0.65$, $\gamma_n = 0.85$ and $x_1 = 1000$. In what follows,

Table 1: Convergence of our iteration (1) for fixed point 0.

No. of iterations	**Agrawal**	**Abbas**	**Thakur**	**K**	**PJ**
1	1000	1000	1000	1000	1000
2	353.7500000	278.4687500	176.8750000	100.9375000	13.34179687
3	125.1390625	78.3705859	31.28476562	10.18837890	0
4	44.26794335	22.3367508	5.533492919	1.028389490	0
5	15.65978496	6.460644850	0.978736560	0.034661323	0
6	5.53964893	1.899990760	0.020473760	0	0
7	1.959650809	0.569010550	0	0	0
8	0.693226473	0.02647376	0	0	0
9	0.034661323	0	0	0	0
10	0	0	0	0	0

we numerically compare PJ iteration process (1) with some existing iteration processes.

Case I: *Taking,* $\alpha_n = \frac{3n}{8n+4}$, $\beta_n = \frac{1}{n+4}$, $\gamma_n = \frac{n}{(5n+2)^2}$.

Case II: *Taking,* $\alpha_n = 1-\frac{1}{2n+8}$, $\beta_n = \frac{n}{16n+1}$, $\gamma_n = \frac{n}{n+5}$.

Case III: *Taking,* $\alpha_n = \frac{2}{2n+1}$, $\beta_n = \frac{1}{\sqrt{n+9}}$, $\gamma_n = \frac{5n}{9n+7}$.

Case IV: *Taking,* $\alpha_n = \frac{n}{2n+1}$, $\beta_n = \frac{1}{\sqrt{n+9}}$, $\gamma_n = \frac{2n}{(7n+2)^2}$.

The observations are given in Figures 1 and 2. We have concluded that the PJ iteration process (1) not only converges faster than the known iterations but also is stable with respect to the parameters α_n, β_n and γ_n. From Figure 1, we also observe that the average number of iterations of the PJ iteration process (1) is the smallest with respect to other processes.

Comparison of various iteration processes for example 1.

Iterations																				
Init. Value	10				100				1000				10000				Iteration Average			
Case	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
Agarwal	6	6	5	5	9	9	9	9	12	12	12	12	16	15	15	15	10.75	10.5	10.25	10.25
Abbas	5	4	4	5	7	6	6	7	9	8	9	10	12	10	12	13	7.75	7.5	7.75	8.75
Thakur	4	3	3	3	5	5	5	5	6	6	6	6	8	8	8	8	5.75	5.5	5.5	5.5
K	3	2	3	3	4	3	4	4	6	4	5	5	7	6	6	7	5	3.75	4.5	4.75
PJ	2	1	2	2	2	2	2	2	3	3	3	3	4	3	4	4	2.75	2.25	2.75	2.75

Figure 1: Table depicting comparison of various iteration process under distinct parameters for example 1.

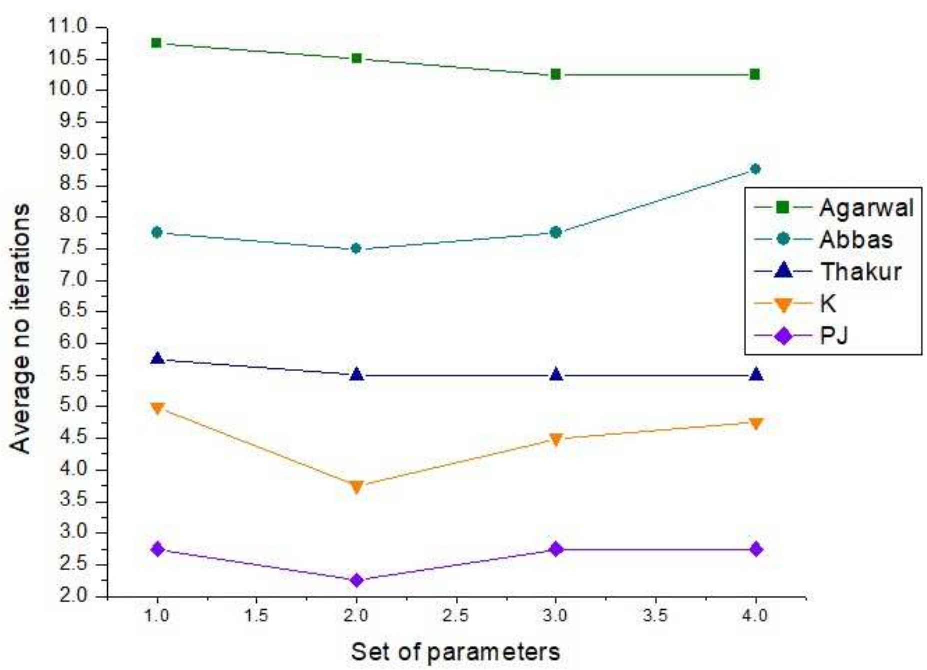

Figure 2: Average number of iterations under distinct parameters for Example 1.

5 Conclusion

In this work, we have presented some fixed point results for a generalized (α, β)-nonexpansive mappings and also proposed a PJ iterative algorithm for approximating the fixed point of this class of mappings in the framework of Banach spaces. Our numerical experiment shows that our iterative algorithm performs well among some existing iterative algorithms in the literature.

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