Applied Mathematics E-Notes, 25(2025), 508-524 © ISSN 1607-2510
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/

Common Fixed Point Results On (ψ, ϕ)-Weak Contraction In Partially Ordered Metric Spaces And Application To Boundary Value Problem*

Joginder Paul†, Umesh Chandra Gairola‡

Received 16 August 2024

Abstract

In this paper, we prove some common fixed point theorems in partially ordered metric spaces for generalized (ψ, ϕ)-weak contraction mappings satisfying the rational type expressions which extend many results of metric spaces established recently. We also generalize various established fixed point theorems from the literature concerning partially ordered metric spaces, which are instrumental for solving the non-linear equations via iterative methods. To illustrate the applicability of our results there is a proper example which supports our result but not to Arya et al. [2]. Additionally, we demonstrate how our results can be applied to prove the existence and uniqueness of solution for first order periodic boundary value problem in the theory of ordinary differential equations.

1 Introduction

The Banach contraction principle, a cornerstone of fixed point theory, plays a critical role in nonlinear analysis and iterative method for solving nonlinear equations across various fields. This principle has been widely generalized, expanding its applicability in numerous mathematical context. Notable contributions include extensions to different types of mappings and spaces as well, for instance, see [7, 18, 20, 23, 25, 31] and references therein. Alber and Guerre-Delabriere [1], in 1997, introduced the notion of weak contraction in a Hilbert spaces, and providing new insights into the existence of fixed point for these mappings. In 2001, B. E. Rhoades [30], demonstrated that the fixed point theorem established by Alber and Guerre-Delabriere is also applicable in complete metric spaces. Subsequently, Dutta and Choudhary [9] extended the result of Alber and Guerre-Delabriere in a complete metric space by introducing the notion of (ψ, ϕ)-weak contraction. Notably, several generalizations of the (ψ, ϕ)-weak contraction have been developed, some of which do not require the continuity or monotonicity of the function ϕ. For further details, we refer to the works cited in [6], [8] and [26]. In 2009, Zhang and Song [32] introduced an extension of the ϕ-weak contraction involving two mappings known as the generalized ϕ-weak contraction, and established conditions for the existence of a common fixed point. Building upon this, Dorić [8] extended their result by defining the generalized (ψ, ϕ)-weak contraction and proving several significant fixed point theorems. In these contexts, the functions ψ and ϕ are referred to as control functions, as they govern the behavior and conditions under which fixed points are guaranteed. The investigation on the existence of fixed point results in partially ordered metric space have been obtained by Ran and Reurings [29] in 2004, and thereafter, Nieto et al. [22] further extended the result of Ran and Reurings to non-decreasing mappings and used to find the solution of first order ordinary differential equation with periodic boundary conditions. Moreover, the study of fixed point theorems and their applications for weakly contractive mapping in partially ordered metric spaces remains a compelling area of research. However, Harjani and Sadarangani (see [12], [13]) also extended the result from Dutta and Choudhary [9] to partially ordered metric spaces and established some fixed point results with applications

*Mathematics Subject Classifications: Primary: 47H10; Secondary: 54H25.

†Department of Mathematics, H. N. B. Garhwal University, B. G. R. Campus, Pauri-246001, Uttarakhand, India
‡Department of Mathematics, H. N. B. Garhwal University, B. G. R. Campus, Pauri-246001, Uttarakhand, India

to differential equations. Additionally, numerous researchers have explored fixed point of weakly contractive mappings in partially ordered metric spaces, as detailed in references [3, 4, 11, 17, 21, 24, 27, 28] and others.

In recent works, Arya et al. [2] in 2023, and Chandra et al. [5] in 2021, established common fixed point results for generalized (ψ, ϕ)-weak contraction mappings that satisfy rational type expressions in metric spaces. Now, our article advances this field by presenting common fixed point theorems in partially ordered metric spaces for generalized (ψ, ϕ)-weak contraction mappings under analogue conditions. These results are derived using the technique developed by Arya et al. [2] and Chandra et al. [5] of iterative schemes. Our results improve, generalize and extend comparable conclusions of the literature in partially ordered metric spaces. Additionally, we provide a non-trivial example demonstrating our results, which do not align with those of Arya et al. [2]. We also discuss the existence and uniqueness of solution for first-order periodic boundary value problem in ordinary differential equations.

2 Preliminaries

Throughout the discussion of the paper, the following definitions will be necessary.

Definition 1 *Let X be a non-empty set. Then a mapping $\varrho : X \times X \to [0, \infty)$ is said to be the metric on X if, for all $\mu, \nu, \eta \in X$, the following properties are satisfied:*

(i) $\varrho(\mu, \nu) = \varrho(\nu, \mu)$*;*

(ii) $\varrho(\mu, \nu) = 0$ *if and only if* $\mu = \nu$*;*

(iii) $\varrho(\mu, \nu) \leq \varrho(\mu, \eta) + \varrho(\eta, \nu)$*.*

Additionally, the ordered pair (X, ϱ) is called a metric space.

Definition 2 ([30]) *Let (X, ϱ) be a metric space. Then a mapping $T : X \to X$ is said to be a ϕ-weak contraction, if $\varrho(T\mu, T\nu) \leq \varrho(\mu, \nu) - \phi(\varrho(\mu, \nu))$ for all $\mu, \nu \in X$, where $\phi : [0, \infty) \to [0, \infty)$ is a continuous and non-decreasing function with $\phi(t) = 0$ if and only if $t = 0$.*

Definition 3 ([14, 15, 16]) *Let (X, ϱ) be a metric space. Then a pair (T, S) of self mappings on X is said to be commuting, if $TS\mu = ST\mu$ for all $\mu \in X$. Also, the pair (T, S) is called compatible, if $\lim_{n\to\infty} \varrho(TS\mu_n, ST\mu_n) = 0$ whenever $\{\mu_n\}$ is a sequence in X such that $\lim_{n\to\infty} T\mu_n = \lim_{n\to\infty} S\mu_n = \mu$ for some $\mu \in X$. Moreover, the pair (T, S) is said to be weakly compatible, if $TS\mu = ST\mu$ whenever $T\mu = S\mu$, i.e., if they commute at their coincidence points.*

Definition 4 ([19]) *Let X be a non-empty set. Then a binary relation $\sqsubseteq$ on X is said to be a partial order if, for all $\mu, \nu, \eta \in X$, the following properties are satisfied:*

(i) $\mu \sqsubseteq \mu$ *(Reflexive);*

(ii) $\mu \sqsubseteq \nu$ *and* $\nu \sqsubseteq \mu$ *imply* $\mu = \nu$ *(Anti-symmetry);*

(iii) $\mu \sqsubseteq \nu$ *and* $\nu \sqsubseteq \eta$ *imply* $\mu \sqsubseteq \eta$ *(Transitivity).*

Additionally, the ordered pair $(X, \sqsubseteq)$ is called a partially ordered set.

Definition 5 ([22]) *Let $(X, \sqsubseteq)$ be a partially ordered set. Then $\mu, \nu \in X$ are said to be comparable, if $\mu \sqsubseteq \nu$ or $\nu \sqsubseteq \mu$, and a non-empty subset A of X is called well ordered set, if any two elements of it are comparable.*

Definition 6 ([10]) *Let $(X, \sqsubseteq)$ be a partially ordered set. A pair (T, S) is said to be weakly increasing if $T\mu \sqsubseteq ST\mu$ and $S\mu \sqsubseteq TS\mu$ for all $\mu \in X$.*

Definition 7 ([22]) *Let $(X, \sqsubseteq)$ be a partially ordered set. Then a mapping $T : X \to X$ is said to be strictly increasing, if $T(\mu) < T(\nu)$ for all $\mu, \nu \in X$ with $\mu < \nu$, and it is called strictly decreasing, if $T(\mu) > T(\nu)$ for all $\mu, \nu \in X$ with $\mu < \nu$.*

Definition 8 ([22]) *Let $(X, \sqsubseteq)$ be a partially ordered set. Then a mapping $T : X \to X$ is said to be monotone non-decreasing, if $\mu \leq \nu$ implies $T\mu \leq T\nu$ for all $\mu, \nu \in X$, and T is said to be monotone non-increasing, if $\mu \leq \nu$ implies $T\mu \geq T\nu$ for all $\mu, \nu \in X$.*

Definition 9 ([4]) *Let $(X, \sqsubseteq)$ be a partially ordered set, and let $T, f : X \to X$ be mappings. Then a mapping T is said to be monotone f-non-decreasing, if $f\mu \leq f\nu$ implies $T\mu \leq T\nu$ for all $\mu, \nu \in X$, and T is said to be monotone f-non-increasing, if $f\mu \leq f\nu$ implies $T\mu \geq T\nu$ for all $\mu, \nu \in X$.*

Definition 10 ([22]) *If $(X, \sqsubseteq)$ is a partially ordered set together with a metric space (X, ϱ), the triple $(X, \varrho, \sqsubseteq)$ is called a partially ordered metric space.*

Definition 11 ([22]) *The triple $(X, \varrho, \sqsubseteq)$ is called a partially ordered complete metric space if (X, ϱ) is a complete metric space.*

Definition 12 ([22]) *A partially ordered metric space $(X, \varrho, \sqsubseteq)$ is said to be the ordered complete, if one of the following condition holds for any sequence $\{\mu_n\} \subseteq X$ with $\mu_n \to \mu$:*

(i) $\mu = \sup\{\mu_n\}$ *whenever $\{\mu_n\}$ is non-decreasing.*

(ii) $\mu = \inf\{\mu_n\}$ *whenever $\{\mu_n\}$ is non-increasing.*

Further, let (T, S) and (f, g) be the pairs of self mappings on a partially ordered metric space $(X, \varrho, \sqsubseteq)$. Then, for any comparable elements $\mu, \nu \in X$, we will be denoted the following notations.

(i) $$M(T\mu, S\nu) = \max\left\{ \begin{array}{l} \varrho(\mu,\nu),\ \varrho(\mu,T\mu),\ \varrho(\nu,S\nu),\ \frac{\varrho(\nu,T\mu)+\varrho(\mu,S\nu)}{2}, \\ \varrho(\nu,S\nu)\left(\frac{1+\varrho(\mu,T\mu)}{1+\varrho(\mu,\nu)}\right),\ \varrho(\mu,T\mu)\left(\frac{1+\varrho(\nu,S\nu)}{1+\varrho(\mu,\nu)}\right) \end{array} \right\}.$$

(ii) $$N(T\mu, S\nu) = \min\left\{ \varrho(\mu,\nu),\ \varrho(\mu,T\mu),\ \varrho(\nu,S\nu),\ \varrho(\nu,T\mu),\ \varrho(\mu,S\nu) \right\}.$$

(iii) $$M_f(T\mu, S\nu) = \max\left\{ \begin{array}{l} \varrho(f\mu,f\nu),\ \varrho(f\mu,T\mu),\ \varrho(f\nu,S\nu),\ \frac{\varrho(f\nu,T\mu)+\varrho(f\mu,S\nu)}{2}, \\ \varrho(f\nu,S\nu)\left(\frac{1+\varrho(f\mu,T\mu)}{1+\varrho(f\mu,f\nu)}\right),\ \varrho(f\mu,T\mu)\left(\frac{1+\varrho(f\nu,S\nu)}{1+\varrho(f\mu,f\nu)}\right) \end{array} \right\}.$$

(iv) $$M_{f,g}(T\mu, S\nu) = \max\left\{ \begin{array}{l} \varrho(f\mu,g\nu),\ \varrho(f\mu,T\mu),\ \varrho(g\nu,S\nu),\ \frac{\varrho(g\nu,T\mu)+\varrho(f\mu,S\nu)}{2}, \\ \varrho(g\nu,S\nu)\left(\frac{1+\varrho(f\mu,T\mu)}{1+\varrho(f\mu,g\nu)}\right),\ \varrho(f\mu,T\mu)\left(\frac{1+\varrho(g\nu,S\nu)}{1+\varrho(f\mu,g\nu)}\right) \end{array} \right\}.$$

3 Main Results

Now, we establish some common fixed point theorems in partially ordered metric spaces for generalized (ψ, ϕ)-weak contraction mappings satisfying the rational type expressions which extend the results of metric spaces proved by Arya et al. [2] and Chandra et al. [5] for analogue conditions. We begin by proving the following theorem:

Theorem 1 *Let $(X, \varrho, \sqsubseteq)$ be a complete partially ordered metric space. Let $T, S : X \to X$ be weakly increasing mappings such that for any comparable elements $\mu, \nu \in X$, we have*

$$\psi(\varrho(T\mu, S\nu)) \leq \psi(M(T\mu, S\nu)) - \phi(\psi(M(T\mu, S\nu))) + \theta(N(T\mu, S\nu)), \tag{1}$$

where

(i) $\psi : [0,\infty) \to [0,\infty)$ *is a continuous and non-decreasing function such that* $\psi(t) = 0$ *if and only if* $t = 0$*, and* $\limsup_{s\to 0} \frac{s}{\psi(s)} < \infty$*;*

(ii) $\phi : [0,\infty) \to [0,\infty)$ *is a lower semi-continuous function such that* $\phi(t) = 0$ *if and only if* $t = 0$*, and for any sequence* $\{t_n\}$ *with* $\lim_{n\to\infty} t_n = 0$*, there are* $k \in (0,1)$ *and* $n_0 \in \mathbb{N}$ *such that* $\phi(t_n) \geq kt_n$ *for each* $n \geq n_0$*;*

(iii) $\theta : [0,\infty) \to [0,\infty)$ *is a continuous function such that* $\theta(t) = 0$ *if and only if* $t = 0$*.*

Then T *and* S *have at least one common fixed point, if one of the following cases satisfies:*

(τ_1*)* T *or* S *is continuous.*

(τ_2*) If a non-decreasing sequence* $\{\mu_n\}$ *converges to* μ*,* $\mu_n \sqsubseteq \mu$ *for all* n*.*

Proof. As we know that the pair (T,S) is weakly increasing, therefore one can construct inductively a sequence $\{\mu_n\}$ starting from an arbitrary element, $\mu_0 \in X$, such that $\mu_n \sqsubseteq \mu_{n+1}$ in the following iterative way:

$$\mu_1 = T\mu_0 \sqsubseteq ST\mu_0 = S\mu_1,\ \mu_2 = S\mu_1 \sqsubseteq TS\mu_1 = T\mu_2,\ \mu_3 = T\mu_2 \sqsubseteq ST\mu_2 = S\mu_3 \ldots,$$

in general, $\mu_{2n+1} = T\mu_{2n}$ and $\mu_{2n+2} = S\mu_{2n+1}$ for all $n \in \mathbb{N} \cup \{0\}$.

Suppose first that $\mu_{n_0} = \mu_{n_0+1}$ for some n_0. Then the sequence $\{\mu_n\}$ is constant for $n \geq n_0$. Indeed, if $n_0 = 2k$ then $\mu_{2k} = \mu_{2k+1}$ and, by using (1), we have the following:

$$\begin{aligned} \psi(\varrho(\mu_{2k+1},\mu_{2k+2})) &= \psi(\varrho(T\mu_{2k},S\mu_{2k+1})) \\ &\leq \psi(M(T\mu_{2k},S\mu_{2k+1})) - \phi(\psi(M(T\mu_{2k},S\mu_{2k+1}))) + \theta(N(T\mu_{2k},S\mu_{2k+1})), \end{aligned}$$

where,

$$\begin{aligned} M(T\mu_{2k},S\mu_{2k+1}) &= \max\left\{ \begin{array}{l} \varrho(\mu_{2k},\mu_{2k+1}),\ \varrho(\mu_{2k},T\mu_{2k}),\ \varrho(\mu_{2k+1},S\mu_{2k+1}),\ \frac{\varrho(\mu_{2k+1},T\mu_{2k})+\varrho(\mu_{2k},S\mu_{2k+1})}{2}, \\ \varrho(\mu_{2k+1},S\mu_{2k+1})\left(\frac{1+\varrho(\mu_{2k},T\mu_{2k})}{1+\varrho(\mu_{2k},\mu_{2k+1})}\right),\ \varrho(\mu_{2k},T\mu_{2k})\left(\frac{1+\varrho(\mu_{2k+1},S\mu_{2k+1})}{1+\varrho(\mu_{2k},\mu_{2k+1})}\right) \end{array} \right\} \\ &= \max\left\{ \begin{array}{l} \varrho(\mu_{2k},\mu_{2k+1}),\ \varrho(\mu_{2k},\mu_{2k+1}),\ \varrho(\mu_{2k+1},\mu_{2k+2}),\ \frac{\varrho(\mu_{2k+1},\mu_{2k+1})+\varrho(\mu_{2k},\mu_{2k+2})}{2}, \\ \varrho(\mu_{2k+1},\mu_{2k+2})\left(\frac{1+\varrho(\mu_{2kn},\mu_{2k+1})}{1+\varrho(\mu_{2k},\mu_{2k+1})}\right),\ \varrho(\mu_{2k},\mu_{2k+1})\left(\frac{1+\varrho(\mu_{2k+1},\mu_{2k+2})}{1+\varrho(\mu_{2k},\mu_{2k+1})}\right) \end{array} \right\} \\ &= \max\left\{(\mu_{2k+1},\mu_{2k+2}),\ \frac{\varrho(\mu_{2k+1},\mu_{2k+1})}{2},\ \varrho(\mu_{2k+1},\mu_{2k+2})\right\} \\ &= \varrho(\mu_{2k+1},\mu_{2k+2}), \end{aligned}$$

and

$$\begin{aligned} N(T\mu_{2k},S\mu_{2k+1}) &= \min\left\{ \begin{array}{l} \varrho(\mu_{2k},\mu_{2k+1}),\ \varrho(\mu_{2k},T\mu_{2k}),\ \varrho(\mu_{2k+1},S\mu_{2k+1}), \\ \varrho(\mu_{2k+1},T\mu_{2k}),\ \varrho(\mu_{2k},S\mu_{2k+1}) \end{array} \right\} \\ &= 0. \end{aligned}$$

Hence $\psi(\varrho(\mu_{2k+1},\mu_{2k+2})) \leq \psi(\varrho(\mu_{2k+1},\mu_{2k+2})) - \phi(\psi(\varrho(\mu_{2k+1},\mu_{2k+2})))$, and so $\phi(\psi(\varrho(\mu_{2k+1},\mu_{2k+2}))) \leq 0$ which implies $\mu_{2k+1} = \mu_{2k+2}$. Similarly, one can easily obtain that $\mu_{2k+2} = \mu_{2k+3}$, if $n_0 = 2k+1$. Thus, the sequence $\{\mu_n\}$ is constant and μ_{n_0} is common fixed point of T and S.

Suppose now that $\varrho(\mu_n,\mu_{n+1}) > 0$ for each n. Then, for each $n \in \mathbb{N} \cup \{0\}$, we shall show that

$$\varrho(\mu_{2n+1},\mu_{2n+2}) \leq M(T\mu_{2n},S\mu_{2n+1}) = \varrho(\mu_{2n},\mu_{2n+1}), \tag{2}$$

and

$$\varrho(\mu_{2n+3},\mu_{2n+2}) \leq M(T\mu_{2n+2},S\mu_{2n+1}) = \varrho(\mu_{2n+2},\mu_{2n+1}). \tag{3}$$

It is clear to observe that $N(T\mu_{2n}, S\mu_{2n+1}) = 0$ and $N(T\mu_{2n+2}, S\mu_{2n+1}) = 0$, for all $n \in \mathbb{N} \cup \{0\}$. Since μ_{2n} and μ_{2n+1} are comparable, by using condition (1) we have

$$\begin{aligned} \psi(\varrho(\mu_{2n+1}, \mu_{2n+2})) &= \psi(\varrho(T\mu_{2n}, S\mu_{2n+1})) \\ &\leq \psi(M(T\mu_{2n}, S\mu_{2n+1})) - \phi(\psi(M(T\mu_{2n}, S\mu_{2n+1}))) + \theta(N(T\mu_{2n}, S\mu_{2n+1})) \\ &\leq \psi(M(T\mu_{2n}, S\mu_{2n+1})), \end{aligned} \tag{4}$$

and as the function ψ is nondecreasing, so it follows that $\varrho(\mu_{2n+1}, \mu_{2n+2}) \leq M(T\mu_{2n}, S\mu_{2n+1})$, where

$$\begin{aligned} M(T\mu_{2n}, S\mu_{2n+1}) &= \max\left\{ \begin{array}{l} \varrho(\mu_{2n}, \mu_{2n+1}),\ \varrho(\mu_{2n}, T\mu_{2n}),\ \varrho(\mu_{2n+1}, S\mu_{2n+1}),\ \frac{\varrho(\mu_{2n+1}, T\mu_{2n}) + \varrho(\mu_{2n}, S\mu_{2n+1})}{2}, \\ \varrho(\mu_{2n+1}, S\mu_{2n+1}) \left(\frac{1+\varrho(\mu_{2n}, T\mu_{2n})}{1+\varrho(\mu_{2n}, \mu_{2n+1})}\right),\ \varrho(\mu_{2n}, T\mu_{2n}) \left(\frac{1+\varrho(\mu_{2n+1}, S\mu_{2n+1})}{1+\varrho(\mu_{2n}, \mu_{2n+1})}\right) \end{array} \right\} \\ &= \max\left\{ \begin{array}{l} \varrho(\mu_{2n}, \mu_{2n+1}),\ \varrho(\mu_{2n}, \mu_{2n+1}),\ \varrho(\mu_{2n+1}, \mu_{2n+2}),\ \frac{\varrho(\mu_{2n+1}, \mu_{2n+1}) + \varrho(\mu_{2n}, \mu_{2n+2})}{2}, \\ \varrho(\mu_{2n+1}, \mu_{2n+2}) \left(\frac{1+\varrho(\mu_{2n}, \mu_{2n+1})}{1+\varrho(\mu_{2n}, \mu_{2n+1})}\right),\ \varrho(\mu_{2n}, \mu_{2n+1}) \left(\frac{1+\varrho(\mu_{2n+1}, \mu_{2n+2})}{1+\varrho(\mu_{2n}, \mu_{2n+1})}\right) \end{array} \right\} \\ &= \max\left\{ \begin{array}{l} \varrho(\mu_{2n}, \mu_{2n+1}),\ \varrho(\mu_{2n+1}, \mu_{2n+2}),\ \frac{\varrho(\mu_{2n}, \mu_{2n+2})}{2}, \\ \varrho(\mu_{2n+1}, \mu_{2n+2}) \left(\frac{1+\varrho(\mu_{2n}, \mu_{2n+1})}{1+\varrho(\mu_{2n}, \mu_{2n+1})}\right),\ \varrho(\mu_{2n}, \mu_{2n+1}) \left(\frac{1+\varrho(\mu_{2n+1}, \mu_{2n+2})}{1+\varrho(\mu_{2n}, \mu_{2n+1})}\right) \end{array} \right\} \\ &= \max\left\{ \varrho(\mu_{2n}, \mu_{2n+1}),\ \varrho(\mu_{2n+1}, \mu_{2n+2}),\ \varrho(\mu_{2n}, \mu_{2n+1}) \left(\frac{1+\varrho(\mu_{2n+1}, \mu_{2n+2})}{1+\varrho(\mu_{2n}, \mu_{2n+1})}\right) \right\}. \end{aligned}$$

Now, if $\varrho(\mu_{2n+1}, \mu_{2n+2}) \geq \varrho(\mu_{2n}, \mu_{2n+1})$ then we have $M(T\mu_{2n}, S\mu_{2n+1}) = \varrho(\mu_{2n+1}, \mu_{2n+2})$, and so (4) implies that

$$\psi(\varrho(\mu_{2n+1}, \mu_{2n+2})) = \psi(\varrho(T\mu_{2n}, S\mu_{2n+1})) \leq \psi(\varrho(\mu_{2n+1}, \mu_{2n+2})) - \phi(\psi(\varrho(\mu_{2n+1}, \mu_{2n+2}))),$$

which is possible only whenever $\varrho(\mu_{2n+1}, \mu_{2n+2}) = 0$, a contradiction. Hence,

$$\varrho(\mu_{2n+1}, \mu_{2n+2}) \leq \varrho(\mu_{2n}, \mu_{2n+1}),\ \ \varrho(\mu_{2n}, \mu_{2n+1}) \left(\frac{1+\varrho(\mu_{2n+1}, \mu_{2n+2})}{1+\varrho(\mu_{2n}, \mu_{2n+1})}\right) \leq \varrho(\mu_{2n}, \mu_{2n+1})$$

and $M(T\mu_{2n}, S\mu_{2n+1}) = \varrho(\mu_{2n}, \mu_{2n+1})$, that is (2) is proved.

In a similar way, one can obtain that $\varrho(\mu_{2n+3}, \mu_{2n+2}) \leq M(T\mu_{2n+2}, S\mu_{2n+1}) = \varrho(\mu_{2n+2}, \mu_{2n+1})$. Thus, (2) and (3) hold for each $n \in \mathbb{N} \cup \{0\}$. Moreover, it also follows that the sequence $\{\varrho(\mu_n, \mu_{n+1})\}$ is non-increasing.

Now, let $\lim_{n\to\infty} \varrho(\mu_n, \mu_{n+1}) = \varrho^*$, for some $\varrho^* \geq 0$. Then,

$$\lim_{n\to\infty} M(T\mu_{2n}, S\mu_{2n+1}) = \varrho^* \ \text{ and } \ \lim_{n\to\infty} M(T\mu_{2n+2}, S\mu_{2n+1}) = \varrho^*.$$

Furthermore, if $\varrho^* > 0$ then we have

$$\psi(\varrho(\mu_{2n+1}, \mu_{2n+2})) \leq \psi(M(T\mu_{2n}, S\mu_{2n+1})) - \phi(\psi(M(T\mu_{2n}, S\mu_{2n+1}))).$$

Therefore, by taking the limit as $n \to \infty$, we get

$$\psi(\varrho^*) \leq \psi(\varrho^*) - \liminf_{n\to\infty} \phi(\psi(M(\mu_{2n}, \mu_{2n+1}))) \leq \psi(\varrho^*) - \phi(\psi(\varrho^*)),$$

i.e., $\psi(\varrho^*) \leq 0$. Using the properties of functions ψ by (i), we get $\varrho^* = 0$, a contradiction. Thus, we conclude that $\lim_{n\to\infty} \varrho(\mu_n, \mu_{n+1}) = 0$.

Next, we show that $\{\mu_n\}$ is a Cauchy sequence in X. Since

$$\lim_{n\to\infty} \psi(M(T\mu_{2n}, S\mu_{2n+1})) = 0 = \lim_{n\to\infty} \psi(M(T\mu_{2n+2}, S\mu_{2n+1})),$$

the property of ϕ by (ii), there exist $k \in (0,1)$ and $n_0 \in \mathbb{N}$ such that

$$\phi(\psi(M(T\mu_{2n}, S\mu_{2n+1}))) \geq k\psi(M(T\mu_{2n}, S\mu_{2n+1}))$$

and

$$\phi(\psi(M(T\mu_{2n+2}, S\mu_{2n+1}))) \geq k\psi(M(T\mu_{2n+2}, S\mu_{2n+1})),$$

for all $n \geq n_0$. Now, if n is an even, by using (1) we get,

$$\psi(\varrho(T\mu_{2n}, S\mu_{2n+1})) \leq \psi(M(T\mu_{2n}, S\mu_{2n+1})) - \phi(\psi(M(T\mu_{2n}, S\mu_{2n+1})))$$

which implies $\psi(\varrho(\mu_{2n+1}, \mu_{2n+2})) \leq (1-k)\psi(\varrho(\mu_{2n}, \mu_{2n+1}))$. Similarly, if n is an odd then one can easily show that $\psi(\varrho(\mu_{2n}, \mu_{2n+1})) \leq (1-k)\psi(\varrho(\mu_{2n}, \mu_{2n-1}))$. Hence, for all $n \geq n_0$, we have $\psi(\varrho(\mu_n, \mu_{n+1})) \leq (1-k)\psi(\varrho(\mu_n, \mu_{n-1}))$. Moreover, we can obtain that

$$\sum_{n=1}^{\infty} \psi(\varrho(\mu_n, \mu_{n+1})) \leq \sum_{n=1}^{n_0} \psi(\varrho(\mu_n, \mu_{n+1})) + \sum_{n=1}^{\infty} (1-k)^n \psi(\varrho(\mu_{n_0}, \mu_{n_0+1})) < \infty.$$

Since

$$\limsup_{n\to\infty} \frac{\varrho(\mu_n, \mu_{n+1})}{\psi(\varrho(\mu_n, \mu_{n+1}))} \leq \limsup_{s\to 0^+} \frac{s}{\psi(s)} < \infty \text{ therefore } \sum_{n=1}^{\infty} \varrho(\mu_n, \mu_{n+1}) < \infty.$$

It shows that $\{\mu_n\}$ is a Cauchy sequence. Also, as (X, ϱ) is a complete metric space, so there exists a $z \in X$ such that $\lim_{n\to\infty} \mu_n = z$.

Finally, we prove that z is a common fixed point of T and S. We distinguish the cases (τ_1) and (τ_2) of the theorem as follow:

Case (τ_1)**.** Let S be a continuous mapping. Then $\mu_{2n+1} \to z$ implies that $\mu_{2n+2} = S\mu_{2n+1} \to Sz$. Also, $\mu_{2n+2} \to z$ because it is a subsequence of $\{\mu_n\}$. It follows that $Sz = z$. Now, to prove $Tz = z$, by the property $z \sqsubseteq z$, we put $\mu = \nu = z$ in (1) and obtain that

$$\psi(\varrho(Tz, Sz)) \leq \psi(M(Tz, Sz)) - \phi(\psi(M(Tz, Sz))) + \theta(N(Tz, Sz)),$$

where

$$\begin{aligned} M(Tz, Sz) &= \max\left\{ \begin{array}{l} \varrho(z,z),\ \varrho(z,Tz),\ \varrho(z,Sz),\ \frac{\varrho(z,Tz)+\varrho(z,Sz)}{2}, \\ \varrho(z,Sz)\left(\frac{1+\varrho(z,Tz)}{1+\varrho(z,z)}\right),\ \varrho(z,Tz)\left(\frac{1+\varrho(z,Sz)}{1+\varrho(z,z)}\right) \end{array} \right\} \\ &= \max\left\{ \begin{array}{l} \varrho(z,z),\ \varrho(z,Tz),\ \varrho(z,z),\ \frac{\varrho(z,Tz)+\varrho(z,z)}{2}, \\ \varrho(z,z)\left(\frac{1+\varrho(z,Tz)}{1+\varrho(z,z)}\right),\ \varrho(z,Tz)\left(\frac{1+\varrho(z,z)}{1+\varrho(z,z)}\right) \end{array} \right\} \\ &= \max\left\{ \varrho(z,Tz),\ \frac{\varrho(z,Tz)}{2},\ \varrho(z,Tz) \right\} \\ &= \varrho(z,Tz), \end{aligned}$$

and

$$N(Tz, Sz) = \min\{\varrho(z,z),\ \varrho(z,Tz),\ \varrho(z,Sz),\ \varrho(z,Tz),\ \varrho(z,Sz)\} = 0.$$

Hence, $\psi(\varrho(Tz,z)) \leq \psi(\varrho(z,Tz)) - \phi(\psi(\varrho(z,Tz)))$ which follows that $z = Tz$. Thus $z = Tz = Sz$, i.e., z is a common fixed point of T and S. Analogously, it can also be shown, if T is continuous.

Case (τ_2)**.** If this case happens, i.e., $\{\mu_n\}$ is a non-decreasing sequence which converges to z, and $\mu_n \sqsubseteq z$ for all n. Then, by taking $\mu = \mu_{2n}$ and $\nu = z$ (which are comparable) in (1), we get

$$\psi(\varrho(T\mu_{2n}, Sz)) \leq \psi(M(T\mu_{2n}, Sz)) - \phi(\psi(M(T\mu_{2n}, Sz))) + \theta(N(T\mu_{2n}, Sz)), \tag{5}$$

where

$$\begin{aligned} M(T\mu_{2n}, Sz) &= \max\left\{ \begin{array}{l} \varrho(\mu_{2n}, z),\ \varrho(\mu_{2n}, T\mu_{2n}),\ \varrho(z, Sz),\ \frac{\varrho(z,T\mu_{2n})+\varrho(\mu_{2n},Sz)}{2}, \\ \varrho(z, Sz)\left(\frac{1+\varrho(\mu_{2n},T\mu_{2n})}{1+\varrho(\mu_{2n},z)}\right),\ \varrho(\mu_{2n}, T\mu_{2n})\left(\frac{1+\varrho(z,Sz)}{1+\varrho(\mu_{2n},z)}\right) \end{array} \right\} \\ &= \max\left\{ \begin{array}{l} \varrho(z, z),\ \varrho(z, z),\ \varrho(z, Sz),\ \frac{\varrho(z,z)+\varrho(z,Sz)}{2}, \\ \varrho(z, Sz)\left(\frac{1+\varrho(z,z)}{1+\varrho(z,z)}\right),\ \varrho(z, z)\left(\frac{1+\varrho(z,Sz)}{1+\varrho(z,z)}\right) \end{array} \right\} \\ &= \varrho(z, Sz), \end{aligned}$$

and

$$N(T\mu_{2n}, Sz) = \min\{\varrho(\mu_{2n}, z),\ \varrho(\mu_{2n}, T\mu_{2n}),\ \varrho(z, Sz),\ \varrho(z, T\mu_{2n}),\ \varrho(\mu_{2n}, Sz)\} = 0.$$

Taking the limit as $n \to \infty$ in (5), we get $\psi(\varrho(z, Sz)) \leq \psi(\varrho(z, Sz)) - \phi(\psi(\varrho(z, Sz)))$, which follows that $z = Sz$. To prove $Tz = z$, using $z \sqsubseteq z$, we can put $\mu = \nu = z$ in (1) and obtain that

$$\psi(\varrho(Tz, Sz)) \leq \psi(M(Tz, Sz)) - \phi(\psi(M(Tz, Sz))) + \theta(N(Tz, Sz)),$$

where

$$\begin{aligned} M(Tz, Sz) &= \max\left\{ \begin{array}{l} \varrho(z, z),\ \varrho(z, Tz),\ \varrho(z, Sz),\ \frac{\varrho(z,Tz)+\varrho(z,Sz)}{2}, \\ \varrho(z, Sz)\left(\frac{1+\varrho(z,Tz)}{1+\varrho(z,z)}\right),\ \varrho(z, Tz)\left(\frac{1+\varrho(z,Sz)}{1+\varrho(z,z)}\right) \end{array} \right\} \\ &= \max\left\{\varrho(z, Tz),\ \frac{\varrho(z, Tz)}{2},\ \varrho(z, Tz)\right\} \\ &= \varrho(z, Tz) \end{aligned}$$

and

$$N(Tz, Sz) = \min\{\varrho(z, z),\ \varrho(z, Tz),\ \varrho(z, Sz),\ \varrho(z, Tz),\ \varrho(z, Sz))\} = 0.$$

Hence, $\psi(\varrho(Tz, z)) \leq \psi(\varrho(z, Tz)) - \phi(\psi(\varrho(z, Tz)))$ which implies $z = Tz$.

■

However, by taking $S = T$ and $\theta(t) = 0$ as the zero function in the above Theorem 1 we obtain the results for a single mapping as the following.

Corollary 1 *Let $(X, \varrho, \sqsubseteq)$ be a complete partially ordered metric space. Let $T : X \to X$ be a non-decreasing mapping such that $\mu_0 \sqsubseteq T\mu_0$ for some $\mu_0 \in X$, and for any comparable elements $\mu, \nu \in X$, we have*

$$\psi(\varrho(T\mu, T\nu)) \leq \psi(M(T\mu, T\nu)) - \phi(\psi(M(T\mu, T\nu))) + \theta(N(T\mu, S\nu)),$$

where

(i) $\psi : [0, \infty) \to [0, \infty)$ is a continuous and non-decreasing function such that $\psi(t) = 0$ if and only if $t = 0$, and $\limsup_{s\to 0} \frac{s}{\psi(s)} < \infty$;

(ii) $\phi : [0, \infty) \to [0, \infty)$ is a lower semi-continuous function such that $\phi(t) = 0$ if and only if $t = 0$, and for any sequence $\{t_n\}$ with $\lim_{n\to\infty} t_n = 0$, there are $k \in (0, 1)$ and $n_0 \in \mathbb{N}$ such that $\phi(t_n) \geq kt_n$ for each $n \geq n_0$.

(iii) $\theta : [0, \infty) \to [0, \infty)$ is a continuous function such that $\theta(t) = 0$ if and only if $t = 0$.

Then, T has a fixed point, if one of the following cases satisfies:

(τ_1) T is continuous.

(τ_2) If a non-decreasing sequence $\{\mu_n\}$ converges to μ, $\mu_n \sqsubseteq \mu$ for all n.

Corollary 2 *Let $(X, \varrho, \sqsubseteq)$ be a complete partially ordered metric space. Let $T : X \to X$ be a non-decreasing mapping such that $\mu_0 \sqsubseteq T\mu_0$ for some $\mu_0 \in X$, and for any comparable elements $\mu, \nu \in X$, we have*

$$\psi(\varrho(T\mu, T\nu)) \leq \psi(M(T\mu, T\nu)) - \phi(\psi(M(T\mu, T\nu))), \tag{6}$$

where

(i) $\psi : [0, \infty) \to [0, \infty)$ is a continuous and non-decreasing function such that $\psi(t) = 0$ if and only if $t = 0$, and $\limsup_{s\to 0} \frac{s}{\psi(s)} < \infty$;

(ii) $\phi : [0, \infty) \to [0, \infty)$ is a lower semi-continuous function such that $\phi(t) = 0$ if and only if $t = 0$, and for any sequence $\{t_n\}$ with $\lim_{n\to\infty} t_n = 0$, there are $k \in (0, 1)$ and $n_0 \in \mathbb{N}$ such that $\phi(t_n) \geq kt_n$ for each $n \geq n_0$.

Then, T has a fixed point, if one of the following cases satisfies:

(τ_1) T is continuous.

(τ_2) If a non-decreasing sequence $\{\mu_n\}$ converges to μ, $\mu_n \sqsubseteq \mu$ for all n.

Next, we give an example in which contractive condition of Theorem 1 is satisfied while the contractive condition appearing in the result, viz. Theorem 2.1, due to Arya et al. [2] is not satisfied.

Example 1 *Let $X = [0, 2]$ be a usual metric space. We define partial order $\sqsubseteq$ on X as follows.*

$$\sqsubseteq := \{(\mu, \mu) \in X \times X \mid \mu \in X\} \cup \left\{(\frac{1}{2}, \frac{3}{2}), (1, 2), (\frac{1}{2}, 2), (\frac{3}{2}, 2)\right\}.$$

We define $T, S : X \to X$ by

$$T\mu = \begin{cases} \mu + \frac{1}{2}, & \mu \in [0, \frac{3}{2}); \\ 2, & \mu \in [\frac{3}{2}, 2], \end{cases} \quad \text{and} \quad S\mu = \begin{cases} \mu + \frac{1}{4} & \mu \in [0, \frac{3}{2}); \\ 2, & \mu \in [\frac{3}{2}, 2]. \end{cases}$$

Also, we define $\phi, \psi, \theta : [0, \infty) \to [0, \infty)$ by $\psi(t) = t^2$, $\phi(t) = \frac{t}{2}$, $\theta(t) = 2t$. Then, we get the following cases:

Case 1. *If $(\mu, \nu) = (\frac{1}{2}, \frac{3}{2})$, we get $(T\mu, S\nu) = (1, 2)$, $M(T\frac{1}{2}, S\frac{3}{2}) = 1$, $N(T\frac{1}{2}, S\frac{3}{2}) = \frac{1}{2}$.*

Case 2. *If $(\mu, \nu) = (1, 2)$, we get $(T\mu, S\nu) = (\frac{3}{2}, 2)$, $M(T1, S2) = 1$, $N(T1, S2) = 0$.*

Case 3. *If $(\mu, \nu) = (\frac{1}{2}, 2)$, we get $(T\mu, S\nu) = (1, 2)$, $M(T\frac{1}{2}, S2) = \frac{3}{2}$, $N(T\frac{1}{2}, S2) = 0$.*

Case 4. *If $(\mu, \nu) = (\frac{3}{2}, 2)$, we get $(T\mu, S\nu) = (2, 2)$, $M(T\frac{3}{2}, S2) = \frac{1}{2}$, $N(T\frac{3}{2}, S2) = 0$.*

Now, all the condition of Theorem 1 are satisfied and 2 is a common fixed point of T, S. But the contractive condition due to Arya et al. [2] is not satisfied, e.g., take $\mu = \frac{1}{2}$ and $\nu = \frac{3}{2}$, then we obtain

$$M_1(T\frac{1}{2}, S\frac{3}{2}) = \max \left\{ \begin{array}{l} \varrho(\frac{1}{2}, \frac{3}{2}),\ \varrho(\frac{1}{2}, T\frac{1}{2}),\ \varrho(\frac{3}{2}, S\frac{3}{2}),\ \frac{\varrho(\frac{3}{2}, T\frac{1}{2}) + \varrho(\frac{1}{2}, S\frac{3}{2})}{2},\ \frac{\varrho(\frac{1}{2}, T\frac{1}{2}) + \varrho(\frac{3}{2}, S\frac{3}{2})}{2}, \\ \varrho(\frac{3}{2}, S\frac{3}{2}) \left(\frac{1+\varrho(\frac{1}{2}, T\frac{1}{2})}{1+\varrho(\frac{1}{2}, \frac{3}{2})}\right),\ \varrho(\frac{1}{2}, T\frac{1}{2}) \left(\frac{1+\varrho(\frac{1}{2}, S\frac{3}{2})}{1+\varrho(\frac{1}{2}, \frac{3}{2})}\right) \end{array} \right\} = 1.$$

Therefore, $\psi(\varrho(T\frac{1}{2}, S\frac{3}{2})) \leq \psi(M_1(T\frac{1}{2}, S\frac{3}{2})) - \phi(M_1(T\frac{1}{2}, S\frac{3}{2}))$ implies $1 \leq \frac{1}{2}$, that is not possible. Thus, Theorem 1 is a proper generalization of the result of Arya et al. [2]

Also, we establish the following result about common fixed point of four self mappings of partially ordered metric spaces.

Theorem 2 *Let $(X,\varrho,\sqsubseteq)$ be a complete partially ordered metric space. Let $f,g,T,S:X\to X$ be mappings such that $T(X)\subseteq g(X)$ and $S(X)\subseteq f(X)$, the pairs (T,f) and (S,g) are weakly compatible, and for any comparable elements $\mu,\nu\in X$, we have*

$$\psi(\varrho(T\mu,S\nu))\leq\psi(M_{f,g}(T\mu,S\nu))-\phi(M_{f,g}(T\mu,S\nu)),\tag{7}$$

where, the functions ψ and ϕ are defined in Theorem 1. Then, f,g,T and S have a unique common fixed point whenever one of the range sets, $T(X)$, $S(X)$, $f(X)$ and $g(X)$, is closed in X.

Proof. Let μ_0 be an arbitrary point in X. As $T(X)\subseteq g(X)$, therefore we can choose $\mu_1\in X$ such that $\nu_0=T\mu_0=g\mu_1$. Similarly, for $S(X)\subseteq f(X)$, we can choose $\mu_2\in X$ such that $\nu_1=S\mu_1=f\mu_2$. Continuing this iterative scheme, one can construct inductively a sequence $\{\nu_n\}$ starting from an arbitrary element, $\nu_0\in X$, such that $\nu_n\sqsubseteq\nu_{n+1}$ with $\nu_{2n}=T\mu_{2n}=g\mu_{2n+1}$ and $\nu_{2n+1}=S\mu_{2n+1}=f\mu_{2n+2}$, for all $n\in\mathbb{N}\cup\{0\}$. Now, we show that $\{\nu_n\}$ is a Cauchy sequence in X by considering the following cases:

Case 1. Suppose that $\nu_{n_0}=\nu_{n_0+1}$ for some n_0, then the sequence $\{\nu_n\}$ is constant for $n\geq n_0$. Indeed, if $n_0=2k$, then $\nu_{2k}=\nu_{2k+1}$ and, by using (7), we get

$$\psi(\varrho(\nu_{2k+2},\nu_{2k+1}))=\psi(\varrho(T\mu_{2k+2},S\mu_{2k+1}))\leq\psi(M_{f,g}(T\mu_{2k+2},S\mu_{2k+1}))-\phi(M_{f,g}(T\mu_{2k+2},S\mu_{2k+1})),$$

where,

$$\begin{aligned}
M_{f,g}(T\mu_{2k+2},S\mu_{2k+1}) &= \max\left\{\begin{array}{l}\varrho(f\mu_{2k+2},g\mu_{2k+1}),\ \varrho(f\mu_{2k+2},T\mu_{2k+2}),\\ \varrho(g\mu_{2k+1},S\mu_{2k+1}),\ \frac{\varrho(g\mu_{2k+1},T\mu_{2k+2})+\varrho(f\mu_{2k+2},S\mu_{2k+1})}{2},\\ \varrho(g\mu_{2k+1},S\mu_{2k+1})\left(\frac{1+\varrho(f\mu_{2k+2},T\mu_{2k+2})}{1+\varrho(f\mu_{2k+2},g\mu_{2k+1})}\right),\\ \varrho(f\mu_{2k+2},T\mu_{2k+2})\left(\frac{1+\varrho(g\mu_{2k+1},S\mu_{2k+1})}{1+\varrho(f\mu_{2k+2},g\mu_{2k+1})}\right)\end{array}\right\}\\
&= \max\left\{\begin{array}{l}\varrho(\nu_{2k+1},\nu_{2k}),\ \varrho(\nu_{2k+1},\nu_{2k+2}),\\ \varrho(\nu_{2k},\nu_{2k+1}),\ \frac{\varrho(\nu_{2k},\nu_{2k+2})+\varrho(\nu_{2k+1},\nu_{2k+1})}{2},\\ \varrho(\nu_{2k},\nu_{2k+1})\left(\frac{1+\varrho(\nu_{2k+1},\nu_{2k+2})}{1+\varrho(\nu_{2k+1},\nu_{2k})}\right),\\ \varrho(\nu_{2k+1},\nu_{2k+2})\left(\frac{1+\varrho(\nu_{2k},\nu_{2k+1})}{1+\varrho(\nu_{2k+1},\nu_{2k})}\right)\end{array}\right\}\\
&= \varrho(\nu_{2k+2},\nu_{2k+1}).
\end{aligned}$$

Hence

$$\psi(\varrho(\nu_{2k+2},\nu_{2k+1})))\leq\psi(\varrho(\nu_{2k+1},\nu_{2k+2}))-\phi(\varrho(\nu_{2k+1},\nu_{2k+2}))<\psi(\varrho(\nu_{2k+1},\nu_{2k+2})),$$

which is a contradiction. Using a similar argument this equality also holds when n is odd. Thus the sequence $\{\nu_n\}$ is constant for $n\geq n_0$, and so it is a Cauchy sequence.

Case 2. Suppose now $\varrho(\nu_{n+1},\nu_n)>0$ for each n. Then, for each $n\in\mathbb{N}\cup\{0\}$, we show that $\psi(\varrho(\nu_{n+1},\nu_n))\leq\psi(\varrho(\nu_n,\nu_{n-1}))$. If $n=2k+1$, where $k\in\mathbb{N}$, by using (7) we have

$$\begin{aligned}
\psi(\varrho(\nu_{2k+2},\nu_{2k+1})) &= \psi(\varrho(T\mu_{2k+2},S\mu_{2k+1}))\\
&\leq \psi(M_{f,g}(T\mu_{2k+2},S\mu_{2k+1}))-\phi(M_{f,g}(T\mu_{2k+2},S\mu_{2k+1})),
\end{aligned}$$

where,

$$M_{f,g}(T\mu_{2k+2}, S\mu_{2k+1}) = \max\left\{\begin{array}{l} \varrho(f\mu_{2k+2}, g\mu_{2k+1}),\ \varrho(f\mu_{2k+2}, T\mu_{2k+2}), \\ \varrho(g\mu_{2k+1}, S\mu_{2k+1}),\ \frac{\varrho(g\mu_{2k+1}, T\mu_{2k+2})+\varrho(f\mu_{2k+2}, S\mu_{2k+1})}{2}, \\ \varrho(g\mu_{2k+1}, S\mu_{2k+1})\left(\frac{1+\varrho(f\mu_{2k+2}, T\mu_{2k+2})}{1+\varrho(f\mu_{2k+2}, g\mu_{2k+1})}\right), \\ \varrho(f\mu_{2k+2}, T\mu_{2k+2})\left(\frac{1+\varrho(g\mu_{2k+1}, S\mu_{2k+1})}{1+\varrho(f\mu_{2k+2}, g\mu_{2k+1})}\right) \end{array}\right\}$$

$$= \max\left\{\begin{array}{l} \varrho(\nu_{2k+1}, \nu_{2k}),\ \varrho(\nu_{2k+1}, \nu_{2k+2}), \\ \varrho(\nu_{2k}, \nu_{2k+1}),\ \frac{\varrho(\nu_{2k}, \nu_{2k+2})+\varrho(\nu_{2k+1}, \nu_{2k+1})}{2}, \\ \varrho(\nu_{2k}, \nu_{2k+1})\left(\frac{1+\varrho(\nu_{2k+1}, \nu_{2k+2})}{1+\varrho(\nu_{2k+1}, \nu_{2k})}\right), \\ \varrho(\nu_{2k+1}, \nu_{2k+2})\left(\frac{1+\varrho(\nu_{2k}, \nu_{2k+1})}{1+\varrho(\nu_{2k+1}, \nu_{2k})}\right) \end{array}\right\}$$

$$= \max\left\{\begin{array}{l} \varrho(\nu_{2k+1}, \nu_{2k}),\ \varrho(\nu_{2k+2}, \nu_{2k+1}, \\ \varrho(\nu_{2k}, \nu_{2k+1}),\ \frac{\varrho(\nu_{2k}, \nu_{2k+2})+\varrho(\nu_{2k+1}, \nu_{2k+1})}{2}, \\ \varrho(\nu_{2k}, \nu_{2k+1})\left(\frac{1+\varrho(\nu_{2k+1}, \nu_{2k+2})}{1+\varrho(\nu_{2k+1}, \nu_{2k})}\right), \\ \varrho(\nu_{2k+1}, \nu_{2k+2})\left(\frac{1+\varrho(\nu_{2k}, \nu_{2k+1})}{1+\varrho(\nu_{2k+1}, \nu_{2k})}\right) \end{array}\right\}.$$

Now, if $M_{f,g}(T\mu_{2k+2}, S\mu_{2k+1}) = \varrho(\nu_{2k+2}, \nu_{2k+1})$ then we have

$$\psi(\varrho(\nu_{2k+2}, \nu_{2k+1})) \leq \psi(\varrho(\nu_{2k+2}, \nu_{2k+1})) - \phi(\varrho(\nu_{2k+2}, \nu_{2k+1})),$$

which is possible only whenever $\varrho(\nu_{2k+2}, \nu_{2k+1}) = 0$, a contradiction. Therefore,

$$\varrho(\nu_{2k+2}, \nu_{2k+1}) \leq \varrho(\nu_{2k+1}, \nu_{2k}), \quad \varrho(\nu_{2k+1}, \nu_{2k})\left(\frac{1+\varrho(\nu_{2k+2}, \nu_{2k+1})}{1+\varrho(\nu_{2k+1}, \nu_{2k})}\right) \leq \varrho(\nu_{2k+1}, \nu_{2k}),$$

and so $M_{f,g}(\nu_{2k+2}, \nu_{2k+1}) = \varrho(\nu_{2k+1}, \nu_{2k})$. Hence, $\varrho(\nu_{2k+2}, \nu_{2k+1}) \leq \varrho(\nu_{2k+1}, \nu_{2k})$ whenever $n = 2k+1$.

In a similar way, one can obtain that $\varrho(\nu_{2k+1}, \nu_{2k}) \leq \varrho(\nu_{2k}, \nu_{2k-1})$ when $n = 2k$, i.e, n is even. Consequently, $\psi(\varrho(\nu_{n+1}, \nu_n)) \leq \psi(\varrho(\nu_n, \nu_{n-1}))$ for all $n \in \mathbb{N} \cup \{0\}$. Moreover, the sequence $\{\varrho(\nu_{n+1}, \nu_n)\}$ is non-increasing and bounded below by zero.

Now, let $\lim_{n\to\infty} \varrho(\nu_{n+1}, \nu_n) = \alpha$, for some $\alpha \geq 0$. Then,

$$\lim_{n\to\infty} M_{f,g}(\nu_{2n+1}, \nu_{2n}) = \alpha \quad \text{and} \quad \lim_{n\to\infty} M_{f,g}(\nu_{2n+2}, \nu_{2n+1}) = \alpha.$$

Furthermore, if $\alpha > 0$, then we have

$$\psi(\varrho(\nu_{2n+1}, \nu_{2n+2})) \leq \psi(M_{f,g}(\nu_{2n+1}, \nu_{2n})) - \phi(M_{f,g}(\nu_{2n+1}, \nu_{2n}))$$

and

$$\lim_{n\to\infty} \psi(M_{f,g}(\nu_{2n+1}, \nu_{2n})) = \psi(\alpha).$$

and now, taking upper limits, we get

$$\limsup_{n\to\infty} \psi(\varrho(\nu_{2n+1}, \nu_{2n})) \leq \limsup_{n\to\infty} \psi(M_{f,g}(\nu_{2n+1}, \nu_{2n})) - \limsup_{n\to\infty} \phi(M_{f,g}(\nu_{2n+1}, \nu_{2n})).$$

Thus, the lower semi-continuity of ϕ gives, $\psi(\alpha) \leq \psi(\alpha) - \phi(\alpha)$ which implies $\phi(\alpha) \leq 0$, and so $\alpha = 0$. Hence, we conclude that $\lim_{n\to\infty} \varrho(\nu_{2n+1}, \nu_{2n}) = 0$.

Since $\lim\limits_{n\to\infty} \psi(M_{f,g}(\nu_{2n+1}, \nu_{2n})) = 0$ and $\lim\limits_{n\to\infty} \psi(M_{f,g}(\nu_{2n+2}, \nu_{2n+1})) = 0$, by the property of ϕ, there exist $k \in (0,1)$ and $n_0 \in \mathbb{N}$ such that

$$\phi(M_{f,g}(\nu_{2n+1}, \nu_{2n})) \geq k\, M_{f,g}(\nu_{2n+1}, \nu_{2n}) \quad \text{and} \quad \phi(M_{f,g}(\nu_{2n+2}, \nu_{2n+1})) \geq k\, M_{f,g}(\nu_{2n+2}, \nu_{2n+1})$$

for all $n \geq n_0$. Now, if n is even, by using (7) we get

$$\psi(\varrho(T\nu_{2n+1}, S\nu_{2n})) \leq \psi(M_{f,g}(\nu_{2n+1}, \nu_{2n})) - \phi(M_{f,g}(\nu_{2n+1}, \nu_{2n}))$$

which implies $\psi(\varrho(\nu_{2n+2}, \nu_{2n+1})) \leq (1-k)\psi(\varrho(\nu_{2n+1}, \nu_{2n}))$. Similarly, if n is an odd then one can easily show that $\psi(\varrho(\nu_{2n+1}, \nu_{2n})) \leq (1-k)\psi(\varrho(\nu_{2n}, \nu_{2n-1}))$. Hence, for all $n \geq n_0$, we have $\psi(\varrho(\nu_{n+1}, \nu_n)) \leq (1-k)\psi(\varrho(\nu_n, \nu_{n-1}))$. Moreover, we can obtain that

$$\sum_{n=1}^{\infty} \psi(\varrho(\nu_n, \nu_{n+1})) \leq \sum_{n=1}^{n_0} \psi(\varrho(\nu_n, \nu_{n+1})) + \sum_{n=1}^{\infty} (1-k)^n \psi(\varrho(\nu_{n_0}, \nu_{n_0+1})) < \infty.$$

Since

$$\limsup_{n\to\infty} \frac{\varrho(\nu_n, \nu_{n+1})}{\psi(\varrho(\nu_n, \nu_{n+1}))} \leq \limsup_{s\to 0^+} \frac{s}{\psi(s)} < \infty \text{ therefore } \sum_{n=1}^{\infty} \varrho(\nu_n, \nu_{n+1}) < \infty.$$

It shows that $\{\nu_n\}$ is a Cauchy sequence.

Also, as (X, ϱ) is a complete metric space, so there exists a $z \in X$ such that $\lim_{n\to\infty} \mu_n = z$. Finally, we prove that z is a common fixed point of T, S, f and g. It is clear that

$$\lim_{n\to\infty} \nu_{2n} = \lim_{n\to\infty} T\mu_{2n} = \lim_{n\to\infty} g\mu_{2n+1} = z$$

and

$$\lim_{n\to\infty} \nu_{2n+1} = \lim_{n\to\infty} S\mu_{2n+1} = \lim_{n\to\infty} f\mu_{2n+2} = z.$$

Assuming that $f(X)$ is closed, then there exists $x \in X$ such that $z = fx$. We claim that $Tx = z$. If not, by using (7), we get

$$\psi(\varrho(Tx, S\mu_{2n+1})) \leq \psi(M_{f,g}(Tx, S\mu_{2n+1})) - \phi(M_{f,g}(Tx, S\mu_{2n+1})),$$

where,

$$\begin{aligned}
M_{f,g}(Tx, S\mu_{2n+1}) &= \max\left\{ \begin{array}{l} \varrho(fx, g\mu_{2n+1}),\ \varrho(fx, Tx),\ \varrho(g\mu_{2n+1}, S\mu_{2n+1}), \\ \frac{\varrho(g\mu_{2n+1}, Tx) + \varrho(fx, S\mu_{2n+1})}{2},\ \varrho(g\mu_{2n+1}, S\mu_{2n+1}) \left(\frac{1+\varrho(fx, Tx)}{1+\varrho(fx, g\mu_{2n+1})} \right), \\ \varrho(fx, Tx) \left(\frac{1+\varrho(g\mu_{2n+1}, S\mu_{2n+1})}{1+\varrho(fx, g\mu_{2n+1})} \right) \end{array} \right\} \\
&= \max\left\{ \begin{array}{l} \varrho(z,z),\ \varrho(z,Tx),\ \varrho(z,z),\ \frac{\varrho(z,Tx)+\varrho(z,z)}{2}, \\ \varrho(z,z)\left(\frac{1+\varrho(z,Tx)}{1+\varrho(z,z)}\right),\ \varrho(z,Tx)\left(\frac{1+\varrho(z,z)}{1+\varrho(z,z)}\right) \end{array} \right\} \\
&= \varrho(z, Tx).
\end{aligned}$$

Taking the limit as $n \to \infty$, we obtain $\psi(\varrho(Tx, z)) \leq \psi(\varrho(z, Tx)) - \phi(\varrho(z, Tx))$, which implies $\phi(\varrho(z, Tx)) \leq 0$, a contradiction. Hence, $Tx = z$ and $Tx = fx = z$. Since the pair (T, f) is weakly compatible, $Tz = Tfx = fTx = fz$.

Furthermore, we claim that $Tz = z$. If not, by Using (7), we get

$$\psi(\varrho(Tz, S\mu_{2n+1})) \leq \psi(M_{f,g}(Tz, S\mu_{2n+1})) - \phi(M_{f,g}(Tz, S\mu_{2n+1})),$$

where,

$$M_{f,g}(Tz, S\mu_{2n+1}) = \max\left\{ \begin{array}{l} \varrho(fz, g\mu_{2n+1}),\ \varrho(fz, Tz),\ \varrho(g\mu_{2n+1}, S\mu_{2n+1}),\ \frac{\varrho(g\mu_{2n+1}, Tz)+\varrho(fz, S\mu_{2n+1})}{2}, \\ \varrho(g\mu_{2n+1}, S\mu_{2n+1})\left(\frac{1+\varrho(fz,Tz)}{1+\varrho(fz, g\mu_{2n+1})}\right),\ \varrho(fz, Tz)\left(\frac{1+\varrho(g\mu_{2n+1}, S\mu_{2n+1})}{1+\varrho(fz, g\mu_{2n+1})}\right) \end{array} \right\}.$$

Taking the limit as $n \to \infty$, we obtain $\lim_{n\to\infty} M_{f,g}(Tz, S\mu_{2n+1}) = \varrho(fz, z) = \varrho(Tz, z)$, which implies $\psi(\varrho(Tz, z)) \leq \psi(\varrho(Tz, z)) - \phi(\varrho(Tz, z))$, a contradiction. Hence, $Tz = z$.

Moreover, we show that z is a fixed point of mappings S and g. Since $T(X) \subseteq g(X)$, there is some $y \in X$ such that $Tz = gy$. Then $Tz = gy = fz = z$. We claim that $Sy = z$. If not, by using (7), we get

$$\begin{aligned}
\psi(\varrho(z, Sy)) &= \psi(\varrho(Tz, Sy)) \\
&\leq \psi(M_{f,g}(Tz, Sy)) - \phi(M_{f,g}(Tz, Sy)) \\
&= \psi \max \left\{ \begin{array}{l} \varrho(fz, gy),\ \varrho(fz, Tz),\ \varrho(gy, Sy),\ \frac{\varrho(gy,Tz)+\varrho(fz,Sy)}{2}, \\ \varrho(gy, Sy)\left(\frac{1+\varrho(fz,Tz)}{1+\varrho(fz,gy)}\right),\ \varrho(fz, Tz)\left(\frac{1+\varrho(gy,Sy)}{1+\varrho(fz,gy)}\right) \end{array} \right\} \\
&\quad -\phi \max \left\{ \begin{array}{l} \varrho(fz, gy),\ \varrho(fz, Tz),\ \varrho(gy, Sy),\ \frac{\varrho(gy,Tz)+\varrho(fz,Sy)}{2}, \\ \varrho(gy, Sy)\left(\frac{1+\varrho(fz,Tz)}{1+\varrho(fz,gy)}\right),\ \varrho(fz, Tz)\left(\frac{1+\varrho(gy,Sy)}{1+\varrho(fz,gy)}\right) \end{array} \right\} \\
&= \psi(\varrho(z, Sy)) - \phi(\varrho(z, Sy)),
\end{aligned}$$

which is a contradiction. Hence, $Sy = z$, and so $Sy = gy = z$. Also, by the weak compatibility of the pair (S, g), we get $Sz = Sgy = gSy = gz$. If $Sz \neq z$, by using(7), we get

$$\begin{aligned}
\psi(\varrho(z, Sz)) &= \psi(\varrho(Tz, Sz)) \\
&= \psi \max \left\{ \begin{array}{l} \varrho(fz, gz),\ \varrho(fz, Tz),\ \varrho(gz, Sz),\ \frac{\varrho(gz,Tz)+\varrho(fz,Sz)}{2}, \\ \varrho(gz, Sz)\left(\frac{1+\varrho(fz,Tz)}{1+\varrho(fz,gz)}\right),\ \varrho(fz, Tz)\left(\frac{1+\varrho(gz,Sz)}{1+\varrho(fz,gz)}\right) \end{array} \right\} \\
&\quad -\phi \max \left\{ \begin{array}{l} \varrho(fz, gz),\ \varrho(fz, Tz),\ \varrho(gz, Sz),\ \frac{\varrho(gz,Tz)+\varrho(fz,Sz)}{2}, \\ \varrho(gz, Sz)\left(\frac{1+\varrho(fz,Tz)}{1+\varrho(fz,gz)}\right),\ \varrho(fz, Tz)\left(\frac{1+\varrho(gz,Sz)}{1+\varrho(fz,gz)}\right) \end{array} \right\} \\
&= \psi(\varrho(z, gz)) - \phi(\varrho(z, gz)) = \psi(\varrho(z, Sz)) - \phi(\varrho(z, Sz)),
\end{aligned}$$

a contradiction. Hence, $Tz = Sz = gz = fz = z$.

Similar analysis can also be used in the cases for which $g(X)$, $T(X)$ or $S(X)$ is closed. Furthermost, the uniqueness of the common fixed point z is obvious. ■

Remark 1 *Theorem 2 is a proper extension of Theorem 2.1 of Chandra et al. [5] to partially ordered metric spaces, and many others in the literature.*

Now we find the following corollaries of above Theorem 2, which are extensions of several theorems of various spaces to partially ordered metric spaces, for instance, see [5, 2, 28, 32] and others.

Corollary 3 *Let $(X, \varrho, \sqsubseteq)$ be a complete partially ordered metric space. Let $f, T, S : X \to X$ be mappings such that $T(X)$ and $S(X)$ are subsets of $f(X)$, the pairs (T, f) and (S, f) are weakly compatible, and for any comparable elements $\mu, \nu \in X$, we have*

$$\psi(\varrho(T\mu, S\nu)) \leq \psi(M_f(T\mu, S\nu)) - \phi(M_f(T\mu, S\nu)), \tag{8}$$

where, the functions ψ and ϕ are defined in Theorem 1. Then, f, T and S have a unique common fixed point whenever one of the range sets, $T(X), S(X)$ and $f(X)$, is closed in X.

Proof. Take $f = g$ in Theorem 2. ■

Corollary 4 *Let $(X, \varrho, \sqsubseteq)$ be a complete partially ordered metric space. Let $T, S : X \to X$ be mappings such that T and S are weakly compatible with identity mapping of X, and for any comparable elements $\mu, \nu \in X$, we have*

$$\psi(\varrho(T\mu, S\nu)) \leq \psi(M(T\mu, S\nu)) - \phi(M(T\mu, S\nu)), \tag{9}$$

where, the functions ψ and ϕ are defined in Theorem 1. Then, T and S have a unique common fixed point whenever one of the range sets, $T(X)$ and $S(X)$, is closed in X.

Proof. Take $f = g = I$ (identity mapping) in Theorem 2. ■

Remark 2 *The above corollaries are the proper extensions of Theorem 2.1 of Arya et al. [2] to partially ordered metric spaces and, of course, others as well.*

4 An Application to First Order Periodic Boundary Value Problem

In this section, we present an example as an application for which our Corollary 2 can be applied to the part of ordinary differential equations. We rigorously establish both the existence and uniqueness of solution for the specified first order periodic boundary value problem:

$$\mu'(p) = q(p, \mu(p)), \qquad p \in I = [0, P] \text{ and } \mu(0) = \mu(P), \tag{10}$$

where $P > 0$ and $q : I \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function. Let $C(I)$ denote the space of all continuous functions defined on I. Clearly, this space with the metric given by $\varrho(\mu, \nu) = \sup\{|\mu(p) - \nu(p)| : p \in I\}$, for $\mu, \nu \in C(I)$, is a complete metric space. Define also a partial order $\sqsubseteq$ on $C(I)$ as follows:

$$\mu, \nu \in C(I), \quad \mu \sqsubseteq \nu \Longleftrightarrow \mu(p) \leq \nu(p) \text{ for all } p \in I.$$

Now, we recall the following definitions:

Definition 13 ([22]) *A function $\beta \in C^1(I)$ is called a lower solution of (10), if $\beta'(p) \leq q(p, \beta(p))$, $p \in I$, $\beta(0) \leq \beta(P)$.*

Definition 14 ([22]) *A function $\beta \in C^1(I)$ is called a upper solution of (10), if $\beta'(p) \geq q(p, \beta(p))$, $p \in I$, $\beta(0) \geq \beta(P)$.*

Theorem 3 *Consider the problem (10) with $q : I \times \mathbb{R} \longrightarrow \mathbb{R}$ continuous and suppose that there exists $\lambda >$ 0 such that for all $x, y \in \mathbb{R}$ with $y \geq x$, we have*

$$0 \leq q(p, y) + \lambda y - [q(p, x) + \lambda x] \leq \lambda \ln(y - x + 1). \tag{11}$$

Then the existence of a lower solution or an upper solution of problem (10) ensures the existence and uniqueness of a solution of problem (10).

Proof. Problem (10) can be rewritten as

$$\begin{cases} \mu'(p) + \lambda\mu(p) = q(p, \mu(p)) + \lambda\mu(p), & p \in I, \\ \mu(0) = \mu(P). \end{cases} \tag{12}$$

The problem (12) is equivalent to the integral equation

$$\mu(p) = \int_0^P G(p, s)[q(s, \mu(s)) + \lambda\mu(s)]ds,$$

where the Green function, $G(p, s)$, is given by

$$G(p, s) = \begin{cases} \dfrac{e^{\lambda(P+s-p)}}{e^{\lambda P} - 1}, & 0 \leq s < p \leq P, \\ \dfrac{e^{\lambda(s-p)}}{e^{\lambda P} - 1}, & 0 \leq p < s \leq P. \end{cases}$$

Now, define a function $B : C(I) \longrightarrow C(I)$ by

$$(B\mu)(p) = \int_0^P G(p, s)[q(s, \mu(s)) + \lambda\mu(s)]ds.$$

Note that if $\mu \in C(I)$ is a fixed point of B then $\mu \in C(I)$ is a solution of (10).

Now, we check that hypothesis in Corollary 2 are satisfied. The mapping B is non-decreasing, since for $\mu \geq \nu$, $q(p,\mu) + \lambda\mu \geq q(p,\nu) + \lambda\nu$ which implies that $G(p,s) > 0$ for $(p,s) \in I \times I$, we give

$$\begin{aligned}(B\mu)(p) &= \int_0^P G(p,s)[q(s,\mu(s)) + \lambda\mu(s)]ds \\ &\geq \int_0^P G(p,s)[q(s,\nu(s)) + \lambda\nu(s)]ds = (B\nu)(p)\end{aligned}$$

for $p \in I$. Besides, for $\mu \geq \nu$, we have

$$\begin{aligned}\ln \varrho((B\mu, B\nu) + 1) &= \ln\left(\sup_{p\in I}|(B\mu)(p) - (B\nu)(p)| + 1\right) \\ &\leq \ln\left(\sup_{p\in I}\int_0^P G(p,s)[q(s,\mu(s)) + \lambda\mu(s) - (q(s,\nu(s)) - \lambda\nu(s)]ds) + 1\right) \\ &\leq \ln\left(\sup_{p\in I}\int_0^P G(p,s)\cdot\lambda\ln(\mu(s) - \nu(s) + 1)ds + 1\right) \\ &\leq \ln\left(\sup_{p\in 1}\int_0^P G(p,s)\cdot\lambda\ \ \ln(\mu(s) - \nu(s) + 1)ds + 1\right) \\ &\leq \ln\left(\lambda\cdot\ln(\varrho(\mu,\nu) + 1)\cdot\sup_{p\in 1}\int_0^P G(p,s)ds + 1\right) \\ &= \ln\left((\lambda\cdot\ln\varrho(\mu,\nu) + 1)\cdot\sup_{p\in 1}\frac{1}{e^{\lambda P} - 1}\left(\frac{1}{\lambda}e^{\lambda(P+s-p)}]_0^p + \frac{1}{\lambda}e^{\lambda(s-p)}]_p^P\right) + 1\right) \\ &= \ln\left(\lambda\cdot\ln(\varrho(\mu,\nu) + 1)\cdot\frac{1}{\lambda(e^{\lambda P} - 1)}(e^{\lambda P} - 1)\right) = \ln\left(\ln(\varrho(\mu,\nu) + 1) + 1\right) \\ &= \ln(M(B\mu, B\nu) + 1) - \big(\ln(M(B\mu, B\nu) + 1) - \ln((\ln M(B\mu, B\nu) + 1) + 1)\big).\end{aligned}$$

Putting $\psi(\mu) = \ln(\mu+1)$ and $\phi(\mu) = \mu - \ln(\mu+1)$. Obviously $\psi : [0,\infty) \to [0,\infty)$ is continuous non-decreasing $(\psi^{'}(\mu) = \frac{1}{1+\mu} > 0))$, positive in $(0,\infty)$, $\psi(0) = 0$ and $\limsup_{\mu\to 0^+}\frac{\mu}{\psi(\mu)} = 1 < \infty$. Also, $\phi : [0,\infty) \to [0,\infty)$ is continuous, positive in $(0,\infty)$ and $\psi(0) = 0$. Now let $\{p_n\}$ be a sequence such that $p_n \to 0$. Since $\lim_{n\to\infty}\frac{\phi(p_n)}{p_n} = \phi^{'}(0) = 0$, for $\epsilon = \frac{1}{4}$, there exists n_0 such that

$$\left|\frac{\phi(p_n)}{p_n} - 0\right| < \frac{1}{4} \text{ for all } n \geq n_0.$$

Hence, $\phi(p_n) \geq \frac{1}{4}p_n$ for all $n \geq n_0$. Therefore the control functions ψ and ϕ satisfying the conditions of Corollary 2.

Finally, if $\beta(p)$ is a lower solution for (10) then we will show that $\beta \leq B\beta$. Now, $\beta'(p) + \lambda\beta(p) \leq q(p,\beta(p)) + \lambda\beta(p)$ for $p \in I$. Multiplying by $e^{\lambda p}$, we get $(\beta(p)e^{\lambda p})'^{\lambda p}$ for $p \in I$, and this gives

$$\beta(p)e^{\lambda p} \leq \beta(0) + \int_0^P [q(s,\beta(s)) + \lambda\beta(s)]e^{\lambda s}ds, for\ p \in I \tag{13}$$

which implies that

$$\beta(0)e^{\lambda p} \leq \beta(P)e^{\lambda p} \leq \beta(0) + \int_0^P [q(s,\beta(s)) + \lambda\beta(s)]e^{\lambda s}ds,$$

and so

$$\beta(0) \leq \int_0^P \frac{e^{\lambda s}}{e^{\lambda P} - 1}[q(s,\beta(s)) + \lambda\beta(s)]ds.$$

From this inequality and (13), we obtain

$$\beta(p)e^{\lambda p} \leq \int_0^p \frac{e^{\lambda(P+s)}}{e^{\lambda P}-1}[q(s,\beta(s))+\lambda\beta(s)]ds + \int_p^P \frac{e^{\lambda s}}{e^{\lambda P}-1}[q(s,\beta(s))+\lambda\beta(s)]ds,$$

and consequently,

$$\beta(p) \leq \int_0^p \frac{e^{\lambda(P+s-p)}}{e^{\lambda P}-1}[q(s,\beta(s))+\lambda\beta(s)]ds + \int_p^P \frac{e^{\lambda(s-p)}}{e^{\lambda P}-1}[q(s,\beta(s))+\lambda\beta(s)]ds.$$

Hence,

$$\beta(p) \leq \int_0^P G(p,s)[q(s,\beta(s))+\lambda\beta(s)]ds = (B\beta)(p),\ p \in I.$$

Using our Corollary 2, we have B has a unique fixed point. ■

5 Conclusion

As discussed above, we have proved some common fixed point results for more generalized (ψ,ϕ)-weak contraction conditions satisfying the rational expression in partially ordered metric spaces. The obtained theorems are the extensions of metric results to partially ordered metric settings, and these are the proper generalizations of various results of several researchers in the literature. It is noteworthy that our result (Theorem 1) can also be applied when $N(T\mu, S\nu)$ has minimum in $\{\varrho(\mu,\nu),\ \varrho(\mu,T\mu),\ \varrho(\nu,S\nu)\}$, but the result due to Gordji et al. [10] is applicable only when it has minimum in $\{\varrho(\nu,T\mu),\ \varrho(\mu,S\nu)\}$. Moreover, we have an example for our results and also we provide an application to the existence and uniqueness of solution for first order periodic boundary value problem involving ordinary differential equations.

Acknowledgment. The authors would like to express their sincere gratitude to the anonymous referees for their insightful comments and constructive suggestions, which have significantly contributed to the improvement of this paper.

References

[1] Y. I. Alber and S. Guerre-Delabriere, Principles of weakly contractive maps in Hilbert spaces, Operator Theory Advances Appl., 98(1997), 7–22.

[2] M. C. Arya, N. Chandra and M. C. Joshi, Common fixed point results for a generalized (ψ,ϕ)-rational contraction, Appl. Gen. Topol., 24(2023), 129–144.

[3] P. Borisut, P. Kumam, V. Gupta and N. Mani, Generalized (ψ,α,β)-weak contractions for initial value problems, Mathematics, 7(2019), 1–14.

[4] S. Chandok and E. Karapinar, Common fixed point of generalized rational type contraction mappings in partially ordered metric spaces, Thai J. Math., 11(2013), 251–260.

[5] N. Chandra, B. Joshi, M. C. Arya and M. C. Joshi, A common fixed point theorem for weakly compatible mappings, Jñānābha, 51(2021), 274–280.

[6] B. S. Choudhury, P. Konar, B. E. Rhoades and N. Metiya, Fixed point theorems for generalized weakly contractive mappings, Nonlinear Anal., 74(2011), 2116–2126.

[7] S. Czerwik, Contraction mappings in b-metric spaces, Acta Math. Inform. Univ. Ostraviensis, 1(1993), 5–11.

[8] D. Dorić, Common fixed point for generalized (ψ, ϕ)-weak contractions, Appl. Math. Lett., 22(2009), 1896–1900.

[9] P. N. Dutta and B. S. Choudhury, A generalisation of contraction principle in metric spaces, Fixed Point Theory Appl., 2008(2008), 1–8.

[10] M. E. Gordji, H. Baghani and G. H. Kim, Common fixed point theorems for (ψ, ϕ)-weak nonlinear contraction in partially ordered sets, Fixed Point Theory Appl., 2012(2012), 1–12.

[11] V. Gupta, N. Mani and J. Jindal, Some new fixed point theorems for generalized weak contraction in partially ordered metric spaces, Comput. Math. Methods, 2(2020), 1–9.

[12] J. Harjani and K. Sadarangani, Fixed point theorems for weakly contractive mappings in partially ordered sets, Nonlinear Anal., 71(2009), 3403–3410.

[13] J. Harjani and K. Sadarangani, Generalized contractions in partially ordered metric spaces and applications to ordinary differential equations, Nonlinear Anal., 72(2010), 1188–1197.

[14] G. Jungck, Commuting mappings and fixed points, Amer. Math. Monthly, 83(1976), 261–263.

[15] G. Jungck, Compatible mappings and common fixed points, Int. J. Math. Math. Sci., 9(1986), 771–779.

[16] G. Jungck and B. E. Rhoades, Fixed points for set valued functions without continuity, Indian J. Pure. Appl. Math., 29(1998), 227–238.

[17] E. Karapinar, I. M. Erhan and U. Aksoy, Weak ψ-contractions on partially ordered metric spaces and applications to boundary value problems, Bound. Value Probl., 1(2014), 1–15.

[18] D. Khantwal, S. Aneja and U. C. Gairola, A fixed point theorem for (ϕ, ψ)-convex contraction in metric spaces, Adv. theory nonlinear anal. appl., 5(2021), 240-245.

[19] S. Lipschutz, Schaum's Outline of Theory and Problems of Set Theory and Related Topics, New York: McGraw-Hill, 1964.

[20] S. G. Matthews, Partial metric topology, Ann. New York Acad. Sci., 728(1994), 183–197.

[21] H. K. Nashine and B. Samet, Fixed point results for mappings satisfying (ψ, ϕ)-weakly contractive condition in partially ordered metric spaces, Nonlinear Anal., 74(2011), 2201–2209.

[22] J. J. Nieto and R. Rodríguez-López, Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations, Order, 22(2005), 223–239.

[23] J. Paul, M. Sajid, N. Chandra and U. C. Gairola, Some common fixed point theorems in bipolar metric spaces and applications, AIMS Math., 8(2023), 19004–19017.

[24] J. Paul and U. C. Gairola, Fixed point theorem in partially ordered metric space for generalized weak contraction mapping satisfying rational type expression, J. Adv. Math. Stud., 16(2023), 57–65.

[25] R. Pant and R. Panicker, Geraghty and Ciric type fixed point theorems in b-metric spaces, J. Nonlinear Sci. Appl., 9(2016), 5741–5755.

[26] O. Popescu, Fixed points for (ψ, ϕ) - weak contractions, Appl. Math. Lett., 24(2011), 1–4.

[27] G. Prasad and R. C. Dimri, Fixed point theorems for weakly contractive mappings in ordered metric spaces with an application, Anal. Theory Appl., 36(2020), 1–11.

[28] S. Radenović and Z. Kadelburg, Generalized weak contractions in partially ordered metric spaces, Comput. Math. Appl., 60(2010), 1776–1783.

[29] A. C. M. Ran and M. C. B. Reurings, A fixed point theorem in partially ordered sets and some applications to matrix equations, Proc. Amer. Math. Soc., 132(2004), 1435–1443.

[30] B. E. Rhoades, Some theorems on weakly contractive maps, Nonlinear Anal., 47(2001), 2683–2693.

[31] S. Shukla, Partial rectangular metric spaces and fixed point theorems, Sci. World J., 2014(2014), 7 pages.

[32] Q. Zhang and Y. Song, Fixed point theory for generalized ϕ-weak contractions, Appl. Math. Lett., 22 (2009), 75–78.