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Asymptotics Of A Sequence Of Improper Integrals*

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Abstract

The asymptotic behavior of a sequence of improper integrals

$$\int_1^\infty f(x^n)\mathrm{d}x, \quad \text{as } n \to \infty$$

is investigated, where $f \in C[1,\infty)$. Using Maclaurin series and Gamma function, we give the asymptotic power series of this sequence of integrals with the assumption that $\lim_{x\to\infty} x^2 f(x)$ exists and finite. And some examples are presented.

1 Introduction

Asymptotic analysis, also known as asymptotics, is a method of describing the behaviour of limits. Asymptotic analysis has a wide range of applications in the fields of number theory, equations, algorithms, and combinatorial mathematics [1].

Suppose that $(x_n)_{n\geq 1}$ is a sequence of real numbers. The power series $\sum_{j=0}^{\infty} c_j n^{-j}$ is called the asymptotic power series of the sequence $(x_n)_{n\geq 1}$ (cf. [2, p. 5, Definition 1]), provided, for every integer $k \geq 0$,

$$x_n = \sum_{j=0}^{k} \frac{c_j}{n^j} + o\left(\frac{1}{n^k}\right), \quad \text{as } n \to \infty.$$

Note that an asymptotic power series $\sum_{j=0}^{\infty} c_j n^{-j}$ does not necessarily converge, but such that taking any initial partial sum provides an asymptotic formula for x_n. The idea is that successive terms provide an increasingly accurate description of the order of growth of x_n. We express the asymptotic power series of x_n in the following notation

$$x_n \sim \sum_{j=0}^{\infty} \frac{c_j}{n^j}, \quad \text{as } n \to \infty.$$

One of the well-known results of asymptotic analysis of integrals is the Riemann-Lebesque lemma [3], which gives the limit of $x_n = \int_a^b f(x)g(nx)\mathrm{d}x$. Andrica and Piticari [4] extended this lemma, and gave the first two terms of asymptotic power series of x_n.

The first two terms of asymptotic power series of $y_n = \int_0^1 f(x^n)\mathrm{d}x$ is obtained by Paltanea [5]. Later, all terms of the asymptotic power series of y_n is obtained by Novac [6] with the assumption that $f \in C[0,1]$ be differentiable at 0.

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The limit of $z_n = \int_0^1 f(x^n)g(x)\mathrm{d}x$ is given by Muşuroia [7] with the assumption that $f, g \in C[0,1]$. Later, the different limit of z_n is presented by Andrica and Piticari [8] with the assumption that $f, g \in C[0,1]$ and $\lim\limits_{x\to 0} \frac{f(x)}{x}$ exists. Recently, the asymptotic power series of z_n is determined by Andrica and Marinescu [9], where f, g satisfy the various conditions.

The limit of the sequence of proper integrals $w_n = \int_1^a f(x^n)\mathrm{d}x$ for any $a > 1$, is provided by Andrica and Piticari [10] with the assumption that $\lim\limits_{x\to\infty} xf(x)$ exists for $f \in C[1,\infty)$. An improper integral $\int_1^\infty f(x^n)\mathrm{d}x$ can be transformed into an integral of the form $\int_0^1 f(x^n)g(x)\mathrm{d}x$. In fact, letting $x = 1/t$, we obtain

$$\int_1^\infty f(x^n)\mathrm{d}x = \int_0^1 \frac{1}{t^2} f\left(\frac{1}{t^n}\right)\mathrm{d}t.$$

Due to the discontinuity of $f\left(\frac{1}{t^n}\right)$ at $t = 0$, it is not possible to use the result of Andrica and Marinescu in [9] for the asymptotic expansion of the improper integral $\int_1^\infty f(x^n)\mathrm{d}x$. The key difficulty is how to extract the parameter n from the integrand function in an asymptotic sequence form. Such an asymptotic expansion intuitively characterizes those improper integrals which vary with n.

The purpose of this paper is to find the asymptotic power series of $\int_1^\infty f(x^n)\mathrm{d}x$.

2 Main Results

Now we present the asymptotic power series of $\int_1^\infty f(x^n)\mathrm{d}x$.

Theorem 1 *Suppose that $f \in C[1,\infty)$, $\lim\limits_{x\to\infty} x^2 f(x)$ exists and it is finite. Then*

$$\int_1^\infty f(x^n)\mathrm{d}x \sim \sum_{j=0}^\infty \frac{c_j}{n^j}, \text{ as } n \to \infty,$$

where $c_0 = 0$, $c_j = \frac{1}{(j-1)!}\int_1^\infty \log^{j-1}(t)\cdot\frac{f(t)}{t}\mathrm{d}t$ for $j = 1, 2, \ldots$.

Proof. Define $h : [1,\infty) \to \mathbb{R}$, $h(x) = x^2 f(x)$. Then $h(x)$ is continuous, bounded and $\lim\limits_{x\to\infty} h(x)$ is finite. And we can find a constant $M > 0$ such that $|h(x)| \le M$ and $|xf(x)| \le M$ for all $x \in [1,\infty)$.

Choose a fixed constant $s > 1$, and put $t = x^n$. Then we have

$$\int_1^s f(x^n)\mathrm{d}x = \frac{1}{n}\int_1^{s^n} t^{\frac{1}{n}}\cdot\frac{f(t)}{t}\mathrm{d}t = \frac{1}{n}\int_1^{s^n} \exp\left(\frac{\log(t)}{n}\right)\cdot\frac{f(t)}{t}\mathrm{d}t.$$

Computing the Maclaurin series with the Lagrange form of the remainder for $\exp\left(\frac{\log(t)}{n}\right)$, then there exists $\theta \in (0,1)$ such that, for $k = 0, 1, \ldots$,

$$\begin{aligned}\int_1^s f(x^n)\mathrm{d}x &= \frac{1}{n}\int_1^{s^n}\left(\sum_{j=0}^k \frac{\log^j(t)}{j!n^j} + \frac{\log^{k+1}(t)}{(k+1)!n^{k+1}}\exp\left(\frac{\theta\log(t)}{n}\right)\right)\cdot\frac{f(t)}{t}\mathrm{d}t\\ &= \sum_{j=0}^k \frac{1}{j!n^{j+1}}\int_1^{s^n}\frac{\log^j(t)f(t)}{t}\mathrm{d}t + \frac{1}{(k+1)!n^{k+2}}\int_1^{s^n}\log^{k+1}(t)\cdot t^{\frac{\theta}{n}}\cdot\frac{f(t)}{t}\mathrm{d}t,\end{aligned}$$

Letting $u = \log(t)$, and by Gamma function [11], we obtain

$$\int_1^\infty \frac{\log^j(t)}{t^2}\mathrm{d}t = \int_0^\infty u^j\cdot\exp(-u)\mathrm{d}u = \Gamma(j+1) = j!.$$

Then

$$\int_1^\infty \frac{\log^j(t)\, f(t)}{t}\mathrm{d}t = \int_1^\infty \frac{\log^j(t)}{t^2}\cdot t\cdot f(t)\mathrm{d}t \le M\int_1^\infty \frac{\log^j(t)}{t^2}\mathrm{d}t = Mj!.$$

Consequently, the improper integral $\int_1^\infty \frac{\log^j(t) f(t)}{t}\mathrm{d}t$ is convergent. Letting $s \to \infty$, one has

$$\int_1^\infty f(x^n)\mathrm{d}x = \sum_{j=0}^{k} \frac{1}{j!n^{j+1}}\int_1^\infty \frac{\log^j(t)\, f(t)}{t}\mathrm{d}t + R_k(n),$$

where the remainder

$$R_k(n) = \frac{1}{(k+1)!n^{k+2}}\int_1^\infty \log^{k+1}(t)\cdot t^{\frac{\theta}{n}}\cdot \frac{f(t)}{t}\mathrm{d}t.$$

Now we should estimate this remainder.

$$\begin{aligned}
|R_k(n)| &= \left|\frac{1}{(k+1)!n^{k+2}}\int_1^\infty \frac{\log^{k+1}(t)}{t^2}\cdot t^{\frac{\theta}{n}-1}\cdot t^2\cdot f(t)\mathrm{d}t\right| \\
&\le \frac{1}{(k+1)!n^{k+2}}\int_1^\infty \left|\frac{\log^{k+1}(t)}{t^2}\cdot t^{\frac{\theta}{n}-1}\cdot h(t)\right|\mathrm{d}t \\
&\le \frac{M}{(k+1)!n^{k+2}}\int_1^\infty \frac{\log^{k+1}(t)}{t^2}\mathrm{d}t \\
&= \frac{M}{n^{k+2}}.
\end{aligned}$$

Thus, we can obtain the asymptotic power series

$$\int_1^\infty f(x^n)\mathrm{d}x = \sum_{j=0}^{k}\frac{c_j}{n^j} + \frac{M}{n^{k+2}}\beta_n,$$

where $c_0 = 0$, $c_j = \frac{1}{(j-1)!}\int_1^\infty \log^{j-1}(t)\cdot\frac{f(t)}{t}\mathrm{d}t$ for $j = 1, 2, \ldots$, and $|\beta_n| \le 1$. ■

3 Examples

In this section we give applications of our main theorem to two specific examples.

Example 1 *Consider the asymptotic power series of*

$$I_n = \int_1^\infty \frac{1}{x^{2n}+2}\mathrm{d}x.$$

Taking $f(x) = \frac{1}{x^2+2}$, $x \in [1, \infty)$, *one can see that* $\lim\limits_{x\to\infty} x^2 f(x) = 1$. *It follows from Theorem 1 that*

$$\int_1^\infty \frac{1}{x^{2n}+2}\mathrm{d}x \sim \sum_{j=0}^{\infty}\frac{c_j}{n^j}, \qquad \text{as } n \to \infty$$

where $c_0 = 0$, $c_j = \frac{1}{(j-1)!}\int_1^\infty \frac{\log^{j-1}(x)}{x(x^2+2)}\mathrm{d}x$ *for* $j = 1, 2, \ldots$.

Example 2 *Consider the asymptotic power series of*

$$J_n = \int_1^\infty \frac{x^n}{x^{3n}+x^{2n}+1}\mathrm{d}x.$$

Taking $f(x) = \frac{x}{x^3+x^2+1}$, *then we have* $\lim\limits_{x\to\infty} x^2 f(x) = 1$. *It follows from Theorem 1 that*

$$\int_1^\infty \frac{x^n}{x^{3n}+x^{2n}+1}\mathrm{d}x \sim \sum_{j=0}^{\infty} \frac{c_j}{n^j}, \qquad \text{as } n\to\infty$$

where $c_0 = 0$, $c_j = \frac{1}{(j-1)!}\int_1^\infty \frac{\log^{j-1}(x)}{x^3+x^2+1}\mathrm{d}x$ *for* $j = 1, 2, \ldots$.

4 Conclusion

Our calculating presents the asymptotic power series of the integral sequence $\int_1^\infty f(x^n)\mathrm{d}x$. This result is a symmetric complement to Novac ([6]) 's of $\int_0^1 f(x^n)\mathrm{d}x$.

Our results are important for those improper integrals that cannot be computed exactly. Our asymptotic power series provides a complete and insightful description of the integrals as the parameter n varies.

In the next work, we will consider the asymptotic power series of the more general sequence of integrals $\int_1^\infty f(x^n)g(x)\mathrm{d}x$.

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