Applied Mathematics E-Notes, 25(2025), 497-503 © ISSN 1607-2510
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/

Power Dominator Coloring Of Line Graph Of Certain Graphs*

Huldah Samuel†, Sathish Kumar Krishnamurthy‡

Received 23 July 2024

Abstract

In the study of the problem of monitoring an electric power system, power domination, a variation on domination in graphs was introduced. The proper coloring of a graph is the assignment of colors to the vertices in which no two vertices receive the same color. The concept of power dominator coloring of graphs has been proposed, based on the concepts of coloring and power domination. A power dominating set is defined as a subset of the vertex set of a graph, wherein the vertices and edges are monitored in accordance with rules of the power dominating monitoring system. For a given graph G, the power dominator chromatic number $\chi_{pd}(G)$ of G is the bare minimum of colors required for power dominator coloring. In this paper, the power dominator chromatic number of different classes of line graphs are obtained.

1 Introduction

Let $G = (V, E)$ be a graph with a finite set of elements, called vertices and a finite set of pairs of vertices, called edges. We consider simple, finite and undirected graphs. A subset $S \subseteq V$ is a dominating set of G if every vertex in $V - S$ has at least one neighbour in S. The concept of domination in graphs has multiple variations [7]. While expressing the problem of monitoring an electric power system in terms of graph theory and minimizing the number of phase measurement units (PMUs) in the system, Haynes et al. [6, 8] developed the concept of power domination. In a graph $G = (V, E)$ depicting an electric power system, a vertex $v \in V(G)$ represents an electrical node, whereas an edge $e \in E(G)$ represents a transmission line that connects two electric nodes. If a set S of vertices monitors every vertex and every edge in the system (according to a set of criteria for power system monitoring), then the set S is said to be a power dominating set of a graph G. The minimum number of vertices required for a power dominating set of a graph G is called the power domination number of G.

Several variations of the concept of graph coloring have been presented and examined by many researchers [1, 10]. A proper coloring of a graph G is a color assignment to a graph's vertices that ensures no two neighboring vertices receive the same color. The chromatic number $\chi(G)$ indicates the minimal number of colors needed to properly color G. The concept of dominator coloring in a graph has been developed and studied, based on the principles of domination and coloring in graphs. A dominator coloring of G is a proper coloring in which each vertex of G dominates every vertex of at least one color class. The chromatic number $\chi(G)$represents the minimum number of colors required to color the vertices of a graph G so that no two neighbouring vertices receive the same color whereas the dominator chromatic number $\chi_d(G)$ represents is the minimum number of colors required for a dominator coloring of the graph [2, 5, 11].

Based on the concepts of coloring and power domination, a new variation of coloring called as power dominator coloring of a graph G has been proposed and investigated [12]. For a vertex v in a graph G, we associate a monitoring set $M(v)$ [13] as follows:

Step (i) $M(v) = N[v]$, the closed neighbourhood of v.

*Mathematics Subject Classifications: 05C15, 05C69.

†Department of Mathematics, Madras Christian College, Affiliated to University of Madras, Chennai 600059, India
‡Department of Mathematics, Madras Christian College, Affiliated to University of Madras, Chennai 600059, India

Step (ii) Add a vertex w to $M(v)$, (which is initially not in $M(v)$) whenever w has a neighbour $u \in M(v)$ such that all the neighbours of u other than w, are already in $M(v)$.

Step (iii) Repeat *Step*(ii) until no more vertices can be added to $M(v)$.

Then we say that v power dominates the vertices in $M(v)$. The power dominator coloring of G is a proper coloring of G in which every vertex in the vertex set V power dominates all vertices in at least one color class. The minimum number of colors required for a power dominator coloring of G is denoted by the power dominator chromatic number $\chi_{\text{pd}}(G)$.

2 Preliminaries

Line graphs [9] form a very basic area of research in graph theory and have been well-investigated. The line graph $L(G)$ of an undirected graph G is a graph such that each vertex of $L(G)$ represents an edge of G and two vertices of $L(G)$ are adjacent if and only if their corresponding edges share a common end vertex in G. Some of the important properties of Line graph are as follows:

- The Line graph of a connected graph is connected.
- The chromatic number of a Line graph $L(G)$ is equal to the edge chromatic number of the graph G.
- If a graph G has an Euler cycle then $L(G)$ is Hamiltonian.

Example 1 *A graph G and its line graph $L(G)$ are shown in Figure 1. The graph G has five edges e_1, e_2, e_3, e_4, e_5 and so the line graph $L(G)$ has five vertices, each one corresponding to an edge. Since the edges e_1, e_2 share a common end vertex at v_2 in G, the vertices e_1, e_2 in $L(G)$ are adjacent in $L(G)$.*

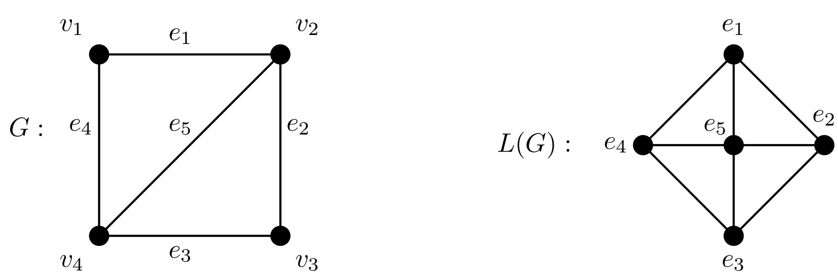

Figure 1: A Graph G and its line graph $L(G)$.

Definition 1 *The wheel graph [1] $W_{1,n}$, $n \geq 3$ is join of the graphs C_n and K_1. That is, $W_{1,n} = C_n + K_1$. Here the vertices corresponding to C_n are called rim vertices and C_n is called rim of $W_{1,n}$ while the vertex corresponding to K_1 is called apex vertex.*

Definition 2 *(i) A bistar graph [4] $B_{n,n}$, $n \geq 2$ is obtained by attaching n pendant edges at each end point of K_2.*

(ii) The graph obtained from the star graph $S_{1,n}$, $n \geq 2$ by attaching a pendant edge to each of the existing n pendant vertices, is called the double star graph and denoted by $S_{1,n,n}$.

Definition 3 *A tadpole graph [14] $T(m,n)$, $m \geq 3$, $n \geq 1$ is the graph obtained by joining a vertex of the cycle C_m, and an end vertex of the path P_n by an edge.*

Definition 4 *The Helm graph [4] H_n with $n \geq 3$, is defined to be the graph obtained from the wheel graph $W_{1,n}$ by attaching a pendant edge at each vertex of the cycle.*

Definition 5 *The n-sunlet graph [15] SL_n, $n \geq 3$, is a graph on $2n$ vertices with a cycle C_n such that each vertex of the cycle is joined to a new pendant vertex.*

Definition 6 ([4]) *The firecracker graph $F_{n,k}$, $n, k \geq 2$ is obtained from n copies $G_1, G_2, \cdots, G_n$ of star graph $S_{1,k}$ by joining a pendant vertex u_i of G_i with a pendant vertex u_{i+1} of G_{i+1} by an edge, for $1 \leq i \leq n-1$.*

We recall some known results.

Theorem 1 ([5]) *Let G be a connected graph. Then*

$$\max\{\chi(G), \gamma(G)\} \leq \chi_d(G) \leq \chi(G) + \gamma(G).$$

Theorem 2 ([5]) *For any graph G, $\chi(G) \leq \chi_d(G)$.*

Theorem 3 ([2]) *Let T be a tree of order $n \geq 2$. Then $\gamma(T) + 1 \leq \chi_d(T) \leq \gamma(T) + 2$.*

Theorem 4 ([12]) *For any graph G, $\chi(G) \leq \chi_{pd}(G) \leq \chi_d(G)$.*

Theorem 5 ([12]) *(i) For a path P_n, $n \geq 2$, $\chi_{pd}(P_n) = 2$.*

(ii) For a cycle C_n, $n \geq 3$,

$$\chi_{pd}(C_n) = \begin{cases} 2 & \text{if } n \text{ is even,} \\ 3 & \text{if } n \text{ is odd.} \end{cases}$$

3 Power Dominator Chromatic Number of Line graph of Certain Graphs

In this section we obtain the power dominator chromatic numbers of line graph of certain classes of graphs.

Theorem 6 *(i) For path P_n, $n \geq 3$, the power dominator chromatic number $\chi_{pd}(L(P_n)) = 2$.*

(ii) For cycle C_n, $n \geq 3$

$$\chi_{pd}(L(C_n)) = \begin{cases} 2 & \text{if } n \text{ is even,} \\ 3 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. (i) For the path P_n on n vertices $n \geq 3$, the line graph is the path P_{n-1} on $n-1$ vertices and $\chi_{pd}(L(P_n)) = \chi_{pd}(P_{n-1}) = 2$ (by Theorem 5).

(ii) For the cycle C_n, $n \geq 3$ with n vertices and n edges, line graph $L(C_n)$ is C_n, so that $\chi_{pd}(L(C_n)) = \chi_{pd}(C_n)$. Hence the required result. ■

Theorem 7 *(i) For the star graph $S_{1,n}$, $n \geq 2$, $\chi_{pd}(L(S_{1,n})) = n$.*

(ii) For the bistar graph $B_{n,n}$, $n \geq 2$, $\chi_{pd}(L(B_{n,n})) = n + 1$.

Proof. (i) For the star graph $S_{1,n}$, $n \geq 2$, the line graph $L(S_{1,n})$ is the complete graph on n vertices. Thus $\chi_{pd}(L(S_{1,n})) = \chi_{pd}(K_n) = n$.

(ii) Let the vertex set of the bistar $B_{n,n}$ be $V = V_1 \cup V_2$, where $V_1 = \{u, v\}$ and $V_2 = \{u_i, v_i \mid 1 \leq i \leq n\}$ such that u and v are adjacent, each u_i is adjacent to u and v_i is adjacent to v, $1 \leq i \leq n$. The line graph of $B_{n,n}$ is the graph $L(B_{n,n})$, each of whose vertices corresponds to an edge in $B_{n,n}$. For $1 \leq i \leq n$, let x_i be the vertex of $L(B_{n,n})$ corresponding to the edge uu_i of $B_{n,n}$. Likewise, for $1 \leq i \leq n$, let y_i be the vertex of $L(B_{n,n})$ corresponding to the edge vv_i of $B_{n,n}$ and z be the vertex of $L(B_{n,n})$ corresponding to

edge uv of $B_{n,n}$. As the edges corresponding to z and x_i, $1 \leq i \leq n$ in $B_{n,n}$ share a common vertex, they form a complete graph of order $n+1$ in $L(B_{n,n})$. Similarly the vertices z and y_i, $1 \leq i \leq n$ in $L(B_{n,n})$ form another complete graph K_{n+1}. Therefore, $L(B_{n,n})$ is obtained by joining two copies of K_{n+1} at a vertex (corresponding to z).
For a proper coloring of $L(B_{n,n})$, assign color i to the vertices x_i and y_i, $1 \leq i \leq n$, and color $n+1$ to the vertex z. Note that this is indeed a power dominator coloring of $L(B_{n,n})$ as the vertices x_i and y_i, $1 \leq i \leq n$, dominate and hence power dominate the color class $\{z\}$. Thus $\chi_{pd}(L(B_{n,n})) = n+1$. ■

Theorem 8 *For the line graph of $S_{1,n,n}$, $n \geq 2$, $\chi_{pd}(L(S_{1,n,n})) = \chi_{pd}(K_n) + 1$.*

Proof. Let $S_{1,n,n}$ be double star graph with vertex set $V = \{v\} \cup V_1 \cup V_2$ where v is the root vertex, $V_1 = \{v_i \mid 1 \leq i \leq n\}$ and $V_2 = \{v'_i \mid 1 \leq i \leq n\}$ and edge set $E = E_1 \cup E_2$ where

$$E_1 = \{x_i = vv_i \mid 1 \leq i \leq n\} \quad \text{and} \quad E_2 = \{y_i = v_iv'_i \mid 1 \leq i \leq n\}.$$

Note that the graph $L(S_{1,n,n})$ has $2n$ vertices and $n(n+1)/2$ edges. In $L(S_{1,n,n})$, the vertices $\{x_i : 1 \leq i \leq n\}$ induce a clique of order n. The vertex y_i is adjacent to the vertex x_i, $1 \leq i \leq n$ in $L(S_{1,n,n})$. For proper coloring, assign color i to x_i, $1 \leq i \leq n$. The vertices y_i, $1 \leq i \leq n$, are colored by $n+1$. Thus $\chi_{pd}(S_{1,n,n}) = n+1 = \chi_{pd}(K_n) + 1$. ■

Example 2 *In Figure 2 a power dominator coloring of the line graph $L(S_{1,5,5})$ of the double star graph $S_{1,5,5}$ is shown.*

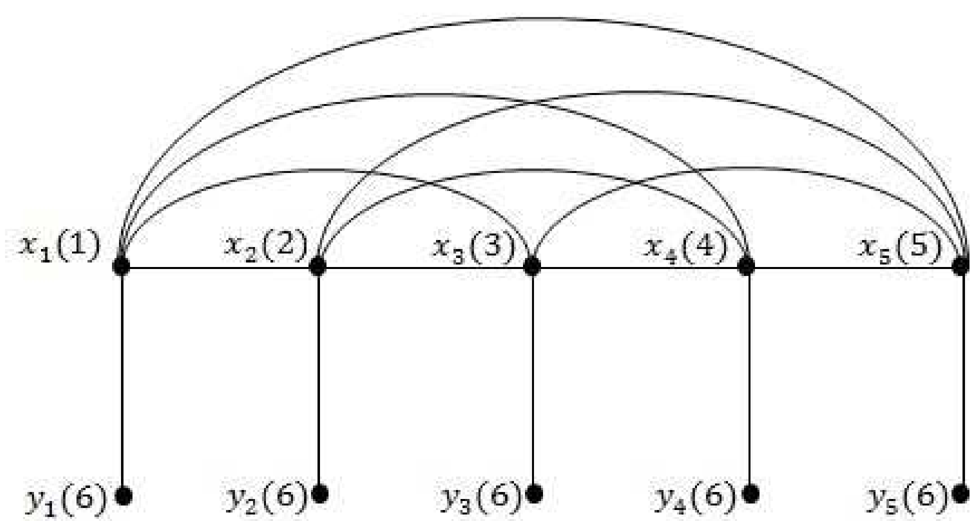

Figure 2: Power dominator coloring of $L(S_{1,5,5})$.

Theorem 9 *Let $T_{m,n}$, $m \geq 3$, $n \geq 1$ be the tadpole graph. Then the power dominator chromatic number of the line graph of $T_{m,n}$ is*

$$\chi_{pd}(L(T_{m,n})) = \begin{cases} 3 & \text{if } m \text{ is even,} \\ 4 & \text{if } m \text{ is odd.} \end{cases}$$

Proof. Let the vertex set and edge set of $T_{m,n}$ be $V(T_{m,n}) = \{v_1, v_2, \cdots, v_m\} \cup \{u_1, u_2, \cdots, u_n\}$ where v_i, $1 \le i \le m$, are on the cycle and u_i, $1 \le i \le n$, are on the path and

$$E(T_{m,n}) = \{x_i = v_i v_{i+1} \mid 1 \le i \le n-1\} \cup \{x_n = v_n v_1\} \cup \{y_j = u_j u_{j+1} \mid 1 \le j \le n-1\} \cup \{z = uv_n\}.$$

The vertex set and edge set of $L(T_{m,n})$ are $V(L(T_{m,n})) = E(T_{m,n})$ and

$$E(L(T_{m,n})) = \{x_i x_{i+1} \mid 1 \le i \le n-1\} \cup \{x_n x_1\} \cup \{y_j y_{j+1} \mid 1 \le j \le n-1\} \cup \{x_n z, x_{n-1} z\}.$$

We assign colors to the vertices in the following way, in order to obtain power dominator coloring of $L(T_{m,n})$. Assign color 1 to the vertex z, color 2 to y_i for odd i, $1 \le i \le n-1$ and color 3 to y_i, for even i, $1 \le i \le n-1$.
Case(i): If m is odd, assign color 2 to each vertex x_i, for odd i, $i \ge 1$ and $i \le m-2$ and color 3 to even i, $1 \le i \le m-1$. The vertex v_m is colored by the color 4. Clearly note that each vertex x_i, $1 \le i \le m-1$ power dominates the color class 4 as well as 1. Also note that the vertices y_i, $1 \le i \le n-1$ power dominate the color class 1. Thus $\chi_{pd}(L(T_{m,n})) = 4$ if m is odd.
Case(ii): If m is even, assign color 2 to each vertex x_i, for odd i, $1 \le i \le m$, and color 3 to even i, $1 \le i \le m$. Hence the vertices x_i, $1 \le i \le m$, and y_i, $1 \le i \le n-1$, dominate and hence power dominate the color class 1. In both the cases , vertex y_i, $1 \le i \le n-1$ power dominates the color class z. Thus $\chi_{pd}(L(T_{m,n})) = 3$ if m is even. ■

Definition 7 ([3]) *A banana tree $B_{n,k}$, $n, k \ge 2$ is obtained from n copies of star graph $S_{1,k}$ by connecting a pendant vertex in each of these n copies with a single root vertex which is distinct from all the vertices in the star graphs.*

Theorem 10 *Let $B_{n,k}$, $n, k \ge 3$ be the banana tree. Then the power dominator chromatic number of its line graph $L(B_{n,k})$ is $n + k$.*

Proof. In the construction of the line graph $L(B_{n,k})$, each of the n-copies of star graph $S_{1,k}$ forms a complete graph K_k of order k and the root vertex of $B_{n,k}$ which is connected to a pendant vertex in each of the n-copies of star $S_{1,k}$ induces a clique of order n. Thus the line graph $L(B_{n,k})$ is obtained by connecting a vertex (which corresponds to the edge joining a pendant vertex in $S_{1,k}$ and the root vertex of $B_{n,k}$) of each of the n-copies of K_k with a distinct vertex of the n-clique induced by the edges at the root vertex. For a proper coloring of $L(B_{n,k})$, assign k distinct colors to the vertices of the n-copies of K_k. The n-clique is colored with distinct additional colors. Thus we require $n+k$ colors (and no less) to yield a power dominator coloring of $L(B_{n,k})$. Hence $\chi_{pd} L(B_{n,k}) = n + k$. ■

Theorem 11 *Let SL_n, $n \ge 3$ be the sunlet graph. Then*

$$\chi_{pd}(L(SL_n)) = \chi_d(L(SL_n)) = \begin{cases} \frac{n}{2} + 2, & \textit{if } n \textit{ is even,} \\ \lceil \frac{n}{2} \rceil + 2, & \textit{if } n \textit{ is odd.} \end{cases}$$

Proof. Let SL_n be the sunlet graph. It is known that [13]

$$\chi_d(L(SL_n)) = \begin{cases} \lceil \frac{n}{2} \rceil + 2, & \text{if } n \text{ is odd,} \\ \frac{n}{2} + 2, & \text{if } n \text{ is even.} \end{cases}$$

and by the definition of power dominator coloring, in this case it can be seen that, $\chi_{pd}(L(SL_n)) = \chi_d(L(SL_n))$. ■

Theorem 12 *For a Helm graph H_n, $n \ge 3$, the power dominator chromatic number of its line graph, $\chi_{pd}(L(H_n)) = n + 3$.*

Proof. Let $V(H_n) = \{v_1\} \cup V_1 \cup V_2$ where v_1 is the apex vertex, $V_1 = \{v_i \mid 2 \leq i \leq n+1\}$ be the rim vertices and $V_2 = \{v_i \mid n+2 \leq i \leq 2n+1\}$ be the pendant vertices incident with the rim vertices such that v_{n+i} is adjacent with v_i, $2 \leq i \leq n+1$ and

$E(H_n) = \{e_i = v_1 v_{i+1} : 1 \leq i \leq n\} \cup \{e'_i = v_{i+1}v_{i+2} : 1 \leq i \leq n-1\} \cup \{e'_{n+1} = v_{n+1 v_2}\} \cup \{f_i = v_{i+1}v_{n+i+1} : 1 \leq i \leq n\}$.

For a proper coloring, assign color i to e_i, $1 \leq i \leq n$, note that these vertices form a complete graph in $L(H_n)$. Here f_i, e'_i are adjacent to e_i. In order to obtain a power dominator coloring, if n-even, assign colors $n+1$ to e'_i, for odd i and $n+2$ to even i and f_i is colored by $n+3$, $1 \leq i \leq n$. If n is odd, assign color $n+1$ to e'_i, for odd i and $n+2$ to even i where $1 \leq i \leq n-1$ and f_i is colored by $n+3$, $1 \leq i \leq n-1$. Then assign $n+3$ color to e'_n and $n+2$ to f_n. Thus we required $n+3$ colors for a power dominator coloring of $L(H_n)$. Hence $\chi_{pd}(L(H_n)) = n+3$. ■

Theorem 13 *For a graph $F_{n,k}$, $n, k \geq 3$ be a graph, the power dominator chromatic number of its line graph* $\chi_{pd}(L(F_{n,k})) = k + \lceil \frac{n}{2} \rceil$.

Proof. Consider the line graph of $F_{n,k}$, $n, k \geq 3$ as in Definition 6. Color the vertices of the part of the line graph corresponding to G_i, $1 \leq i \leq n$, by the distinct colors $1, 2, \cdots, k$. The vertices w_i, $1 \leq i \leq n-1$, in the part of the line graph corresponding to the edges $u_i u_{i+1}$ are colored as follows: color w_1 and w_{n-1} by the new colors $k+1$ and $k+2$ respectively. In the remaining vertices $w_2, \cdots, w_{n-2}$, each of the vertices $w_3, w_5, \cdots$ is given a new color, thus requiring $\lceil \frac{n-4}{2} \rceil$ colors. The remaining w_i's are given the colors from $1, 2, \cdots, k$ to ensure proper coloring. A little reflection will tell that their cannot be any other proper coloring requiring lesser number of colors. Thus

$$\chi_{pd}(L(F_{n,k})) = k + 2 + \lceil \frac{n-4}{2} \rceil = k + \lceil \frac{n}{2} \rceil.$$

■

Acknowledgment. The authors are grateful to the referees for their attentive reading, useful comments, and recommendations to enhance this paper.

References

[1] J. A. Bondy and U. S. R. Murty, Graph Theory with Applications, Elsevier, North Holland, New York, 1986.

[2] M. Chellai and F. Maffray, Dominator colorings in some classes of graphs, Graphs combin., 28 (2012), 97–107.

[3] W. C. Chen, H. I. Lu and Y. N. Yeh, Operations of interlaced trees and graceful trees, Southeast Asian Bull. Math., 21(1997), 337–348.

[4] J. Gallian, A dynamic survey of graph labeling, Electron. J. Combin., 5(1998), 43 pp.

[5] R. M. Gera, On dominator coloring in graphs, Graph Theory Notes N. Y., 52(2007), 25–30.

[6] S. Guze, An application of the selected graph theory domination concepts to transportation networks modelling, Scientific Journals of the Maritime University of Szczecin, 52(2017), 97–102.

[7] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, Fundamentals of Domination in Graphs, Monographs and Textbooks in Pure and Applied Mathematics, 208. Marcel Dekker, Inc., New York, 1998.

[8] T. W. Haynes, S. M. Hedetniemi, S. T. Hedetniemi and M. A. Henning, Domination is graphs applied to electic power networks, SIAM J. Discrete Math., 5(2002), 519–529.

[9] R. L. Hemminger and L. W. Beineke, Line graphs and line digraphs, in: L.W. Beineke and R.J. Wilson, eds., Selected Topics in Graph Theory, (Academic Press, London, New York) (1978), 271–305.

[10] T. R. Jensen and B. Toft, Graph Coloring Problems, Wiley-Interscience, New York, (1995).

[11] K. Kavitha and N. G. David, Dominator and strong dominator coloring on Bistar graph families, International Journal of Applied Mathematics, 14(2012), 444–448.

[12] K. S. Kumar, N. G. David and K. G. Subramanian, Graphs and power dominator colorings, Annals of Pure and Applied Mathematics, 11(2016), 67–71.

[13] K. S. Kumar, N. G. David, A. K. Nagar and K. G. Subramanian, Power dominator chromatic numbers of splitting graphs of certain classes of graph, Commun. Comb. Optim., 9(2024), 317–327.

[14] W. C. Shiu, P. K. Sun and R. M. Low, Integer-antimagic spectra of tadpole and lollipop graphs. Congr. Numer., 225(2015), 5–22.

[15] J. V. Vivin and M. Vekatachalam, On b-chromatic number of Sun let graph and wheel graph families, Journal of the Egyptian Mathematical Society, 23(2015), 215–18.