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Improvements Of Slater's Inequality And Its Applications For Power Means And Divergences*

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Abstract

The study of mathematical inequalities has significantly influenced numerous scientific disciplines due to their wide-ranging applications in science and technology. This highlights their importance in both theoretical and practical contexts, leading to increased attention from the research community. This research work aims to improve Slater's inequality within the continuous framework using techniques from convex analysis applied to twice differentiable functions. To highlight their theoretical importance, the applications of these improvements will be discussed, particularly in the context of power means and divergences. It is expected that these findings will have practical applications and will simulate further research in this area.

1 Introduction

The concept of inequality serves as a foundational pillar in mathematics, with numerous applications across various branches of scientific fields [20, 29, 39, 69, 70]. Without inequalities, solving certain mathematical problems would be much more difficult or even impossible in many cases [36,53,59,66]. Inequalities are also fundamental because they allow us to estimate the range of many problems that require approximation [22,37, 48,49,51]. It has been observed that inequalities and convex functions have a very strong relation [2,19,64,70]. Many inequalities have been successfully derived and improved using the properties and applications of convex functions [3,23,46,65,67,68]. Among the important inequalities in mathematics, the Jensen inequality holds a pivotal position due to its broad applicability and inventive formulation. The impact of the Jensen inequality extends across multiple fields of study, from engineering [24] to information theory [57], economics [33], fractional calculus [18], statistics [35], and epidemiology [58], shaping foundational concepts and applications. Additionally, Jensen's inequality is highly regarded for its role in establishing a wide range of classical inequalities, exploring its fundamental importance in mathematical analysis. Furthermore, one of the most significant aspects of Jensen's inequality is its ability to generalize the definition of convex functions, thereby providing a powerful framework for analyzing and extending properties across mathematical disciplines. Now, we express Jensen's inequality in its standard continuous form:

Theorem 1 *Assume that $\psi : [\alpha_1, \alpha_2] \to \mathbb{R}$ is convex function and $Y_1, Y_2 : I \to [\alpha_1, \alpha_2]$ are integrable functions with $Y_1 \geq 0$ and $T := \int_{\alpha_1}^{\alpha_2} Y_1(\sigma)d\sigma > 0$. If the function $\psi \circ Y_2$ is integrable, then*

$$\psi\left(\frac{\int_{\alpha_1}^{\alpha_2} Y_1(\sigma)Y_2(\sigma)d\sigma}{T}\right) \leq \frac{\int_{\alpha_1}^{\alpha_2} Y_1(\sigma)\psi(Y_2(\sigma))d\sigma}{T}. \tag{1}$$

When ψ is concave, (1) can be observed in the revers sense.

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Many researchers have dedicated significant efforts to explore the wide-ranging applicability and crucial significance of Jensen's inequality, due to its interesting properties and ability to simplify problem-solving [8, 16, 27, 45]. In recent years, new versions of refinements [26, 50], improvements [6, 7], generalizations [28, 38, 56] of Jensen's inequality, as well as extensions and variations of other inequalities using Jensen's inequalities, have been developed using new techniques and principles [12–15]. In 1981, Slater [54] introduced a complementary inequality to Jensen's inequality, widely recognized in the literature as Slater's inequality. The following theorem presents the classic statement of Slater's inequality:

Theorem 2 *Let $\varrho_i \geq 0$ and $\sigma_i \in (\alpha_1, \alpha_2)$ for $i \in \{1, 2, \ldots, n\}$ with $\varrho^* := \sum\limits_{i=1}^{n} \varrho_i > 0$ and $\sum\limits_{i=1}^{n} \varrho_i \psi_+^{'}(\sigma_i) \neq 0$ and also assuming $\psi : (\alpha_1, \alpha_2) \to \mathbb{R}$ is an increasing and convex function. Then*

$$\frac{1}{\varrho^*} \sum_{i=1}^{n} \varrho_i \psi(\sigma_i) \leq \psi\left(\frac{\sum\limits_{i=1}^{n} \varrho_i \sigma_i \psi_+^{'}(\sigma_i)}{\sum\limits_{i=1}^{n} \varrho_i \psi_+^{'}(\sigma_i)} \right). \tag{2}$$

Inequality (2) becomes true in opposite direction, if ψ is concave.

The integral form of the Slater is stated in below theorem.

Theorem 3 *Assume that $Y_1, Y_2 : (a, b) \to (c, d)$ be integrable functions with $Y_1(\sigma) \geq 0$ for $\sigma \in (a, b)$ and $Y := \int_a^b Y_1(\sigma) d\sigma > 0$. Further, let $\psi : (c, d) \to \mathbb{R}$ be an increasing convex function with $\psi \circ Y_2$ and $\psi_+^{'} \circ Y_2$ are integrable. If*

$$Y^* := \int_a^b \psi_+^{'}\big(Y_1(\sigma)\big) d\sigma \neq 0,$$

then

$$\frac{1}{Y} \int_a^b Y_1(\sigma) \psi\big(Y_2(\sigma)\big) d\sigma \leq \psi\left(\frac{\int_a^b Y_1(\sigma) Y_2(\sigma) \psi_+^{'}\big(Y_2(\sigma)\big) d\sigma}{Y^*} \right). \tag{3}$$

For the concave function ψ, inequality (3) holds positive in reverse sense.

Over the past few decades, Slater's inequality has been extensively examined by researchers, resulting in notable progressions including extensions, refinements, improvements, and generalizations. In 1985, Pečarić [47] extended Slater's inequality by introducing a generalization that relaxed the requirement of convexity. This was achieved by imposing a condition on the tuples, ensuring that

$$\frac{1}{\sum\limits_{i=1}^{n} \varrho_i \psi_+^{'}(\sigma_i)} \sum_{i=1}^{n} \varrho_i \sigma_i \psi_+^{'}(\sigma_i) \in (\alpha_1, \alpha_2).$$

In 2000, Matić and Pečarić [44] introduced generalized companion inequalities to Jensen's inequality in discrete settings. They subsequently applied these inequalities to derive Slater's inequality and other related results. In 2017, Delavar and Dragomir [25] extended inequalities of the Slater type to classes of η-convex functions and differential η-convex functions. Additionally, they derived inequalities of the Jensen, Hermite-Hadamard, and Fajer types. In 2022, You et al. [63] utilized a 4-convex function to improve Slater's inequality in both discrete and continuous contexts. Their research extended to establishing connections with power means and providing bounds for divergences, Shannon entropy, Bhattacharyya coefficient, and Zipf-Mandelbrot entropy. Adil Khan et al. [4] developed new estimations for the Slater difference through the join applications of convexity and Jensen's and related inequalities, exploring their applications in domains of power means and information theory. In 2022, Adil Khan et al. [5] developed a method to determine new estimates for the Slater difference using the convexity of a differentiable function in the absolute sense. They also discussed the applications of their findings to derive bounds for divergences and power means.

2 Improvements of Slater's Inequality

This section is dedicated to discussing improvements to the Slater inequality, specifically in the context of continuous Riemannian integrals. The following theorem introduces a refined version of the Slater inequality, developed using concepts from convex analysis and inequalities such as the triangular inequality, the definition of convex functions, and the classical Hölder inequality.

Theorem 4 *Let $Y_1, Y_2 : (\alpha_1, \alpha_2) \to (\beta_1, \beta_2)$ be integrable functions, and $\psi : (\beta_1, \beta_2) \to \mathbb{R}$ be a twice differentiable function. Further, Assuming $T := \int_{\alpha_1}^{\alpha_2} Y_1(\sigma)d\sigma$ and $\int_{\alpha_1}^{\alpha_2} Y_1(\sigma)\psi^{'}(Y_2(\sigma))d\sigma$ are non-zero and for $q > 1$ the function $|\psi''|^q$ is convex. If*

$$\overline{T} := \frac{\int_{\alpha_1}^{\alpha_2} Y_1(\sigma)Y_2(\sigma)\psi^{'}(Y_2(\sigma))d\sigma}{\int_{\alpha_1}^{\alpha_2} Y_1(\sigma)\psi^{'}(Y_2(\sigma))d\sigma} \in (\beta_1, \beta_2),$$

then

$$\begin{aligned}\left|\psi(\overline{T}) - \frac{1}{T}\int_{\alpha_1}^{\alpha_2} Y_1(\sigma)\psi(Y_2(\sigma))d\sigma\right| \quad \leq \quad & \int_{\alpha_1}^{\alpha_2} \left|\frac{Y_1(\sigma)}{T}\right|\Big(Y_2(\sigma) - \overline{T}\Big)^2 \\ & \times\Big(\frac{(q+1)|\psi''(Y_2)|^q + |\psi''(\overline{T})|^q}{(q+1)(q+2)}\Big)^{\frac{1}{q}} d\sigma. \end{aligned} \tag{4}$$

Proof. *Assuming $\overline{T} \neq Y_2(\sigma)$ holds for $\sigma \in (\alpha_1, \alpha_2)$ without loss of generality. We apply integration by parts to obtain:*

$$\begin{aligned} & \int_{\alpha_1}^{\alpha_2} \left|\frac{Y_1(\sigma)}{T}\right|\Big(Y_2(\sigma) - \overline{T}\Big)^2 \int_0^1 t\psi^{''}\Big(tY_2(\sigma) + (1-t)\overline{T}\Big)dtdy \\ = \quad & \frac{1}{T}\int_{\alpha_1}^{\alpha_2} Y_1(\sigma)\Big(Y_2(\sigma) - \overline{T}\Big)^2 \\ & \times\Big(\frac{t\psi^{'}(tY_2(\sigma) + (1-t)\overline{T})}{Y_2(\sigma) - \overline{T}}\Big|_0^1 - \int_0^1 \frac{\psi^{'}(tY_2(\sigma) + (1-t)\overline{T})}{Y_2(\sigma) - \overline{T}}dt\Big)d\sigma \\ = \quad & \frac{1}{T}\int_{\alpha_1}^{\alpha_2} Y_1(\sigma)\Big(Y_2(\sigma) - \overline{T}\Big)^2\Big(\frac{\psi^{'}(Y_2(\sigma)}{Y_2(\sigma) - \overline{T}} - \frac{\psi(tY_2(\sigma) + (1-t)\overline{T})}{(Y_2(\sigma) - \overline{T})^2}\Big|_0^1\Big)d\sigma \\ = \quad & \psi(\overline{T}) - \frac{1}{T}\int_{\alpha_1}^{\alpha_2} Y_1(\sigma)\psi(Y_2(\sigma))d\sigma. \end{aligned} \tag{5}$$

We can rewrite identity (5) as:

$$\begin{aligned} & \psi(\overline{T}) - \frac{1}{T}\int_{\alpha_1}^{\alpha_2} Y_1(\sigma)\psi(Y_2(\sigma))d\sigma \\ = \quad & \frac{1}{T}\int_{\alpha_1}^{\alpha_2} \frac{Y_1(\sigma)}{T}\Big(Y_2(\sigma) - \overline{T}\Big)^2 \int_0^1 t\psi^{''}\Big(tY_2(\sigma) + (1-t)\overline{T}\Big)dtd\sigma. \end{aligned} \tag{6}$$

We deduce the below inequality by applying the triangle inequality after taking the absolute value of (6):

$$\begin{aligned} & \left|\psi(\overline{T}) - \frac{1}{T}\int_{\alpha_1}^{\alpha_2} Y_1(\sigma)\psi(Y_2(\sigma))d\sigma\right| \\ \leq \quad & \int_{\alpha_1}^{\alpha_2} \left|\frac{Y_1(\sigma)}{T}\right|\Big(Y_2(\sigma) - \overline{T}\Big)^2 \int_0^1 \left|t\psi^{''}\Big(tY_2(\sigma) + (1-t)\overline{T}\Big)\right|dtd\sigma. \end{aligned} \tag{7}$$

By employing the well-known Hölder inequality in (7), we achieve:

$$\begin{aligned}
&\Big|\psi(\overline{T}) - \frac{1}{T}\int_{\alpha_1}^{\alpha_2} Y_1(\sigma)\psi(Y_2(\sigma))d\sigma\Big| \\
\leq\ & \int_{\alpha_1}^{\alpha_2} \Big|\frac{Y_1(\sigma)}{T}\Big|\Big(Y_2(\sigma)-\overline{T}\Big)^2\Big(\int_0^1 \Big|t\psi^{''}\Big(tY_2(\sigma)+(1-t)\overline{T}\Big)\Big|^q dt\Big)^{\frac{1}{q}} d\sigma. \qquad (8)
\end{aligned}$$

Applying convexity of the function $|\psi''|^q$ to the right side of (8), we acquire (4). ■

In next theorem, we obtain an improvement of the Slater inequality by applying the Hölder inequality and the famous Jensen inequality for concave functions.

Theorem 5 *Suppose that $\psi : (\beta_1, \beta_2) \to \mathbb{R}$ is any function such that ψ'' exists, and the functions $Y_1, Y_2(\alpha_1, \alpha_2) \to (\beta_1, \beta_2)$ are integrable and for $q > 1$, the function $|\psi''|^q$ is concave. Also, let $T := \int_{\alpha_1}^{\alpha_2} Y_1(\sigma)d\sigma \neq 0$ and $\int_{\alpha_1}^{\alpha_2} Y_1(\sigma)\psi^{'}(Y_2(\sigma))d\sigma \neq 0$ and*

$$\overline{T} := \frac{\int_{\alpha_1}^{\alpha_2} Y_1(\sigma)Y_2(\sigma)\psi^{'}(Y_2(\sigma))d\sigma}{\int_{\alpha_1}^{\alpha_2} Y_1(\sigma)\psi^{'}(Y_2(\sigma))d\sigma} \in (\beta_1, \beta_2).$$

Then

$$\begin{aligned}
&\Big|\psi(\overline{T}) - \frac{1}{T}\int_{\alpha_1}^{\alpha_2} Y_1(\sigma)\psi(Y_2(\sigma))d\sigma\Big| \\
\leq\ & \Big(\frac{1}{q+1}\Big)^{\frac{1}{q}} \int_{\alpha_1}^{\alpha_2} \Big|\frac{Y_1(\sigma)}{T}\Big|\Big(Y_2(\sigma)-\overline{T}\Big)^2\Big|\psi''\Big(\frac{(q+1)Y_2(\sigma)+\overline{T}}{q+2}\Big)\Big| d\sigma. \qquad (9)
\end{aligned}$$

Proof. *From inequality (8), we can formulate that,*

$$\begin{aligned}
\Big|\psi(\overline{T}) - \frac{1}{T}\int_{\alpha_1}^{\alpha_2} Y_1(\sigma)\psi(Y_2(\sigma))d\sigma\Big| \leq\ & \Big(\frac{1}{q+1}\Big)^{\frac{1}{q}} \int_{\alpha_1}^{\alpha_2} \Big|\frac{Y_1(\sigma)}{T}\Big|\Big(Y_2(\sigma)-\overline{T}\Big)^2 \\
& \times\Big(\frac{\int_0^1 \Big|t\psi^{''}\Big(tY_2(\sigma)+(1-t)\overline{T}\Big)\Big|^q dt}{\int_0^1 t^q dt}\Big)^{\frac{1}{q}} d\sigma. \qquad (10)
\end{aligned}$$

Utilize Jensen's inequality in (10), we receive

$$\begin{aligned}
\Big|\psi(\overline{T}) - \frac{1}{T}\int_{\alpha_1}^{\alpha_2} Y_1(\sigma)\psi(Y_2(\sigma))d\sigma\Big| \leq\ & \Big(\frac{1}{q+1}\Big)^{\frac{1}{q}} \int_{\alpha_1}^{\alpha_2} \Big|\frac{Y_1(\sigma)}{T}\Big|\Big(Y_2(\sigma)-\overline{T}\Big)^2 \\
& \times\Big(\Big|\psi''\Big(\frac{\Big|\int_0^1 t\big(tY_2(\sigma)+(1-t)\overline{T}\big)dt}{\int_0^1 t^q dt}\Big)\Big|^q\Big)^{\frac{1}{q}} d\sigma. \qquad (11)
\end{aligned}$$

By simplifying (11), we deduce (9). ■

The following theorem gives an improvement for the Slater inequality which has been proved by working with the Hölder inequality and definition of convex function.

Theorem 6 *Assume that, all the conditions of Theorem 4 are fulfilled. Moreover, if $p > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\begin{aligned}
\Big|\psi(\overline{T}) - \frac{1}{T}\int_{\alpha_1}^{\alpha_2} Y_1(\sigma)\psi(Y_2(\sigma))d\sigma\Big| \leq\ & \Big(\frac{1}{p+1}\Big)^{\frac{1}{p}} \int_{\alpha_1}^{\alpha_2} \Big|\frac{Y_1(\sigma)}{T}\Big|\Big(Y_2(\sigma)-\overline{T}\Big)^2 \\
& \times\Big(\frac{|\psi''(Y_2)|^q + |\psi''(\overline{T})|^q}{2}\Big)^{\frac{1}{q}} d\sigma. \qquad (12)
\end{aligned}$$

Proof. *Inequality can also be written in the following form,*

$$\begin{aligned}\left|\psi(\overline{T})-\frac{1}{T}\int_{\alpha_1}^{\alpha_2}Y_1(\sigma)\psi(Y_2(\sigma))d\sigma\right| &\leq \int_{\alpha_1}^{\alpha_2}\left|\frac{Y_1(\sigma)}{T}\right|\Big(Y_2(\sigma)-\overline{T}\Big)^2\\ &\quad\times\int_0^1 t\left|\psi''\Big(tY_2(\sigma)+(1-t)\overline{T}\Big)\right|dtd\sigma. \quad (13)\end{aligned}$$

Applying the Hölder inequality on the right side of (7), we arrive

$$\begin{aligned}\left|\psi(\overline{T})-\frac{1}{T}\int_{\alpha_1}^{\alpha_2}Y_1(\sigma)\psi(Y_2(\sigma))d\sigma\right| &\leq \Big(\frac{1}{p+1}\Big)^{\frac{1}{p}}\int_{\alpha_1}^{\alpha_2}\left|\frac{Y_1(\sigma)}{T}\right|\Big(Y_2(\sigma)-\overline{T}\Big)^2\\ &\quad\times\bigg(\int_0^1\left|\psi''\Big(tY_2(\sigma)+(1-t)\overline{T}\Big)\right|^q dt\bigg)^{\frac{1}{q}}d\sigma. \quad (14)\end{aligned}$$

Now, using the convexity of $|\psi''|^q$ *on the right side of* (14), *we receive*

$$\begin{aligned}\left|\psi(\overline{T})-\frac{1}{T}\int_{\alpha_1}^{\alpha_2}Y_1(\sigma)\psi(Y_2(\sigma))d\sigma\right| &\leq \Big(\frac{1}{p+1}\Big)^{\frac{1}{p}}\int_{\alpha_1}^{\alpha_2}\left|\frac{Y_1(\sigma)}{T}\right|\Big(Y_2(\sigma)-\overline{T}\Big)^2\\ &\quad\times\bigg(\int_0^1\Big(t\big|\psi''(Y_2(\sigma)\big|^q+(1-t)\big|(\overline{T})\big|^q\Big)dt\bigg)^{\frac{1}{q}}d\sigma. \quad (15)\end{aligned}$$

Simplifying (15), we acquire

$$\begin{aligned}\left|\psi(\overline{T})-\frac{1}{T}\int_{\alpha_1}^{\alpha_2}Y_1(\sigma)\psi(Y_2(\sigma))d\sigma\right| &\leq \Big(\frac{1}{p+1}\Big)^{\frac{1}{p}}\int_{\alpha_1}^{\alpha_2}\left|\frac{Y_1(\sigma)}{T}\right|\Big(Y_2(\sigma)-\overline{T}\Big)^2\\ &\quad\times\bigg(\big|\psi''(Y_2(\sigma)\big|^q\int_0^1 tdt+\big|(\overline{T})\big|^q\Big)\int_0^1(1-t)dt\bigg)^{\frac{1}{q}}d\sigma. \quad (16)\end{aligned}$$

Since, $\int_0^1 tdt=\frac{1}{2}$ *and* $\int_0^1(1-t)dt=\frac{1}{2}$. *Therefore (16) is equivalent (12).* ■

The below theorem provides another improvement for the Slater inequality, which easily be deduced by using the Hölder inequality and definition of convex functions.

Theorem 7 *Let the hypotheses of Theorem 5 be true. Furthermore, if* $\frac{1}{p}+\frac{1}{q}=1$ *for* $p>1$, *then*

$$\begin{aligned}&\left|\psi(\overline{T})-\frac{1}{T}\int_{\alpha_1}^{\alpha_2}Y_1(\sigma)\psi(Y_2(\sigma))d\sigma\right|\\ &\leq \Big(\frac{1}{p+1}\Big)^{\frac{1}{p}}\int_{\alpha_1}^{\alpha_2}\left|\frac{Y_1(\sigma)}{T}\right|\Big(Y_2(\sigma)-\overline{T}\Big)^2\left|\psi''\Big(\frac{Y_2(\sigma)+\overline{T}}{2}\Big)\right|d\sigma. \quad (17)\end{aligned}$$

Proof. *Utilizing Hölder inequality on the right side of (7), we get*

$$\begin{aligned}\left|\psi(\overline{T})-\frac{1}{T}\int_{\alpha_1}^{\alpha_2}Y_1(\sigma)\psi(Y_2(\sigma))d\sigma\right| &\leq \Big(\frac{1}{p+1}\Big)^{\frac{1}{p}}\int_{\alpha_1}^{\alpha_2}\left|\frac{Y_1(\sigma)}{T}\right|\Big(Y_2(\sigma)-\overline{T}\Big)^2\\ &\quad\times\left(\frac{\int_0^1\left|\psi''\Big(tY_2(\sigma)+(1-t)\overline{T}\Big)\right|^q dt}{\int_0^1 dt}\right)^{\frac{1}{q}}d\sigma. \quad (18)\end{aligned}$$

Instantly, applying Jensen's inequality to (18), we receive

$$\begin{aligned}\left|\psi(\overline{T})-\frac{1}{T}\int_{\alpha_1}^{\alpha_2}Y_1(\sigma)\psi(Y_2(\sigma))d\sigma\right| &\leq \Big(\frac{1}{p+1}\Big)^{\frac{1}{p}}\int_{\alpha_1}^{\alpha_2}\left|\frac{Y_1(\sigma)}{T}\right|\Big(Y_2(\sigma)-\overline{T}\Big)^2\\ &\quad\times\left|\psi''\bigg(\int_0^1\Big(tY_2(\sigma)+(1-t)\overline{T}\Big)dt\bigg)\right|. \quad (19)\end{aligned}$$

The inequality (17) can easily be deduced by just finding the integrals in (19). ■

We utilized the power mean inequality and definition of convex function to present an improvement for the Slater inequality.

Theorem 8 *Suppose that, the assumptions of Theorem 4 are valid, then*

$$\Big|\psi(\overline{T})-\frac{1}{T}\int_{\alpha_1}^{\alpha_2}Y_1(\sigma)\psi(Y_2(\sigma))d\sigma\Big| \leq \Big(\frac{1}{2}\Big)^{1-\frac{1}{p}}\int_{\alpha_1}^{\alpha_2}\Big|\frac{Y_1(\sigma)}{T}\Big|\Big(Y_2(\sigma)-\overline{T}\Big)^2 \times\Big(\frac{2|\psi''(Y_2)|^q+|\psi''(\overline{T})|^q}{6}\Big)^{\frac{1}{q}}d\sigma. \tag{20}$$

Proof. *By exploiting the power mean inequality on the right side of (7), we acquire*

$$\Big|\psi(\overline{T})-\frac{1}{T}\int_{\alpha_1}^{\alpha_2}Y_1(\sigma)\psi(Y_2(\sigma))d\sigma\Big| \leq \Big(\frac{1}{2}\Big)^{1-\frac{1}{p}}\int_{\alpha_1}^{\alpha_2}\Big|\frac{Y_1(\sigma)}{T}\Big|\Big(Y_2(\sigma)-\overline{T}\Big)^2 \times\Big(\int_0^1 t\Big|\psi''\Big(tY_2(\sigma)+(1-t)\overline{T}\Big)\Big|^q dt\Big)^{\frac{1}{q}}d\sigma. \tag{21}$$

Recently, applying the definition of convex function in (21), we obtain

$$\Big|\psi(\overline{T})-\frac{1}{T}\int_{\alpha_1}^{\alpha_2}Y_1(\sigma)\psi(Y_2(\sigma))d\sigma\Big| \leq \Big(\frac{1}{2}\Big)^{1-\frac{1}{p}}\int_{\alpha_1}^{\alpha_2}\Big|\frac{Y_1(\sigma)}{T}\Big|\Big(Y_2(\sigma)-\overline{T}\Big)^2 \times\Big(\int_0^1 t\Big(t|\psi''(Y_2(\sigma))|^q+(1-t)|\psi''(\overline{T})|^q\Big)dt\Big)^{\frac{1}{q}}d\sigma. \tag{22}$$

Now, simplifying (22), we catch (20). ■

In the coming theorem, we propose an improvement of the Slater inequality by consuming the power mean inequality and the Jensen inequality for concave functions.

Theorem 9 *Let all the conditions of Theorem 5 be true. Then*

$$\Big|\psi(\overline{T})-\frac{1}{T}\int_{\alpha_1}^{\alpha_2}Y_1(\sigma)\psi(Y_2(\sigma))d\sigma\Big| \leq \frac{1}{2}\int_{\alpha_1}^{\alpha_2}\Big|\frac{Y_1(\sigma)}{T}\Big|\Big(Y_2(\sigma)-\overline{T}\Big)^2\times\Big|\psi''\Big(\frac{Y_2(\sigma)+\overline{T}}{3}\Big)\Big|. \tag{23}$$

Proof. *Using (21) we can write that,*

$$\Big|\psi(\overline{T})-\frac{1}{T}\int_{\alpha_1}^{\alpha_2}Y_1(\sigma)\psi(Y_2(\sigma))d\sigma\Big| \leq \frac{1}{2}\int_{\alpha_1}^{\alpha_2}\Big|\frac{Y_1(\sigma)}{T}\Big|\Big(Y_2(\sigma)-\overline{T}\Big)^2 \times\Big(\frac{\int_0^1 t\Big|\psi''\Big(tY_2(\sigma)+(1-t)\overline{T}\Big)\Big|^q dt}{\int_0^1 tdt}\Big)^{\frac{1}{q}}d\sigma. \tag{24}$$

By utilizing the Jensen inequality on the right side of (24), we deduce

$$\Big|\psi(\overline{T})-\frac{1}{T}\int_{\alpha_1}^{\alpha_2}Y_1(\sigma)\psi(Y_2(\sigma))d\sigma\Big| \leq \Big(\frac{1}{p+1}\Big)^{\frac{1}{p}}\int_{\alpha_1}^{\alpha_2}\Big|\frac{Y_1(\sigma)}{T}\Big|\Big(Y_2(\sigma)-\overline{T}\Big)^2 \times\Big|\psi''\Big(\frac{\int_0^1\Big(tY_2(\sigma)+(1-t)\overline{T}\Big)dt}{\int_0^1 tdt}\Big)\Big|. \tag{25}$$

Evaluating integrals in (25), we obtain (23). ■

3 Applications for the Means

The power means is one of the most consequential, noteworthy and valuable tools for solving a variety of problems [21,42,62]. One of the important fact about the power means is that, it has an estimable structure, and due this circumstance, power means have a very dominant place in many fields of science [11,34,61]. There are exist a lot of results in the literature that have found by numerous researchers of different areas [10,55,60]. In recent decades, the already established have been refined, improved, extended and generalized in many ways according to the interest [56,57,63]. Due to the great importance and wide significance of power means, we have also show an interest to provide some results for this means.

In this section, we establish a number of relations for the power means with the help of main results. The desired relations will give different estimations for the power means. To forward ahead, first we recall the definition of power mean.

Definition 1 *Let $Y_1, Y_2 : (\alpha_1, \alpha_2) \to \mathbb{R}^+$ be integrable functions with $T := \int_{\alpha_1}^{\alpha_2} Y_1(\sigma)d\sigma$. Then, the power mean of order $r \in \mathbb{R}$ is given by:*

$$M_r(Y_1, Y_2) = \begin{cases} \left(\frac{\int_{\alpha_1}^{\alpha_2} Y_1(\sigma)Y_2^r(\sigma)d\sigma}{T}\right)^{\frac{1}{r}}, & r \neq 0, \\ \exp\left(\frac{\int_{\alpha_1}^{\alpha_2} Y_1(\sigma)\log Y_2(\sigma)d\sigma}{T}\right), & r = 0. \end{cases}$$

The first consequence of Theorem 4 for power means is given in the next corollary.

Corollary 1 *Assume that $g_1, g_2 : (\alpha_1, \alpha_2) \to (0, \infty)$ are integrable functions with $\overline{g} := \int_{\alpha_1}^{\alpha_2} g_1(\sigma)d\sigma$ and $q > 1$. Further, let r and t be non-zero real numbers such that $t \leq r$.*

(i) If r and t are positive, then

$$\begin{aligned} & M_t^t(g_1, g_2) - \left(\frac{\overline{g}M_t^t(g_1, g_2)}{\int_{\alpha_1}^{\alpha_2} g_1(\sigma)g_2^{t-r}(\sigma)d\sigma}\right)^{\frac{t}{r}} \\ \leq\ & \frac{t(r-t)}{\overline{g}r^2}\int_{\alpha_1}^{\alpha_2} g_1(\sigma)\left(g_2^r(\sigma) - \frac{\overline{g}M_t^t(g_1, g_2)}{\int_{\alpha_1}^{\alpha_2} g_1(\sigma)g_2^{t-r}(\sigma)d\sigma}\right)^2 \\ & \times\left(\frac{(q+1)g_2^{q(t-2r)}(\sigma) + \left(\frac{\overline{g}M_t^t(g_1,g_2)}{\int_{\alpha_1}^{\alpha_2} g_1(\sigma)g_2^{t-r}(\sigma)d\sigma}\right)^{q(\frac{t}{r}-2)}}{(q+1)(q+2)}\right)^{\frac{1}{q}} d\sigma. \end{aligned} \tag{26}$$

(ii) f r and t are negative with $\frac{t}{r} \notin \left(2, 2+\frac{1}{q}\right)$, then

$$\begin{aligned} & \left(\frac{\overline{g}M_t^t(g_1, g_2)}{\int_{\alpha_1}^{\alpha_2} g_1(\sigma)g_2^{t-r}(\sigma)d\sigma}\right)^{\frac{t}{r}} - M_t^t(g_1, g_2) \\ \leq\ & \frac{t(t-r)}{\overline{g}r^2}\int_{\alpha_1}^{\alpha_2} g_1(\sigma)\left(g_2^r(\sigma) - \frac{\overline{g}M_t^t(g_1, g_2)}{\int_{\alpha_1}^{\alpha_2} g_1(\sigma)g_2^{t-r}(\sigma)d\sigma}\right)^2 \\ & \times\left(\frac{(q+1)g_2^{q(t-2r)}(\sigma) + \left(\frac{\overline{g}M_t^t(g_1,g_2)}{\int_{\alpha_1}^{\alpha_2} g_1(\sigma)g_2^{t-r}(\sigma)d\sigma}\right)^{q(\frac{t}{r}-2)}}{(q+1)(q+2)}\right)^{\frac{1}{q}} d\sigma. \end{aligned} \tag{27}$$

(iii) If r is positive and t is negative, then (27) holds.

Proof. *(i) To prove inequality (26), consider $\psi(\sigma) = \sigma^{\frac{t}{r}}$, $\sigma > 0$. Then certainly*

$$\psi''(\sigma) = \frac{t}{r}\big(\frac{t}{r} - 1\big)\sigma^{\frac{t}{r}-2}$$

and

$$\big(|\psi''(\sigma)|^q\big)'' = \big|\frac{t}{r}\big(\frac{t}{r} - 1\big)\big|^q q^2\big(\frac{t}{r} - 1\big)\big(\big(\frac{t}{r} - 1\big) - 1\big)\sigma^{q\left(\frac{t}{r}\right)-2}.$$

Clearly, $\psi''(\sigma)$ is negative and $\big(|\psi''(\sigma)|^q\big)''$ is positive for the specified conditions, which conclude that $\psi(\sigma)$ is concave and $|\psi''(\sigma)|^q$ is convex. Therefore, to acquire (26), just put $\psi(\sigma) = \sigma^{\frac{t}{r}}$, $Y_1(\sigma) = g_1(\sigma)$, and $Y_2(\sigma) = g_2^r(\sigma)$ in (4).

(ii)–(iii) For the mentioned values of r, t, and q in cases (ii) and (iii) respectively, both the functions $\psi(\sigma)$ and $|\psi''(\sigma)|^q$ are convex. Therefore, inequality (27) can easily be deduced by following the procedure of (i). ■

We obtain a relation for the power means with the help of Theorem 5, which is stated in the below corollary.

Corollary 2 *Assume that $g_1, g_2 : (\alpha_1, \alpha_2) \to (0, \infty)$ are integrable functions with $\overline{g} := \int_{\alpha_1}^{\alpha_2} g_1(\sigma)d\sigma$ and $q > 1$. Further, let r and t be non-zero real numbers such that $t \le r$ and $\frac{t}{r} \in \big(2, 2 + \frac{1}{q}\big)$, Then*

$$\begin{aligned}
&\left(\frac{\overline{g}M_t^t(g_1, g_2)}{\int_{\alpha_1}^{\alpha_2} g_1(\sigma)g_2^{t-r}(\sigma)d\sigma}\right)^{\frac{t}{r}} - M_t^t(g_1, g_2)\\
\le\ &\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\frac{t(t-r)}{\overline{g}r^2}\int_{\alpha_1}^{\alpha_2} g_1(\sigma)\left(g_2^r(\sigma) - \frac{\overline{g}M_t^t(g_1, g_2)}{\int_{\alpha_1}^{\alpha_2} g_1(\sigma)g_2^{t-r}(\sigma)d\sigma}\right)^2\\
&\times\left(\frac{(q+1)g_2^r(\sigma) + \frac{\overline{g}M_t^t(g_1,g_2)}{\int_{\alpha_1}^{\alpha_2} g_1(\sigma)g_2^{t-r}(\sigma)d\sigma}}{q+2}\right)^{\frac{t}{r}-2} d\sigma. \qquad (28)
\end{aligned}$$

Proof. *Consider $\psi(\sigma) = \sigma^{\frac{t}{r}}$ defined on $(0, \infty)$. Then, obviously $\psi(\sigma)$ is convex and $|\psi''(\sigma)|^q$ is concave. Therefore, use $\psi(\sigma) = \sigma^{\frac{t}{r}}$, $Y_1(\sigma) = g_1(\sigma)$, and $Y_2(\sigma) = g_2^r(\sigma)$ in (9), we receive (28).* ■

Some more relations for the power are stated in coming corollary which have been deduced from Theorem 6.

Corollary 3 *Let r and t be non-zero real numbers with $t \le r$ and $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Also, assume that $g_1, g_2 : (\alpha_1, \alpha_2) \to (0, \infty)$ are integrable functions with $\overline{g} := \int_{\alpha_1}^{\alpha_2} g_1(\sigma)d\sigma$.*

(i) If r and t are positive, then

$$\begin{aligned}
&M_t^t(g_1, g_2) - \left(\frac{\overline{g}M_t^t(g_1, g_2)}{\int_{\alpha_1}^{\alpha_2} g_1(\sigma)g_2^{t-r}(\sigma)d\sigma}\right)^{\frac{t}{r}}\\
\le\ &\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\frac{t(r-t)}{\overline{g}r^2}\int_{\alpha_1}^{\alpha_2} g_1(\sigma)\left(g_2^r(\sigma) - \frac{\overline{g}M_t^t(g_1, g_2)}{\int_{\alpha_1}^{\alpha_2} g_1(\sigma)g_2^{t-r}(\sigma)d\sigma}\right)^2\\
&\times\left(\frac{g_2^{q(t-2r)}(\sigma) + \left(\frac{\overline{g}M_t^t(g_1,g_2)}{\int_{\alpha_1}^{\alpha_2} g_1(\sigma)g_2^{t-r}(\sigma)d\sigma}\right)^{q\left(\frac{t}{r}-2\right)}}{2}\right)^{\frac{1}{q}} d\sigma. \qquad (29)
\end{aligned}$$

(ii) If r and t are negative with $\frac{t}{r} \notin \left(2, 2+\frac{1}{q}\right)$, then

$$\begin{aligned}
&\left(\frac{\overline{g} M_t^t(g_1, g_2)}{\int_{\alpha_1}^{\alpha_2} g_1(\sigma) g_2^{t-r}(\sigma) d\sigma}\right)^{\frac{t}{r}} - M_t^t(g_1, g_2) \\
\leq\ & \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \frac{t(t-r)}{\overline{g} r^2} \int_{\alpha_1}^{\alpha_2} g_1(\sigma)\left(g_2^r(\sigma) - \frac{\overline{g} M_t^t(g_1, g_2)}{\int_{\alpha_1}^{\alpha_2} g_1(\sigma) g_2^{t-r}(\sigma) d\sigma}\right)^2 \\
&\times\left(\frac{g_2^{q(t-2r)}(\sigma) + \left(\frac{\overline{g} M_t^t(g_1, g_2)}{\int_{\alpha_1}^{\alpha_2} g_1(\sigma) g_2^{t-r}(\sigma) d\sigma}\right)^{q\left(\frac{t}{r}-2\right)}}{2}\right)^{\frac{1}{q}} d\sigma. \qquad (30)
\end{aligned}$$

(iii) If r is positive and t is negative, then (30) holds.

Proof. *(i) Inequality (29) can certainly be achieved by taking $\psi(\sigma) = \sigma^{\frac{t}{r}}$, $\sigma > 0$, $Y_1(\sigma) = g_1(\sigma)$, and $Y_2(\sigma) = g_2^r(\sigma)$ in (12).*

(ii)–(iii) For inequality (30) follows the method of (i). ■

Another relation is obtained from Theorem 7 for the power means which is given in the following theorem.

Corollary 4 *Presume that, the suppositions of Corollary 2 are valid. Moreover, if $p > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\begin{aligned}
&\left(\frac{\overline{g} M_t^t(g_1, g_2)}{\int_{\alpha_1}^{\alpha_2} g_1(\sigma) g_2^{t-r}(\sigma) d\sigma}\right)^{\frac{t}{r}} - M_t^t(g_1, g_2) \\
\leq\ & \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \frac{t(t-r)}{\overline{g} r^2} \int_{\alpha_1}^{\alpha_2} g_1(\sigma)\left(g_2^r(\sigma) - \frac{\overline{g} M_t^t(g_1, g_2)}{\int_{\alpha_1}^{\alpha_2} g_1(\sigma) g_2^{t-r}(\sigma) d\sigma}\right)^2 \\
&\times\left(\frac{g_2^r(\sigma) + \frac{\overline{g} M_t^t(g_1, g_2)}{\int_{\alpha_1}^{\alpha_2} g_1(\sigma) g_2^{t-r}(\sigma) d\sigma}}{2}\right)^{\frac{t}{r}-2} d\sigma. \qquad (31)
\end{aligned}$$

Proof. *Putting $\psi(\sigma) = \sigma^{\frac{t}{r}}$, $\sigma \in (0, \infty)$, $Y_1(\sigma) = g_1(\sigma)$, and $Y_2(\sigma) = g_2^r(\sigma)$ in (17), we obtain (31).* ■

By utilizing Theorem 8, we establish some relations for the power means highlighted in the coming corollary.

Corollary 5 *Let the positive valued functions g_1 and g_2 be integrable over (α_1, α_2) with $\overline{g} := \int_{\alpha_1}^{\alpha_2} g_1(\sigma) d\sigma$. Also, suppose that r and t are non-zero real numbers such that $t \leq r$. Assume that $q > 1$.*

(i) If r and t are positive, then

$$\begin{aligned}
&M_t^t(g_1, g_2) - \left(\frac{\overline{g} M_t^t(g_1, g_2)}{\int_{\alpha_1}^{\alpha_2} g_1(\sigma) g_2^{t-r}(\sigma) d\sigma}\right)^{\frac{t}{r}} \\
\leq\ & \left(\frac{1}{2}\right)^{1-\frac{1}{q}} \frac{t(r-t)}{\overline{g} r^2} \int_{\alpha_1}^{\alpha_2} g_1(\sigma)\left(g_2^r(\sigma) - \frac{\overline{g} M_t^t(g_1, g_2)}{\int_{\alpha_1}^{\alpha_2} g_1(\sigma) g_2^{t-r}(\sigma) d\sigma}\right)^2 \\
&\times\left(\frac{2 g_2^{q(t-2r)}(\sigma) + \left(\frac{\overline{g} M_t^t(g_1, g_2)}{\int_{\alpha_1}^{\alpha_2} g_1(\sigma) g_2^{t-r}(\sigma) d\sigma}\right)^{q\left(\frac{t}{r}-2\right)}}{6}\right)^{\frac{1}{q}} d\sigma. \qquad (32)
\end{aligned}$$

(ii) If r and t are negative with $\frac{t}{r} \notin \left(2, 2+\frac{1}{q}\right)$, then

$$\begin{aligned} &\left(\frac{\overline{g}M_t^t(g_1,g_2)}{\int_{\alpha_1}^{\alpha_2} g_1(\sigma)g_2^{t-r}(\sigma)d\sigma}\right)^{\frac{t}{r}} - M_t^t(g_1,g_2) \\ \leq\ & \left(\frac{1}{2}\right)^{1-\frac{1}{q}} \frac{t(t-r)}{\overline{g}r^2} \int_{\alpha_1}^{\alpha_2} g_1(\sigma)\left(g_2^r(\sigma) - \frac{\overline{g}M_t^t(g_1,g_2)}{\int_{\alpha_1}^{\alpha_2} g_1(\sigma)g_2^{t-r}(\sigma)d\sigma}\right)^2 \\ & \times\left(\frac{2g_2^{q(t-2r)}(\sigma) + \left(\frac{\overline{g}M_t^t(g_1,g_2)}{\int_{\alpha_1}^{\alpha_2} g_1(\sigma)g_2^{t-r}(\sigma)d\sigma}\right)^{q(\frac{t}{r}-2)}}{6}\right)^{\frac{1}{q}} d\sigma. \end{aligned} \tag{33}$$

(iii) If r is positive and t is negative, then (33) holds.

Proof. *(i) To achieve (32), consider $\psi(\sigma) = \sigma^{\frac{t}{r}}$, $\sigma > 0$, $Y_1(\sigma) = g_1(\sigma)$, and $Y_2(\sigma) = g_2^r(\sigma)$ in (20).*

(ii)–(iii) Now, to deduce (33) select $\psi(\sigma) = \sigma^{\frac{t}{r}}$, $\sigma \in (0,\infty)$, $Y_1(\sigma) = g_1(\sigma)$, and $Y_2(\sigma) = g_2^r(\sigma)$ in (20). ■

The next corollary gives a relation for the power means as a consequence of Theorem 9.

Corollary 6 *Presume that the hypotheses of Corollary 2 are satisfied, then*

$$\begin{aligned} &\left(\frac{\overline{g}M_t^t(g_1,g_2)}{\int_{\alpha_1}^{\alpha_2} g_1(\sigma)g_2^{t-r}(\sigma)d\sigma}\right)^{\frac{t}{r}} - M_t^t(g_1,g_2) \\ \leq\ & \frac{t(t-r)}{2\overline{g}r^2} \int_{\alpha_1}^{\alpha_2} g_1(\sigma)\left(g_2^r(\sigma) - \frac{\overline{g}M_t^t(g_1,g_2)}{\int_{\alpha_1}^{\alpha_2} g_1(\sigma)g_2^{t-r}(\sigma)d\sigma}\right)^2 \\ & \times\left(\frac{2g_2^r(\sigma) + \frac{\overline{g}M_t^t(g_1,g_2)}{\int_{\alpha_1}^{\alpha_2} g_1(\sigma)g_2^{t-r}(\sigma)d\sigma}}{3}\right)^{\frac{t}{r}-2} d\sigma. \end{aligned} \tag{34}$$

Proof. *Inequality (34) can easily be acquired by using $\psi(\sigma) = \sigma^{\frac{t}{r}}$, $\sigma \in (0,\infty)$, $Y_1(\sigma) = g_1(\sigma)$, and $Y_2(\sigma) = g_2^r(\sigma)$ in (23).* ■

Some relations are deduced for the power means by utilizing Theorem 4, which are given in the below corollary.

Corollary 7 *Let g_1, g_2 be positive valued functions on (α_1, α_2) and $q > 1$. Then*

(i)

$$\begin{aligned} \frac{M_0(g_1,g_2)}{M_{-1}(g_1,g_2)} \leq\ & \exp\left(\frac{1}{\overline{g}}\int_{\alpha_1}^{\alpha_2} g_1(\sigma)\Big(g_2(\sigma) - M_{-1}(g_1,g_2)\Big)^2\right. \\ & \left.\times\left(\frac{(q+1)g_2^{-2r}(\sigma) + M_{-1}^{-2p}(g_1,g_2)}{(q+1)(q+2)}\right)^{\frac{1}{p}} dy\right). \end{aligned} \tag{35}$$

(ii)

$$\begin{aligned} \exp\left(\frac{M_2^2(g_1,g_2)}{M_1(g_1,g_2)}\right) - M_1(g_1,g_2) \leq\ & \frac{1}{\overline{g}}\int_{\alpha_1}^{\alpha_2} g_1(\sigma)\left(\log g_2(\sigma) - \frac{M_2^2(g_1,g_2)}{M_1(g_1,g_2)}\right)^2 \\ & \times\left(\frac{(q+1)g_2^q(\sigma) + \exp\left(\frac{M_2^2(g_1,g_2)}{M_1(g_1,g_2)}\right)}{(q+1)(q+2)}\right)^{\frac{1}{p}} d\sigma. \end{aligned} \tag{36}$$

Proof. *(i) First, we prove inequality (26). For this, take* $\psi(\sigma) = \log\sigma$, $\sigma \in (0,\infty)$. *Then, clearly, both the functions* ψ *and* $|\psi''|^q$ *are convex. Therefore, to obtain (35), use* $\psi(\sigma) = \log\sigma$, $Y_1(\sigma) = g_1(\sigma)$, *and* $Y_2(\sigma) = g_2(\sigma)$ *in (4).*

(ii) Now, to achieve (36), let us take $\psi(\sigma) = \exp y$, $\sigma \in (-\infty,\infty)$. *Then surely, the functions* ψ *and* $|\psi''|^q$ *both are convex. Thus, applying (4) by choosing* $\psi(\sigma) = \exp y$, $Y_1(\sigma) = g_1(\sigma)$, *and* $Y_2(\sigma) = g_2(\sigma)$, *we obtain (36).* ■

The following corollary provides relations for the power means, which have been acquired from Theorem 6.

Corollary 8 *Suppose that, the conditions of Corollary 7 are fulfilled. Additionally, if* $p > 1$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$, *then*

(i)

$$\frac{M_0(g_1,g_2)}{M_{-1}(g_1,g_2)} \leq \exp\Bigg(\Big(\frac{1}{p+1}\Big)^{\frac{1}{p}} \frac{1}{\bar{g}} \int_{\alpha_1}^{\alpha_2} g_1(\sigma)\Big(g_2(\sigma) - M_{-1}(g_1,g_2)\Big)^2 \times\Bigg(\frac{g_2^{-2r}(\sigma) + M_{-1}^{-2p}(g_1,g_2)}{2}\Bigg)^{\frac{1}{p}} dy\Bigg). \tag{37}$$

(ii)

$$\exp\Big(\frac{M_2^2(g_1,g_2)}{M_1(g_1,g_2)}\Big) - M_1(g_1,g_2) \leq \Big(\frac{1}{p+1}\Big)^{\frac{1}{p}} \frac{1}{\bar{g}} \int_{\alpha_1}^{\alpha_2} g_1(\sigma)\Bigg(\log g_2(\sigma) - \frac{M_2^2(g_1,g_2)}{M_1(g_1,g_2)}\Bigg)^2 \times\Bigg(\frac{g_2^q(\sigma) + \exp\Big(\frac{M_2^2(g_1,g_2)}{M_1(g_1,g_2)}\Big)}{2}\Bigg)^{\frac{1}{p}} d\sigma. \tag{38}$$

Proof. *(i) Use* $\psi(\sigma) = \log\sigma$, $\sigma \in (0,\infty)$, $Y_1(\sigma) = g_1(\sigma)$, *and* $Y_2(\sigma) = g_2(\sigma)$ *in (12), we receive (37).*

(ii) Take $\psi(\sigma) = \exp\sigma$, $\sigma \in (-\infty,\infty)$, $Y_1(\sigma) = g_1(\sigma)$, *and* $Y_2(\sigma) = g_2(\sigma)$ *in (12), we get (38).* ■

The following relations are the consequences of Theorem 8 for the power means.

Corollary 9 *Suppose that, the conditions of Corollary 7 are fulfilled. Additionally, if* $p > 1$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$, *then*

(i)

$$\frac{M_0(g_1,g_2)}{M_{-1}(g_1,g_2)} \leq \exp\Bigg(\Big(\frac{1}{2}\Big)^{1-\frac{1}{q}} \frac{1}{\bar{g}} \int_{\alpha_1}^{\alpha_2} g_1(\sigma)\Big(g_2(\sigma) - M_{-1}(g_1,g_2)\Big)^2 \times\Bigg(\frac{2g_2^{-2r}(\sigma) + M_{-1}^{-2p}(g_1,g_2)}{6}\Bigg)^{\frac{1}{p}} dy\Bigg). \tag{39}$$

(ii)

$$\exp\Big(\frac{M_2^2(g_1,g_2)}{M_1(g_1,g_2)}\Big) - M_1(g_1,g_2) \leq \Big(\frac{1}{2}\Big)^{1-\frac{1}{q}} \frac{1}{\bar{g}} \int_{\alpha_1}^{\alpha_2} g_1(\sigma)\Bigg(\log g_2(\sigma) - \frac{M_2^2(g_1,g_2)}{M_1(g_1,g_2)}\Bigg)^2 \times\Bigg(\frac{2g_2^q(\sigma) + \exp\Big(\frac{M_2^2(g_1,g_2)}{M_1(g_1,g_2)}\Big)}{6}\Bigg)^{\frac{1}{p}} d\sigma. \tag{40}$$

Proof. *(i) To acquire inequality (39), select $\psi(\sigma)=\log\sigma,\ \sigma\in(0,\infty)$, $Y_1(\sigma)=g_1(\sigma)$, and $Y_2(\sigma)=g_2(\sigma)$ in (20).*

(ii) For inequality (40), apply (20) by putting $\psi(\sigma)=\exp\sigma$, $\sigma\in(-\infty,\infty)$, $Y_1(\sigma)=g_1(\sigma)$, and $Y_2(\sigma)=g_2(\sigma)$. ■

4 Applications for Divergences

During the last few decades, the divergences have been utilized very abundantly in the various areas of science, particularly in communication domain [1,9]. The divergences can be used to remove faults in communication or transmission of signals over channels [30,31]. Moreover, the pragmatic information processing in computers, in the internet, and in other networks can be studied and developed with the help of divergences [17, 32, 40]. In the recent years, there are many theoretical and conceptual approaches have been organized in very systematical manners, how divergence should be used to solve many problems in the communication networks [41,43,52]. Due to the great importance and real applicability of divergence in several real world problems, we have also made an attempt to contribute some literature to this area. Here, we provide some estimations for the Csiszár and Kullback–Leibler divergences, Shannon entropy, and Bhattacharyya coefficient as applications of the main results. In order to proceed to the desire estimations, first we state the definition of Csiszár divergence.

Definition 2 *Let $g_1,g_2:(\alpha_1,\alpha_2\to(0,\infty)$ be integrable functions and ψ be a real valued function defined on $(0,\infty)$. Then, the Csiszár divergence is defined as:*

$$D_\psi(g_1,g_2)=\int_{\alpha_1}^{\alpha_2} g_1(\sigma)\psi\left(\frac{g_2(\sigma)}{g_1(\sigma)}\right)d\sigma.$$

Corollary 10 *Assume that, the functions g_1 and g_2 are positive such that their integrals exists over (α_1,α_2). Also, let*

$$\overline{g}:=\int_{\alpha_1}^{\alpha_2} g_1(\sigma)d\sigma,\ \frac{\int_{\alpha_1}^{\alpha_2} g_2(\sigma)\psi'\big(\frac{g_2(\sigma)}{g_1(\sigma)}\big)d\sigma}{\int_{\alpha_1}^{\alpha_2} g_1(\sigma)\psi'\big(\frac{g_2(\sigma)}{g_1(\sigma)}\big)d\sigma}>0,$$

and ψ be a real valued twice differentiable function over $(0,\infty)$ such that $|\psi''|^q$ is convex for $q>1$. Then

$$\begin{aligned}&\left|\psi\left(\frac{\int_{\alpha_1}^{\alpha_2} g_2(\sigma)\psi'\big(\frac{g_2(\sigma)}{g_1(\sigma)}\big)d\sigma}{D_{\psi'}(g_1,g_2)}\right)-\frac{D_\psi(g_1,g_2)}{\overline{g}}\right|\\ \leq\ &\frac{1}{\overline{g}}\int_{\alpha_1}^{\alpha_2} g_1(\sigma)\left(\frac{g_2(\sigma)}{g_1(\sigma)}-\frac{\int_{\alpha_1}^{\alpha_2} g_2(\sigma)\psi'\big(\frac{g_2(\sigma)}{g_1(\sigma)}\big)d\sigma}{D_{\psi'}(g_1,g_2)}\right)^2\\ &\times\left(\frac{(q+1)\big|\psi''\big(\frac{g_2(\sigma)}{g_1(\sigma)}\big)\big|^q+\left|\psi''\left(\frac{\int_{\alpha_1}^{\alpha_2} g_2(\sigma)\psi'\big(\frac{g_2(\sigma)}{g_1(\sigma)}\big)d\sigma}{D_{\psi'}(g_1,g_2)}\right)\right|^q}{(q+1)(q+2)}\right)^{\frac{1}{q}}d\sigma. \end{aligned}\tag{41}$$

Proof. *Inequality (41) can be easily deduced by taking $Y_1(\sigma)=g_1(\sigma)$ and $Y_2(\sigma)=\frac{g_2(\sigma)}{g_1(\sigma)}$ in (4).* ■

Corollary 11 *Let $g_1,g_2:(\alpha_1,\alpha_2)\to(0,\infty)$ be integrable function with*

$$\overline{g}:=\int_{\alpha_1}^{\alpha_2} g_1(\sigma)d\sigma\quad and\quad \frac{\int_{\alpha_1}^{\alpha_2} g_2(\sigma)\psi'\big(\frac{g_2(\sigma)}{g_1(\sigma)}\big)d\sigma}{\int_{\alpha_1}^{\alpha_2} g_1(\sigma)\psi'\big(\frac{g_2(\sigma)}{g_1(\sigma)}\big)d\sigma}>0.$$

Further, assume that $\psi:(0,\infty)\to\mathbb{R}$ is a twice differentiable function such that $|\psi''|^q$ is concave for $q>1$, then

$$\left|\psi\left(\frac{\int_{\alpha_1}^{\alpha_2} g_2(\sigma)\psi^{'}\left(\frac{g_2(\sigma)}{g_1(\sigma)}\right)d\sigma}{D_{\psi'}(g_1,g_2)}\right)-\frac{D_\psi(g_1,g_2)}{\bar{g}}\right|$$
$$\leq \frac{1}{\bar{g}(q+1)^{\frac{1}{q}}}\int_{\alpha_1}^{\alpha_2} g_1(\sigma)\left(\frac{g_2(\sigma)}{g_1(\sigma)}-\frac{\int_{\alpha_1}^{\alpha_2} g_2(\sigma)\psi^{'}\left(\frac{g_2(\sigma)}{g_1(\sigma)}\right)d\sigma}{D_{\psi'}(g_1,g_2)}\right)^2$$
$$\times\left|\psi''\left(\frac{(q+1)\frac{g_2(\sigma)}{g_1(\sigma)}+\frac{\int_{\alpha_1}^{\alpha_2} g_2(\sigma)\psi^{'}\left(\frac{g_2(\sigma)}{g_1(\sigma)}\right)d\sigma}{D_{\psi'}(g_1,g_2)}}{q+2}\right)\right|d\sigma. \tag{42}$$

Proof. *Use $Y_1(\sigma)=g_1(\sigma)$ and $Y_2(\sigma)=\frac{g_2(\sigma)}{g_1(\sigma)}$ in (9), we acquire (42).* ■

Corollary 12 *Suppose that the assumptions of Corollary 10 are valid. Moreover, if $p>1$ such that $\frac{1}{p}+\frac{1}{q}=1$, then*

$$\left|\psi\left(\frac{\int_{\alpha_1}^{\alpha_2} g_2(\sigma)\psi^{'}\left(\frac{g_2(\sigma)}{g_1(\sigma)}\right)d\sigma}{D_{\psi'}(g_1,g_2)}\right)-\frac{D_\psi(g_1,g_2)}{\bar{g}}\right|$$
$$\leq \frac{1}{\bar{g}(p+1)^{\frac{1}{p}}}\int_{\alpha_1}^{\alpha_2} g_1(\sigma)\left(\frac{g_2(\sigma)}{g_1(\sigma)}-\frac{\int_{\alpha_1}^{\alpha_2} g_2(\sigma)\psi^{'}\left(\frac{g_2(\sigma)}{g_1(\sigma)}\right)d\sigma}{D_{\psi'}(g_1,g_2)}\right)^2$$
$$\times\left(\frac{\left|\psi''\left(\frac{g_2(\sigma)}{g_1(\sigma)}\right)\right|^q+\left|\psi''\left(\frac{\int_{\alpha_1}^{\alpha_2} g_2(\sigma)\psi^{'}\left(\frac{g_2(\sigma)}{g_1(\sigma)}\right)d\sigma}{D_{\psi'}(g_1,g_2)}\right)\right|^q}{2}\right)^{\frac{1}{q}}d\sigma. \tag{43}$$

Proof. *To obtain (43), just put $Y_1(\sigma)=g_1(\sigma)$ and $Y_2(\sigma)=\frac{g_2(\sigma)}{g_1(\sigma)}$ in (12).* ■

Corollary 13 *Suppose that, all the hypotheses of Corollary 11 are satisfied. Furthermore, if the relation $\frac{1}{p}+\frac{1}{q}$ is true for $p>1$, then*

$$\left|\psi\left(\frac{\int_{\alpha_1}^{\alpha_2} g_2(\sigma)\psi^{'}\left(\frac{g_2(\sigma)}{g_1(\sigma)}\right)d\sigma}{D_{\psi'}(g_1,g_2)}\right)-\frac{D_\psi(g_1,g_2)}{\bar{g}}\right|$$
$$\leq \frac{1}{\bar{g}(p+1)^{\frac{1}{p}}}\int_{\alpha_1}^{\alpha_2} g_1(\sigma)\left(\frac{g_2(\sigma)}{g_1(\sigma)}-\frac{\int_{\alpha_1}^{\alpha_2} g_2(\sigma)\psi^{'}\left(\frac{g_2(\sigma)}{g_1(\sigma)}\right)d\sigma}{D_{\psi'}(g_1,g_2)}\right)^2$$
$$\times\left|\psi''\left(\frac{\frac{g_2(\sigma)}{g_1(\sigma)}+\frac{\int_{\alpha_1}^{\alpha_2} g_2(\sigma)\psi^{'}\left(\frac{g_2(\sigma)}{g_1(\sigma)}\right)d\sigma}{D_{\psi'}(g_1,g_2)}}{2}\right)\right|d\sigma. \tag{44}$$

Proof. *By utilizing $Y_1(\sigma)=g_1(\sigma)$ and $Y_2(\sigma)=\frac{g_2(\sigma)}{g_1(\sigma)}$ in (17), we receive (44).* ■

Corollary 14 *Let the conditions of Corollary 10 be fulfilled. then*

$$\left|\psi\left(\frac{\int_{\alpha_1}^{\alpha_2} g_2(\sigma)\psi^{'}\left(\frac{g_2(\sigma)}{g_1(\sigma)}\right)d\sigma}{D_{\psi'}(g_1,g_2)}\right)-\frac{D_\psi(g_1,g_2)}{\bar{g}}\right|$$
$$\leq \frac{1}{2^{1-\frac{1}{p}}\bar{g}}\int_{\alpha_1}^{\alpha_2} g_1(\sigma)\left(\frac{g_2(\sigma)}{g_1(\sigma)}-\frac{\int_{\alpha_1}^{\alpha_2} g_2(\sigma)\psi^{'}\left(\frac{g_2(\sigma)}{g_1(\sigma)}\right)d\sigma}{D_{\psi'}(g_1,g_2)}\right)^2$$
$$\times\left(\frac{2\left|\psi''\left(\frac{g_2(\sigma)}{g_1(\sigma)}\right)\right|^q+\left|\psi''\left(\frac{\int_{\alpha_1}^{\alpha_2} g_2(\sigma)\psi^{'}\left(\frac{g_2(\sigma)}{g_1(\sigma)}\right)d\sigma}{D_{\psi'}(g_1,g_2)}\right)\right|^q}{6}\right)^{\frac{1}{q}}d\sigma. \tag{45}$$

Proof. *Applying (20) by choosing $Y_1(\sigma) = g_1(\sigma)$ and $Y_2(\sigma) = \frac{g_2(\sigma)}{g_1(\sigma)}$, we get (45).* ■

Corollary 15 *Assume that, the hypotheses Corollary 11 are positive, then*

$$\begin{aligned} &\left|\psi\left(\frac{\int_{\alpha_1}^{\alpha_2} g_2(\sigma)\psi'\left(\frac{g_2(\sigma)}{g_1(\sigma)}\right)d\sigma}{D_{\psi'}(g_1,g_2)}\right) - \frac{D_\psi(g_1,g_2)}{\overline{g}}\right| \\ \leq\ & \frac{1}{2\overline{g}}\int_{\alpha_1}^{\alpha_2} g_1(\sigma)\left(\frac{g_2(\sigma)}{g_1(\sigma)} - \frac{\int_{\alpha_1}^{\alpha_2} g_2(\sigma)\psi'\left(\frac{g_2(\sigma)}{g_1(\sigma)}\right)d\sigma}{D_{\psi'}(g_1,g_2)}\right)^2 \\ &\times\left|\psi''\left(\frac{2\frac{g_2(\sigma)}{g_1(\sigma)} + \frac{\int_{\alpha_1}^{\alpha_2} g_2(\sigma)\psi'\left(\frac{g_2(\sigma)}{g_1(\sigma)}\right)d\sigma}{D_{\psi'}(g_1,g_2)}}{3}\right)\right|d\sigma. \end{aligned} \tag{46}$$

Proof. *To acquire (46), use (23) for $Y_1(\sigma) = g_1(\sigma)$ and $Y_2(\sigma) = \frac{g_2(\sigma)}{g_1(\sigma)}$.* ■

Definition 3 *For any probability density function $g_1, g_2 : (\alpha_1, \alpha_2) \to (0, \infty)$, then Kullback–Leibler divergence is defined as:*

$$D_k(g_1,g_2) = \int_{\alpha_1}^{\alpha_2} g_1(\sigma)\psi\left(\frac{g_1(\sigma)}{g_2(\sigma)}\right)d\sigma$$

Corollary 16 *Suppose that g_1 and g_2 are positive probability distributions on (α_1, α_2) and $q > 1$, then*

$$\begin{aligned} \log\left(\int_{\alpha_1}^{\alpha_2}\frac{g_1^2(\sigma)}{g_2(\sigma)}dy\right) - D_k(g_1,g_2) \leq \int_{\alpha_1}^{\alpha_2} g_1(\sigma)\left(\frac{g_2(\sigma)}{g_1(\sigma)} - \left(\int_{\alpha_1}^{\alpha_2}\frac{g_1^2(\sigma)}{g_2(\sigma)}dy\right)^{-1}\right)^2 \\ \times\left(\frac{(q+1)\left(\frac{g_1(\sigma)}{g_2(\sigma)}\right)^{2q} + \left(\int_{\alpha_1}^{\alpha_2}\frac{g_1^2(\sigma)}{g_2(\sigma)}dy\right)^{2q}}{(q+1)(q+2)}\right)^{\frac{1}{q}}d\sigma. \end{aligned} \tag{47}$$

Proof. *Consider $\psi(\sigma) = -\log\sigma$ defined on $(0,\infty)$, then by successive differentiations, we have $\psi''(\sigma) = \sigma^{-2}$ and $\left(|\psi''(\sigma)|^q\right)'' = 2q(q2+1)\sigma^{-(2q+2)}$. From these expressions, it is clear that both the functions ψ'' and $\left(|\psi''|^q\right)''$ are positive with the given conditions. Which reflects the convexity of ψ and $|\psi''(\sigma)|^q$. Thus, to achieve (47) use $\psi(\sigma) = -\log\sigma$ in (41).* ■

Corollary 17 *Assume that the hypotheses of Corollary 16 are true. Moreover, if for $p > 1$ the relation $\frac{1}{p} + \frac{1}{q} = 1$ is valid, then*

$$\begin{aligned} &\log\left(\int_{\alpha_1}^{\alpha_2}\frac{g_1^2(\sigma)}{g_2(\sigma)}dy\right) - D_k(g_1,g_2) \\ \leq\ & \left(\frac{1}{p+1}\right)^{\frac{1}{p}}\int_{\alpha_1}^{\alpha_2} g_1(\sigma)\left(\frac{g_2(\sigma)}{g_1(\sigma)} - \left(\int_{\alpha_1}^{\alpha_2}\frac{g_1^2(\sigma)}{g_2(\sigma)}dy\right)^{-1}\right)^2 \\ &\times\left(\frac{\left(\frac{g_1(\sigma)}{g_2(\sigma)}\right)^{2q} + \left(\int_{\alpha_1}^{\alpha_2}\frac{g_1^2(\sigma)}{g_2(\sigma)}dy\right)^{2q}}{6}\right)^{\frac{1}{q}}d\sigma. \end{aligned} \tag{48}$$

Proof. *To get (48), take $\psi(\sigma) = -\log\sigma$, $\sigma \in (0,\infty)$ in (43).* ■

Corollary 18 *Let the conditions of Corollary 16 are fulfilled, then*

$$\begin{aligned}&\log\left(\int_{\alpha_1}^{\alpha_2}\frac{g_1^2(\sigma)}{g_2(\sigma)}dy\right)-D_k(g_1,g_2)\\ \leq\ &\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\int_{\alpha_1}^{\alpha_2}g_1(\sigma)\left(\frac{g_2(\sigma)}{g_1(\sigma)}-\left(\int_{\alpha_1}^{\alpha_2}\frac{g_1^2(\sigma)}{g_2(\sigma)}dy\right)^{-1}\right)^2\\ &\times\left(\frac{2\left(\frac{g_1(\sigma)}{g_2(\sigma)}\right)^{2q}+\left(\int_{\alpha_1}^{\alpha_2}\frac{g_1^2(\sigma)}{g_2(\sigma)}dy\right)^{2q}}{6}\right)^{\frac{1}{q}}d\sigma. \qquad (49)\end{aligned}$$

Proof. *Apply (45) for $\psi(\sigma)=-\log\sigma,\ \sigma\in(0,\infty)$, we obtain (49).* ∎

Definition 4 *For the probability density function $\psi:(\alpha_1,\alpha_2)\to(0,\infty)$, the Shannon entropy is defined by:*

$$S(g_1)=\int_{\alpha_1}^{\alpha_2}g_1(\sigma)\log\big(g_1(\sigma)\big)d\sigma.$$

Corollary 19 *Presume that g_1 is a positive probability density function on (α_1,α_2) and $q>1$, then*

$$\begin{aligned}\log\left(\int_{\alpha_1}^{\alpha_2}g_1^2(\sigma)d\sigma\right)-S(g_1)\ \leq\ &\int_{\alpha_1}^{\alpha_2}g_1(\sigma)\left(\frac{1}{g_1(\sigma)}-\frac{1}{\int_{\alpha_1}^{\alpha_2}g_1^2(\sigma)d\sigma}\right)^2\\ &\times\left(\frac{(q+1)g_1^{2q}(\sigma)+\left(\int_{\alpha_1}^{\alpha_2}g_1^2(\sigma)d\sigma\right)^{2q}}{(q+1)(q+2)}\right)^{\frac{1}{q}}d\sigma. \qquad (50)\end{aligned}$$

Proof. *Inequality (50) can smoothly be gained by taking $\psi(\sigma)=-\log\sigma,\ \sigma\in(0,\infty)$ in (41).* ∎

Corollary 20 *Assume that the conditions of Corollary 20 are true with further addition that, the relation $\frac{1}{p}+\frac{1}{q}=1$ is valid for $p>1$, then*

$$\begin{aligned}\log\left(\int_{\alpha_1}^{\alpha_2}g_1^2(\sigma)d\sigma\right)-S(g_1)\ \leq\ &\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\int_{\alpha_1}^{\alpha_2}g_1(\sigma)\left(\frac{1}{g_1(\sigma)}-\frac{1}{\int_{\alpha_1}^{\alpha_2}g_1^2(\sigma)d\sigma}\right)^2\\ &\times\left(\frac{g_1^{2q}(\sigma)+\left(\int_{\alpha_1}^{\alpha_2}g_1^2(\sigma)d\sigma\right)^{2q}}{2}\right)^{\frac{1}{q}}d\sigma. \qquad (51)\end{aligned}$$

Proof. *Utilizing (43) for $\psi(\sigma)=-\log\sigma,\ \sigma\in(0,\infty)$, we receive (51).* ∎

Corollary 21 *Let the hypotheses of Corollary 20 hold. Then*

$$\begin{aligned}\log\left(\int_{\alpha_1}^{\alpha_2}g_1^2(\sigma)d\sigma\right)-S(g_1)\ \leq\ &\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\int_{\alpha_1}^{\alpha_2}g_1(\sigma)\left(\frac{1}{g_1(\sigma)}-\frac{1}{\int_{\alpha_1}^{\alpha_2}g_1^2(\sigma)d\sigma}\right)^2\\ &\times\left(\frac{2g_1^{2q}(\sigma)+\left(\int_{\alpha_1}^{\alpha_2}g_1^2(\sigma)d\sigma\right)^{2q}}{6}\right)^{\frac{1}{q}}d\sigma. \qquad (52)\end{aligned}$$

Proof. *To acquire (52), apply (45) for $\psi(\sigma)=-\log\sigma,\ \sigma\in(0,\infty)$.* ∎

Definition 5 *The Bhattacharyya coefficient for the positive Probability density functions g_1 and g_2 over (α_1,α_2) is defined by:*

$$B(g_1,g_2)=\int_{\alpha_1}^{\alpha_2}\sqrt{g_1(\sigma)g_2(\sigma)}\,d\sigma.$$

Corollary 22 *Assume that, all the conditions of Corollary 16 are fulfilled, then*

$$B(g_1,g_2)-\sqrt{\frac{B(g_1,g_2)}{\int_{\alpha_1}^{\alpha_2} g_1^{\frac{3}{2}}(\sigma)g_2^{-\frac{1}{2}}(\sigma)d\sigma}} \leq \frac{1}{4}\int_{\alpha_1}^{\alpha_2} g_1(\sigma)\bigg(\frac{g_2(\sigma)}{g_1(\sigma)}-\frac{B(g_1,g_2)}{\int_{\alpha_1}^{\alpha_2} g_1^{\frac{3}{2}}(\sigma)g_2^{-\frac{1}{2}}(\sigma)d\sigma}\bigg)^2 \times\left(\frac{(q+1)\Big(\frac{g_1(\sigma)}{g_2(\sigma)}\Big)^{\frac{2q}{3}}+\Big(\frac{\int_{\alpha_1}^{\alpha_2} g_1^{\frac{3}{2}}(\sigma)g_2^{-\frac{1}{2}}(\sigma)d\sigma}{B(g_1,g_2)}\Big)^{\frac{2q}{3}}}{(q+1)(q+2)}\right)^{\frac{1}{q}}d\sigma. \tag{53}$$

Proof. *Consider $\psi(\sigma)=-\sqrt{\sigma},\ \sigma\in(0,\infty)$, then distinctly $\psi''(\sigma)=\frac{1}{4}\sigma^{-\frac{3}{2}}$ and*

$$\big(|\psi''(\sigma)|^q\big)''=\big(\frac{1}{4}\big)^q\frac{3q}{2}\big(\frac{3q}{2}+1\big)\sigma^{-\left(\frac{3q}{2}+2\right)}.$$

Which implies that both $\psi''(\sigma)$ and $\big(|\psi''(\sigma)|^q\big)''$ are positive on $(0,\infty)$ for $q>1$. This admit that the functions $\psi(\sigma)$ and $|\psi''(\sigma)|^q$ are convex. Hence, to acquire (53) put $\psi(\sigma)=-\sqrt{\sigma}$ in (41). ■

Corollary 23 *Suppose that the hypotheses of Corollary 17 are true, then*

$$\begin{aligned}&B(g_1,g_2)-\sqrt{\frac{B(g_1,g_2)}{\int_{\alpha_1}^{\alpha_2} g_1^{\frac{3}{2}}(\sigma)g_2^{-\frac{1}{2}}(\sigma)d\sigma}}\\ \leq\ &\frac{1}{4(p+1)^{\frac{1}{p}}}\int_{\alpha_1}^{\alpha_2} g_1(\sigma)\bigg(\frac{g_2(\sigma)}{g_1(\sigma)}-\frac{B(g_1,g_2)}{\int_{\alpha_1}^{\alpha_2} g_1^{\frac{3}{2}}(\sigma)g_2^{-\frac{1}{2}}(\sigma)d\sigma}\bigg)^2\\ &\times\left(\frac{\Big(\frac{g_1(\sigma)}{g_2(\sigma)}\Big)^{\frac{2q}{3}}+\Big(\frac{\int_{\alpha_1}^{\alpha_2} g_1^{\frac{3}{2}}(\sigma)g_2^{-\frac{1}{2}}(\sigma)d\sigma}{B(g_1,g_2)}\Big)^{\frac{2q}{3}}}{2}\right)^{\frac{1}{q}}d\sigma.\end{aligned} \tag{54}$$

Proof. *Using $\psi(\sigma)=-\sqrt{\sigma},\ \sigma\in(0,\infty)$ in (43), we achieve (54).* ■

Corollary 24 *Let the assumptions of Corollary 16 are come true. Then*

$$\begin{aligned}&B(g_1,g_2)-\sqrt{\frac{B(g_1,g_2)}{\int_{\alpha_1}^{\alpha_2} g_1^{\frac{3}{2}}(\sigma)g_2^{-\frac{1}{2}}(\sigma)d\sigma}}\\ \leq\ &\Big(\frac{1}{2}\Big)^{3-\frac{1}{q}}\int_{\alpha_1}^{\alpha_2} g_1(\sigma)\bigg(\frac{g_2(\sigma)}{g_1(\sigma)}-\frac{B(g_1,g_2)}{\int_{\alpha_1}^{\alpha_2} g_1^{\frac{3}{2}}(\sigma)g_2^{-\frac{1}{2}}(\sigma)d\sigma}\bigg)^2\\ &\times\left(\frac{2\Big(\frac{g_1(\sigma)}{g_2(\sigma)}\Big)^{\frac{2q}{3}}+\Big(\frac{\int_{\alpha_1}^{\alpha_2} g_1^{\frac{3}{2}}(\sigma)g_2^{-\frac{1}{2}}(\sigma)d\sigma}{B(g_1,g_2)}\Big)^{\frac{2q}{3}}}{6}\right)^{\frac{1}{q}}d\sigma.\end{aligned} \tag{55}$$

Proof. *Inequality (55) can be obtained by utilizing (45) while taking $\psi(\sigma)=-\sqrt{\sigma},\ \sigma\in(0,\infty)$.* ■

5 Conclusion

The theory of mathematical inequalities is a crucial and influential area of research. It provides a valuable framework for introducing and developing mathematical tools to solve and manage various real-world problems. An interesting fact about mathematical inequalities is that many of them are based on convex functions. Among these inequalities, Jensen's and Slater's inequalities are particularly noteworthy. This article aims to presented novel improvements to Slater's inequality through a new methodological approach. The intended improvements were proven by utilizing twice-differentiable functions and other mathematical tools from convex analysis and mathematical inequalities. The applications of the main findings were explored for power means and in information theory, providing new relations for power means, Csiszár and Kullback–Leibler divergence, Shannon entropy, and Bhattacharyya coefficient.

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