

On Implicit Neutral Differential System With Conformable Fractional Derivatives With State-Dependent Delay And Impulses*

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Abstract

This paper examines into the existence and uniqueness of solutions for a nonlinear implicit neutral fractional differential system (INFDS) incorporating state-dependent delay (SDD) and impulsive perturbations (IPs). We utilize the conformable fractional derivative ($\mathcal{CFD}$), which offers computational benefits and preserves key properties. Our results are derived based on the assumption that the involved functions satisfy local Lipschitz continuity conditions. To illustrate the applicability of our theoretical contributions, we present an illustrative example.

1 Introduction

Fractional calculus is a fascinating area of mathematics that goes beyond regular differentiation and integration, exploring operators of non-integer orders. It provides a powerful way to model systems with memory and hereditary behaviors, helping us better understand complex systems [1–3]. The idea of fractional calculus dates back to the 17th century, when mathematicians like Leibniz and L'Hopital first thought about fractional derivatives. Over time, other great mathematicians such as Euler, Riemann, and Grunwald made important contributions to its theoretical development [4–6].

Impulsive differential equations (IDEs) have emerged as a powerful tool for modeling dynamical systems that experience abrupt state changes. Unlike classical differential equations, where changes occur continuously, IDEs incorporate impulsive terms representing sudden jumps or discontinuities in state variables [7]. This framework is particularly effective for capturing phenomena such as impacts in mechanics, control systems with sudden parameter shifts, population dynamics with rapid migrations, and biological systems with discrete events like heartbeats [8–11].

State-dependent delay (SDD) is a significant concept in the study of differential equations, where the delay in the system's response depends on the current or past states of the system. This type of delay more accurately reflects real-world scenarios where the timing of events or actions is influenced by the system's current condition. For instance, in biological systems, the reaction time can vary depending on the current health state, or in control systems, the response delay can depend on the current operating conditions [10–13].

The $\mathcal{CFD}$ is a relatively new definition of a fractional derivative introduced by Abdel-Rahman Khalil et al. [14] in 2014. Unlike some classical fractional derivatives (e.g., Riemann-Liouville, Caputo), the conformable derivative offers several advantages, see for instance [9, 15–17]. Conformable fractional differential equations (CFDEs) represent a relatively new area of investigation within the broader field of fractional calculus. In 2017, Bayour and Torres [18] examined the existence results for CFDEs of order $\rho \in (0, 1]$ under fixed point techniques. Later, in [19], authors discussed the existence and stability results for the system discussed in [18] with non-local conditions under appropriate conditions. More recently, Hannabou et al. [9] studied a

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novel class of non-local IDEs involving the $\mathcal{CFD}$. Their approach leverages the theory of operator semigroups and fractional calculus to define a solution concept within this domain. The analysis hinges on the powerful tools of fixed-point theory, where the authors employ appropriate theorems to guarantee the existence and uniqueness of solutions. Li et al. [20] focus on conformable neutral systems, establishing existence results using fixed-point theorems. Abbas and Benchohra [13] considered the system as in [18], addressing existence and uniqueness for systems with both finite and infinite delays using fixed-point theory, and further extend their results to neutral-type systems. Hilal et al. [21] delve into fractional conformable neutral type systems with a non-local condition, proving existence and uniqueness of mild solutions via fixed-point theorems. Based on Xiao et al. [22], another work explores optimal control for conformable fractional neutral stochastic integro-differential systems [23]. Most recently, Krim et al. [10] analyzed implicit CFDEs, establishing existence results using a specific contraction mapping in b -metric spaces. Even with these advancements, there has been little research on the existence, uniqueness, and stability of solutions for implicit neutral-type CFDEs with SDD, especially for systems like model (1.1). This gap in the literature provides a valuable opportunity for future research.

Inspired by the above mentioned works [10, 13], this paper investigates the existence and uniqueness of a novel class of nonlinear implicit neutral CFDEs. The system under investigation is formulated as:

$$\begin{cases} {}^{\mathcal{CFD}}D_{t_p}^\rho [\omega(t) - H(t, \omega_t)] = F\left(t, \omega_{\kappa(t, \omega_t)}, {}^{\mathcal{CFD}}D_{t_p}^\rho \omega(t)\right), & t \in I_p, p = 0, 1, \dots, \ell, \\ \omega(t_p^+) = \omega(t_p^-) + L_p(\omega_{t_p^-}), & p = 1, \dots, \ell, \\ \omega(t) = \xi(t), & t \in (-\infty, 0], \end{cases} \quad (1.1)$$

where $F : I \times \mathcal{W} \times \mathbb{R} \rightarrow \mathbb{R}$, $H, \kappa : I \times \mathcal{W} \rightarrow \mathbb{R}$ are functions with $H(0, \omega_0) = 0$, $L_p : \mathcal{W} \rightarrow \mathbb{R}$, $p = 1, \dots, \ell$, $\xi : (-\infty, 0] \rightarrow \mathbb{R}$, $0 = t_0 < t_1 < \dots < t_\ell < t_{\ell+1} = Z < \infty$, where $I = [0, Z]$, $I_0 = [0, t_1]$, $I_p = (t_p, t_{p+1}]$, ${}^{\mathcal{CFD}}D_{t_p}^\rho$ is the CFD of order $\rho \in (0, 1)$, and $\mathcal{W}$ represents a space with properties relevant to the system, which will be formally introduced as the phase space later.

For any $t \in I$, we define $\omega_t \in \mathcal{W}$ by

$$\omega_t(\theta) = \omega(t + \theta), \quad \text{for } \theta \in (-\infty, 0].$$

Our work is carefully structured to explain the concepts and results in a clear order. Section 2 provides the basics, including the notation and key ideas from fractional calculus. It also includes important supporting results to prepare the reader for the analysis that follows. Section 3 focuses on the main part of the paper: proving the existence and uniqueness of solutions for problem (1.1) using the Banach contraction principle. Finally, Section 4 includes an example to demonstrate the importance of our main findings.

2 Preliminaries

This section lays the groundwork by providing a comprehensive introduction to the concept of the conformable derivative. It delves into the fundamental definitions, terminology, and essential background information necessary for understanding the applications of this derivative throughout the paper.

Let $\mathcal{C}(I, \mathbb{R})$ represent the collection of continuous functions $\omega : I \rightarrow \mathbb{R}$. This particular space, labeled as $\mathcal{C}(I, \mathbb{R})$, forms a Banach space characterized by the supremum norm $\|\cdot\|$, defined as:

$$\|\omega\|_\infty = \sup\{|\omega(t)| : t \in I\}.$$

Now, we construct the subsequent Banach space

$$\begin{aligned} \mathcal{PC}(I, \mathbb{R}) = & \left\{ \omega : I \rightarrow \mathbb{R}, \omega \in \mathcal{C}(I_p, \mathbb{R}) \text{ for } p = 0, 1, \dots, \ell, \text{ and there exist } \omega_p^- \text{ and } \omega_p^+ \right. \\ & \left. \text{with } \omega(t_p) = \omega(t_p^-), p = 1, 2, \dots, \ell \right\} \end{aligned}$$

and is normed by

$$\|\omega\|_{\mathcal{PC}} = \sup_{t \in I} |\omega(t)|.$$

Definition 2.1 ([14]) The $\mathcal{CFD}$ of a function $f : [0, \infty) \rightarrow \mathbb{R}$ of order $0 < \rho \leq 1$ is defined by

$$D^\rho f(t) = \lim_{\varepsilon \rightarrow 0} \frac{f(t + \varepsilon t^{1-\rho}) - f(t)}{\varepsilon},$$

provided the limit exists. In the case where $t = 0$, we modify the definition as follows:

$${}^{\mathcal{CFD}}D^\rho f(0) = \lim_{t \rightarrow 0^+} {}^{\mathcal{CFD}}D^\rho f(t).$$

Theorem 2.1 ([14]) Let $\rho \in (0, 1]$ and f_1, f_2 be ρ -differentiable at a point $t > 0$. As a result, we have

$$(i) \quad {}^{\mathcal{CFD}}D^\rho(f_1 f_2) = f_1 {}^{\mathcal{CFD}}D^\rho(f_2) + f_2 {}^{\mathcal{CFD}}D^\rho(f_1).$$

$$(ii) \quad {}^{\mathcal{CFD}}D^\rho f(t) = t^{1-\rho} Df(t), \text{ where } f \text{ is differentiable and } D = \frac{d}{dt}.$$

Definition 2.2 ([15]) Let $\rho \in (0, 1]$. The $\mathcal{CF}$ integral starting from a point d of a function $f : [0, \infty) \rightarrow \mathbb{R}$ of order ρ is described as

$$I^\rho(f)(t) = \int_0^t \sigma^{\rho-1} f(\sigma) d\sigma.$$

Theorem 2.2 ([14]) If $f(\cdot)$ is a continuous function in the domain of $I^\rho(\cdot)$, then for all $t > 0$, we have

$${}^{\mathcal{CFD}}D^\rho(I^\rho f(t)) = f(t).$$

Proposition 2.1 ([14]) If $f(\cdot)$ is a differentiable function, then for all $t > 0$, we have

$$I^\rho({}^{\mathcal{CFD}}D^\rho f(\cdot)) = f(t) - f(0).$$

Now, we construct the phase space axioms. Let $(\mathcal{W}, \|\cdot\|_{\mathcal{W}})$ be a seminormed linear space consisting of functions mapping $(-\infty, 0]$ into $\mathbb{R}$, defined by

$$\mathcal{W} = \left\{ \omega : (-\infty, 0] \rightarrow \mathbb{R} : \theta \mapsto \omega(\theta) \in \mathcal{C}((\theta_p, \theta_{p+1}], \mathbb{R}) \text{ for } p = 0, 1, \dots, \ell, \text{ and there exist } \omega(\theta_p^-) \text{ and } \omega(\theta_p^+) \right. \\ \left. \text{with } \omega(\theta_p) = \omega(\theta_p^-), p = 1, 2, \dots, \ell \text{ and } \theta_p = t_p - t \text{ for each } t \in (t_p, t_{p+1}] \right\}$$

and satisfy the subsequent axioms, which are derived from the original formulations by Hale and Kato [24].

(B1) If $\omega : (-\infty, 0] \rightarrow \mathbb{R}$, and $\omega_0 \in \mathcal{W}$, then there exist constants $\eta_1, \eta_2, \eta_3 > 0$ such that for each $t \in I$, the following hold:

$$(a) \quad \omega_t \in \mathcal{W},$$

$$(b) \quad \|\omega_t\|_{\mathcal{W}} \leq \eta_1 \|\omega_0\|_{\mathcal{W}} + \eta_2 \sup_{\psi \in [0, t]} |\omega(\psi)|,$$

$$(c) \quad \|\omega(t)\| \leq \eta_3 \|\omega_t\|_{\mathcal{W}}.$$

(B2) For the function $\omega(\cdot)$ as described in (B1), ω_t is continuous on I and maps into the space $\mathcal{W}$.

(B3) The space $\mathcal{W}$ possesses the property of completeness.

Consider the space

$$\Theta = \left\{ \omega : (-\infty, 0] \rightarrow \mathbb{R}, \omega|_{(-\infty, 0]} \in \mathcal{W}, \omega|_I \in \mathcal{PC}(I, \mathbb{R}) \right\}.$$

Definition 2.3 A function $\omega \in \Theta$ constitutes a solution to system (1.1) if and only if it adheres to the subsequent integral equation:

$$\omega(t) = \begin{cases} \xi(0) + H(t, \omega_t) + \int_0^t \sigma^{\rho-1} \Xi(\sigma) d\sigma, & t \in [0, t_1], \\ \xi(0) + H(t, \omega_t) + \sum_{p=1}^{\ell} L_p(\omega_{t_p^-}) + \sum_{p=1}^{\ell} \int_{t_{p-1}}^{t_p} \sigma^{\rho-1} \Xi(\sigma) d\sigma + \int_{t_p}^t \sigma^{\rho-1} \Xi(\sigma) d\sigma, & t \in (t_p, t_{p+1}], p = 1, 2, \dots, \ell, \\ \xi(t), & t \in (-\infty, 0], \end{cases} \quad (2.1)$$

where $\Xi \in \mathcal{C}(I)$ in a way that $\Xi(t) = F(t, \omega_{\kappa(t, \omega_t)}, \Xi(t))$.

3 Existence and Uniqueness Results

This section is dedicated to proving existence and uniqueness results for system (1.1). To begin, we establish an existence theorem for the system (1.1) by employing Banach's fixed-point theorem [25].

In the subsequent analysis, we assume that $0 \leq \kappa(t, \chi) \leq t$, $\chi \in \mathcal{W}$ and adopt the further assumptions:

(HF) The function $F : I \times \mathcal{W} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and $\exists \mathcal{M}_F > 0$ and $0 < \widetilde{\mathcal{M}}_F < 1$ such that

$$|F(t, \xi, v) - F(t, \bar{\xi}, \bar{v})| \leq \mathcal{M}_F \|\xi - \bar{\xi}\|_{\mathcal{W}} + \widetilde{\mathcal{M}}_F |v - \bar{v}|$$

for each $t \in I$, $\xi, \bar{\xi} \in \mathcal{W}$, $v, \bar{v} \in \mathbb{R}$.

(HF1) For every $q > 0$, we can find a constant $\overline{\mathcal{M}}_F > 0$ such that

$$|F(t, \omega_{t_2}, v) - F(t, \omega_{t_1}, \bar{v})| \leq \overline{\mathcal{M}}_F(q) |t_2 - t_1| + \widetilde{\mathcal{M}}_F |v - \bar{v}|, \quad t, t_1, t_2 \in I, v, \bar{v} \in \mathbb{R},$$

for all function $\omega : (-\infty, Z] \rightarrow \mathbb{R}$ such that $\omega_0 = \chi \in \mathcal{B}$, $\omega : [0, Z] \rightarrow \mathbb{R}$ is continuous and $\max_{0 \leq s \leq Z} \|\omega(s)\| \leq q$.

(HS) The function $\kappa : I \times \mathcal{W} \rightarrow [0, \infty)$ is such that

- (i) the function $t \mapsto \kappa(t, \chi)$ is continuous for every $\chi \in \mathcal{W}$;
- (ii) there exists a constant $\mathcal{M}_\kappa > 0$ such that

$$|\kappa(t, u) - \kappa(t, \bar{u})| \leq \mathcal{M}_\kappa \|u - \bar{u}\|_{\mathcal{W}}, \quad u, \bar{u} \in \mathcal{W}, \quad \forall t \in I.$$

(HH) The function $H : I \times \mathcal{W} \rightarrow \mathbb{R}$ is continuous and $\exists$ a constant $\mathcal{M}_H > 0$ such that

$$|H(t, u) - H(t, \bar{u})| \leq \mathcal{M}_H \|u - \bar{u}\|_{\mathcal{W}}, \quad \text{for each } t \in I, u, \bar{u} \in \mathcal{W}.$$

(HI) The functions $L_p : \mathcal{W} \rightarrow \mathbb{R}$, $p = 1, 2, \dots, \ell$ are continuous and $\exists$ a constant $\mathcal{M}_{L_p} > 0$ such that

$$|L_p(u) - L_p(\bar{u})| \leq \mathcal{M}_{L_p} \|u - \bar{u}\|_{\mathcal{W}}, \quad \text{for all } u, \bar{u} \in \mathcal{W}$$

and $\mathcal{M}_L = \max\{\mathcal{M}_{L_1}, \mathcal{M}_{L_2}, \dots, \mathcal{M}_{L_\ell}\}$.

Theorem 3.1 Suppose F, κ, H and L_p , $p = 1, 2, \dots, \ell$ are satisfy the conditions (HF), (HF1), (HS), (HH) and (HI). If

$$\Lambda_1 = \left[\eta_2 \left(\mathcal{M}_H + \sum_{p=1}^{\ell} \mathcal{M}_{L_p} + \frac{(\mathcal{M}_F + \overline{\mathcal{M}}_F(Q) \mathcal{M}_\kappa)}{\rho(1 - 2\widetilde{\mathcal{M}}_F)} \{\ell A^\rho + B^\rho\} \right) \right] < 1, \quad (3.1)$$

where $A = \max\{t_1 - t_0, t_2 - t_1, t_3 - t_2, \dots, t_p - t_{p-1}, p = 1, 2, \dots, \ell\}$ and $B = \max\{t - t_1, t - t_2, \dots, t - t_p\}$, then the system (1.1) has a unique solution on $(-\infty, Z]$.

Proof. Define the operator $\Upsilon : \Theta \rightarrow \Theta$ by

$$(\Upsilon\omega)(t) = \begin{cases} \xi(0) + H(t, \omega_t) + \int_0^t \sigma^{\rho-1} \Xi(\sigma) d\sigma, & t \in [0, t_1], \\ \xi(0) + H(t, \omega_t) + \sum_{p=1}^{\ell} L_p(\omega_{t_p^-}) + \sum_{p=1}^{\ell} \int_{t_{p-1}}^{t_p} \sigma^{\rho-1} \Xi(\sigma) d\sigma \\ + \int_{t_p}^t \sigma^{\rho-1} \Xi(\sigma) d\sigma, & t \in (t_p, t_{p+1}], p = 1, 2, \dots, \ell, \\ \xi(t), & t \in (-\infty, 0]. \end{cases} \quad (3.2)$$

Now, we show that $\Upsilon B_Q \subset B_Q$. To do this, let $\max_{t \in I} F(t, 0, 0) = \mathcal{M}_F^*$; $\max_{t \in I} H(t, 0) = \mathcal{M}_H^*$; $\max\{L_p(0), p = 1, 2, \dots, \ell\} = \mathcal{M}_L^*$ and let $B_Q = B(0, Q) = \{\omega \in \Theta : \|\omega\|_{\Theta} \leq Q, 0 \leq t \leq Z\}$ be the ball centered at the origin with radius

$$Q > \frac{\Omega_1}{1 - \left(\mathcal{M}_H \eta_2 + \frac{\mathcal{M}_F \eta_2 t_1^{\rho}}{\rho(1 - \widetilde{\mathcal{M}}_F)} \right)}. \quad (3.3)$$

For each $t \in [0, t_1]$ and $\omega \in B_Q$, we sustain

$$|(\Upsilon\omega)(t)| = \left| \xi(0) + H(t, \omega_t) + \int_0^t \sigma^{\rho-1} \Xi(\sigma) d\sigma \right| \leq |\xi(0)| + |H(t, \omega_t)| + \int_0^t \sigma^{\rho-1} |\Xi(\sigma)| d\sigma. \quad (3.4)$$

Note 1 From the phase space axioms and conditions (HF)–(HI), we have the following estimations:

$$|\xi(0)| \leq \eta_3 \|\xi\|_{\mathcal{W}}, \quad (3.5)$$

$$\begin{aligned} |H(t, \omega_t)| &\leq |H(t, \omega_t) - H(t, 0)| + |H(t, 0)| = \mathcal{M}_H \|\omega_t\|_{\mathcal{W}} + \mathcal{M}_H^* \\ &\leq \mathcal{M}_H \left[\eta_1 \|\xi\|_{\mathcal{W}} + \eta_2 \sup_{0 \leq \psi \leq t} |\omega(\psi)| \right] + \mathcal{M}_H^* \\ &\leq \mathcal{M}_H [\eta_1 \|\xi\|_{\mathcal{W}} + \eta_2 Q] + \mathcal{M}_H^* \\ &= \mathcal{M}_H \eta_1 \|\xi\|_{\mathcal{W}} + \mathcal{M}_H \eta_2 Q + \mathcal{M}_H^*, \end{aligned} \quad (3.6)$$

$$\begin{aligned} |\Xi(t)| &\leq |F(t, \omega_{\kappa(t, \omega_t)}, \Xi(t)) - F(t, 0, 0)| + |F(t, 0, 0)| \\ &\leq \mathcal{M}_F \|\omega_{\kappa(t, \omega_t)}\|_{\mathcal{W}} + \widetilde{\mathcal{M}}_F |\Xi(t)| + \mathcal{M}_F^* \\ &\leq \mathcal{M}_F \left[\eta_2 \max_{0 \leq \tau \leq \kappa(t, \omega_t)} |\omega(\tau)| + \eta_1 \|\xi\|_{\mathcal{W}} \right] + \widetilde{\mathcal{M}}_F |\Xi(t)| + \mathcal{M}_F^* \\ &= \mathcal{M}_F \eta_1 \|\xi\|_{\mathcal{W}} + \mathcal{M}_F \eta_2 Q + \widetilde{\mathcal{M}}_F |\Xi(t)| + \mathcal{M}_F^* \\ &\leq \frac{\mathcal{M}_F (\eta_1 \|\xi\|_{\mathcal{W}} + \eta_2 Q) + \mathcal{M}_F^*}{1 - \widetilde{\mathcal{M}}_F}. \end{aligned} \quad (3.7)$$

By incorporating the results of equations (3.5)–(3.7), equation (3.4) is transformed into the following expression:

$$|(\Upsilon\omega)(t)| \leq \eta_3 \|\xi\|_{\mathcal{W}} + \mathcal{M}_H \eta_1 \|\xi\|_{\mathcal{W}} + \mathcal{M}_H \eta_2 Q + \mathcal{M}_H^* + \frac{\mathcal{M}_F (\eta_1 \|\xi\|_{\mathcal{W}} + \eta_2 Q) + \mathcal{M}_F^*}{1 - \widetilde{\mathcal{M}}_F} \int_0^t \sigma^{\rho-1} d\sigma$$

$$\leq \Omega_1 + \left(\mathcal{M}_H \eta_2 + \frac{\mathcal{M}_F \eta_2 t_1^\rho}{\rho(1 - \widetilde{\mathcal{M}}_F)} \right) Q \leq Q,$$

where

$$\Omega_1 = (\mathcal{M}_H \eta_1 + \eta_3) \|\xi\|_{\mathcal{W}} + \mathcal{M}_H^* + \frac{(\mathcal{M}_F \eta_1 \|\xi\|_{\mathcal{W}} + \mathcal{M}_F^*) t_1^\rho}{\rho(1 - \widetilde{\mathcal{M}}_F)}.$$

Further, for each $t \in I_p$, $p = 1, 2, \dots, \ell$ and $\omega \in B_Q$, we obtain

$$\begin{aligned} |(\Upsilon\omega)(t)| &\leq |\xi(0)| + |H(t, \omega_t)| + \left| \sum_{p=1}^{\ell} L_p(\omega_{t_p}^-) \right| + \sum_{p=1}^{\ell} \int_{t_{p-1}}^{t_p} \sigma^{\rho-1} |\Xi(\sigma)| d\sigma + \int_{t_p}^t \sigma^{\rho-1} |\Xi(\sigma)| d\sigma \\ &\leq \Omega_2 + \left(\mathcal{M}_H \eta_2 + \eta_2 \sum_{p=1}^{\ell} \mathcal{M}_{L_p} + \frac{\mathcal{M}_F \eta_2}{\rho(1 - \widetilde{\mathcal{M}}_F)} \{\ell A^\rho + B^\rho\} \right) Q, \end{aligned}$$

where

$$\Omega_2 = \left(\eta_3 + \mathcal{M}_H \eta_1 + \sum_{p=1}^{\ell} \mathcal{M}_{L_p} \eta_1 \right) \|\xi\|_{\mathcal{W}} + \mathcal{M}_H^* + \ell \mathcal{M}_L^* + \frac{(\mathcal{M}_F \eta_1 \|\xi\|_{\mathcal{W}} + \mathcal{M}_F^*)}{\rho(1 - \widetilde{\mathcal{M}}_F)} \{\ell A^\rho + B^\rho\}.$$

Then for all $t \in I$, and $\omega \in B_Q$, we have

$$\|\Upsilon\omega\|_{\Theta} \leq \Omega_2 + \eta_2 \left(\mathcal{M}_H + \sum_{p=1}^{\ell} \mathcal{M}_{L_p} + \frac{\mathcal{M}_F}{\rho(1 - \widetilde{\mathcal{M}}_F)} \{\ell A^\rho + B^\rho\} \right) Q \leq Q.$$

This proves that Υ transforms the ball B_Q into itself. Next, for $\omega, \bar{\omega} \in \mathcal{PC}$ and $t \in [0, t_1]$, we sustain

$$|(\Upsilon\omega)(t) - (\Upsilon\bar{\omega})(t)| \leq |H(t, \omega_t) - H(t, \bar{\omega}_t)| + \int_0^t \sigma^{\rho-1} |\Xi(\sigma) - \bar{\Xi}(\sigma)| d\sigma, \quad (3.8)$$

where $\Xi, \bar{\Xi} \in \mathcal{C}(I, \mathbb{R})$ are such that

$$\Xi(t) = F(t, \omega_{\kappa(t, \omega_t)}, \Xi(t)) \quad \text{and} \quad \bar{\Xi}(t) = F(t, \bar{\omega}_{\kappa(t, \bar{\omega}_t)}, \bar{\Xi}(t)).$$

From (HH) and phase space axioms, we have

$$|H(t, \omega_t) - H(t, \bar{\omega}_t)| \leq \mathcal{M}_H \|\omega_t - \bar{\omega}_t\|_{\mathcal{W}} \leq \mathcal{M}_H \eta_2 \max_{0 \leq \tau \leq t} \|\omega(\tau) - \bar{\omega}(\tau)\|. \quad (3.9)$$

In view of (HF), (HF1) and (HS), we have

$$\begin{aligned} |\Xi(t) - \bar{\Xi}(t)| &= |F(t, \omega_{\kappa(t, \omega_t)}, \Xi(t)) - F(t, \bar{\omega}_{\kappa(t, \bar{\omega}_t)}, \bar{\Xi}(t))| \\ &\leq |F(t, \omega_{\kappa(t, \omega_t)}, \Xi(t)) - F(t, \bar{\omega}_{\kappa(t, \omega_t)}, \bar{\Xi}(t))| + |F(t, \bar{\omega}_{\kappa(t, \omega_t)}, \Xi(t)) - F(t, \bar{\omega}_{\kappa(t, \bar{\omega}_t)}, \bar{\Xi}(t))| \\ &\leq \mathcal{M}_F \|\omega_{\kappa(t, \omega_t)} - \bar{\omega}_{\kappa(t, \omega_t)}\|_{\mathcal{W}} + \widetilde{\mathcal{M}}_F |\Xi(t) - \bar{\Xi}(t)| + \overline{\mathcal{M}}_F(Q) |\kappa(t, \omega_t) - \kappa(t, \bar{\omega}_t)| \\ &\quad + \widetilde{\mathcal{M}}_F |\Xi(t) - \bar{\Xi}(t)| \\ &\leq \mathcal{M}_F \eta_2 \max_{0 \leq \tau \leq t} \|\omega(\tau) - \bar{\omega}(\tau)\| + \overline{\mathcal{M}}_F(Q) \mathcal{M}_{\kappa} \|\omega_t - \bar{\omega}_t\|_{\mathcal{W}} + 2\widetilde{\mathcal{M}}_F |\Xi(t) - \bar{\Xi}(t)| \\ &\leq \mathcal{M}_F \eta_2 \max_{0 \leq \tau \leq t} \|\omega(\tau) - \bar{\omega}(\tau)\| + \overline{\mathcal{M}}_F(Q) \mathcal{M}_{\kappa} \eta_2 \max_{0 \leq \tau \leq t} \|\omega(\tau) - \bar{\omega}(\tau)\| + 2\widetilde{\mathcal{M}}_F |\Xi(t) - \bar{\Xi}(t)| \\ &\leq \frac{\eta_2 (\mathcal{M}_F + \overline{\mathcal{M}}_F(Q) \mathcal{M}_{\kappa})}{1 - 2\widetilde{\mathcal{M}}_F} \max_{0 \leq \tau \leq t} \|\omega(\tau) - \bar{\omega}(\tau)\|. \end{aligned} \quad (3.10)$$

Utilizing (3.9) and (3.10) in (3.8), we have

$$|(\Upsilon\omega)(t) - (\Upsilon\bar{\omega})(t)| \leq \eta_2 \left(\mathcal{M}_H + \frac{(\mathcal{M}_F + \overline{\mathcal{M}}_F(Q)\mathcal{M}_\kappa)t_1^\rho}{\rho(1 - 2\widetilde{\mathcal{M}}_F)} \right) \max_{0 \leq \tau \leq t} \|\omega(\tau) - \bar{\omega}(\tau)\|.$$

Further, for each $t \in I_p$, $p = 1, 2, \dots, \ell$ and $\omega \in B_Q$, we obtain

$$\begin{aligned} |(\Upsilon\omega)(t) - (\Upsilon\bar{\omega})(t)| &\leq \mathcal{M}_H |\omega_t - \bar{\omega}_t|_{\mathcal{W}} + \sum_{p=1}^{\ell} |L_p(\omega(t_p^-)) - L_p(\bar{\omega}(t_p^-))| \\ &\quad + \sum_{p=1}^{\ell} \int_{t_{p-1}}^{t_p} \sigma^{\rho-1} |\Xi(\sigma) - \bar{\Xi}(\sigma)| d\sigma + \int_{t_p}^t \sigma^{\rho-1} |\Xi(\sigma) - \bar{\Xi}(\sigma)| d\sigma \\ &\leq \eta_2 \left(\mathcal{M}_H + \sum_{p=1}^{\ell} \mathcal{M}_{L_p} + \frac{(\mathcal{M}_F + \overline{\mathcal{M}}_F(Q)\mathcal{M}_\kappa)}{\rho(1 - 2\widetilde{\mathcal{M}}_F)} \{\ell A^\rho + B^\rho\} \right) \max_{0 \leq \tau \leq t} \|\omega(\tau) - \bar{\omega}(\tau)\|. \end{aligned}$$

Thus, for all $t \in I$, we obtain

$$\begin{aligned} \|\Upsilon\omega - \Upsilon\bar{\omega}\|_{\Theta} &\leq \eta_2 \left(\mathcal{M}_H + \sum_{p=1}^{\ell} \mathcal{M}_{L_p} + \frac{(\mathcal{M}_F + \overline{\mathcal{M}}_F(Q)\mathcal{M}_\kappa)}{\rho(1 - 2\widetilde{\mathcal{M}}_F)} \{\ell A^\rho + B^\rho\} \right) \|\omega - \bar{\omega}\|_{\Theta} \\ &\leq \Lambda_1 \|\omega - \bar{\omega}\|_{\Theta} \end{aligned}$$

which implies that Υ is a contraction. The fixed point $\omega(\cdot)$ of Υ is the unique solution of system (1.1) in B_Q . ■

4 Example

Consider the following impulsive conformable fractional infinite delay system

$$\begin{aligned} {}^{CFD}D_{t_p}^{\frac{1}{2}} \left[\omega(t) - \frac{t\omega_t e^{-\kappa t+t}}{180(e^t - e^{-t})(1 + |\omega_t|)} \right] &= \frac{\omega(t) - \tau(\omega(t))e^{-\kappa t+t}}{190(e^t - e^{-t})(1 + |\omega(t - \tau(\omega(t)))|)} \\ &\quad + \frac{\omega(t)e^{-\kappa t+t}}{256(e^t - e^{-t}) \left(1 + \left| {}^{CFD}D_{t_p}^{\frac{1}{2}} \omega(t) \right| \right)}, \\ t &\in \left[0, \frac{1}{2} \right] \cup \left(\frac{1}{2}, 1 \right], \end{aligned} \quad (4.1)$$

$$\omega \left(\frac{1}{2}^+ \right) - \omega \left(\frac{1}{2}^- \right) = \frac{|\omega_{\frac{1}{2}}^-|}{20 + |\omega_{\frac{1}{2}}^-|}, \quad (4.2)$$

$$\omega(t) = t + 1, \quad t \in (-\infty, 0], \quad (4.3)$$

where $\tau \in \mathcal{C}(\mathbb{R}, [0, 1])$. Let $\bar{\kappa} > 0$ and

$$C_{\bar{\kappa}} = \left\{ \omega \in \mathcal{PC}((-\infty, 1], \mathbb{R}) : \lim_{\lambda \rightarrow -\infty} e^{\bar{\kappa}\lambda} \omega(\lambda) \text{ exists in } \mathbb{R} \right\}.$$

The norm of $C_{\bar{\kappa}}$ is defined as follows:

$$\|\omega\|_{\bar{\kappa}} = \sup_{\lambda \in (-\infty, 1]} e^{\bar{\kappa}\lambda} |\omega(\lambda)|.$$

It is easy to verify that $C_{\bar{\kappa}}$ satisfies the axioms of a phase space. Consequently, we can establish $C_{\bar{\kappa}}$ as a phase space itself.

Set

$$\begin{aligned}\kappa(t, \xi) &= t - \tau(\xi(0)), \quad (t, \xi) \in [0, 1] \times \mathcal{C}((-\infty, 0], \mathbb{R}); \\ F(t, u, v) &= \frac{e^{-\bar{\kappa}t+t}}{190(e^t - e^{-t})(1 + \|u\|_{C_{\bar{\kappa}}})} + \frac{e^{-\bar{\kappa}t+t}}{256(e^t - e^{-t})(1 + |v|)}, \quad t \in [0, 1], u \in C_{\bar{\kappa}}, v \in \mathbb{R}; \\ H(t, u) &= \frac{te^{-\bar{\kappa}t+t}}{180(e^t - e^{-t})(1 + \|u\|_{C_{\bar{\kappa}}})}, \quad t \in [0, 1], u \in C_{\bar{\kappa}}.\end{aligned}$$

It is important to note that $H(0, u) = 0$ for any $u \in C_{\bar{\kappa}}$. For any $u, \bar{u} \in C_{\bar{\kappa}}, v, \bar{v} \in \mathbb{R}$ and $t \in [0, 1]$, we have

$$\begin{aligned}|F(t, u, v) - F(t, \bar{u}, \bar{v})| &\leq \frac{1}{190}\|u - \bar{u}\|_{C_{\bar{\kappa}}} + \frac{1}{256}|v - \bar{v}|; \\ |H(t, u) - H(t, \bar{u})| &\leq \frac{1}{180}\|u - \bar{u}\|_{C_{\bar{\kappa}}}; \\ |L_1(u) - L_1(\bar{u})| &\leq \frac{1}{20}\|u - \bar{u}\|_{C_{\bar{\kappa}}}.\end{aligned}$$

Hence, assumptions (HF), (HH) and (HI) hold with $\mathcal{M}_F = \frac{1}{190}$, $\widetilde{\mathcal{M}}_F = \frac{1}{256}$, $\mathcal{M}_H = \frac{1}{180}$, and $\mathcal{M}_{L_1} = \frac{1}{20}$. Set $\rho = \frac{1}{2}$, $\ell = Z = \eta_2 = \mathcal{M}_{\kappa} = 1$ and $\overline{\mathcal{M}}_F(Q) = \frac{1}{190}$. At $t_1 = \frac{1}{2}$, then $A = \frac{1}{2}$ and $B = \frac{1}{2}$.

Furthermore

$$\Lambda_1 = \eta_2 \left(\mathcal{M}_H + \sum_{p=1}^{\ell} \mathcal{M}_{L_p} + \frac{(\mathcal{M}_F + \overline{\mathcal{M}}_F(Q)\mathcal{M}_{\kappa})}{\rho(1 - 2\widetilde{\mathcal{M}}_F)} \{\ell A^{\rho} + B^{\rho}\} \right) = 0.0856 < 1.$$

Hence, in view of Theorem 3.1, the given impulsive conformable fractional system (4.1)–(4.3) has a unique solution on $(-\infty, 1]$.

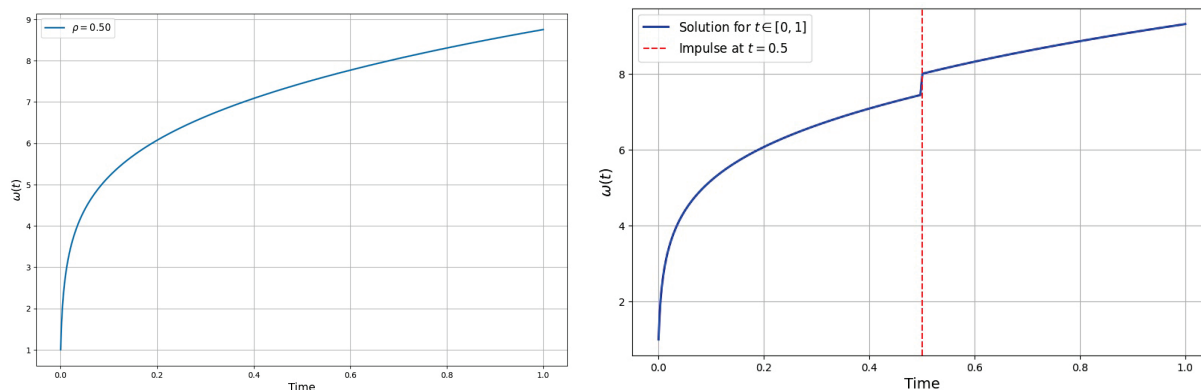


Figure 1: Trajectory of the solution (4.1)–(4.3) with $\rho = 0.5$ and $L_1 = 0.5$.

Figure 1 illustrates the solution of the system (4.1)–(4.3) without and with impulses for $\rho = 0.5$. The red dashed line indicates the impulsive jump applied at $t = 0.5$.

Figure 2 shows the solutions $\omega(t)$ of the system (4.1)–(4.3) for different fractional orders $\rho = 0.8, 0.85, 0.90, 0.95$, with impulsive jumps at $t = 0.25, 0.5$, and 0.75 . The solutions exhibit distinct characteristics influenced by impulsive jumps, fractional orders, and smooth dynamics between impulses. The red dashed lines at $t = 0.25, 0.5$, and 0.75 mark the locations of impulsive jumps, each causing a sudden increase in $\omega(t)$, with the magnitude dictated by the impulsive condition. Fractional orders significantly impact the growth rate, with lower order ($\rho = 0.8$) resulting in slower overall growth, while higher order

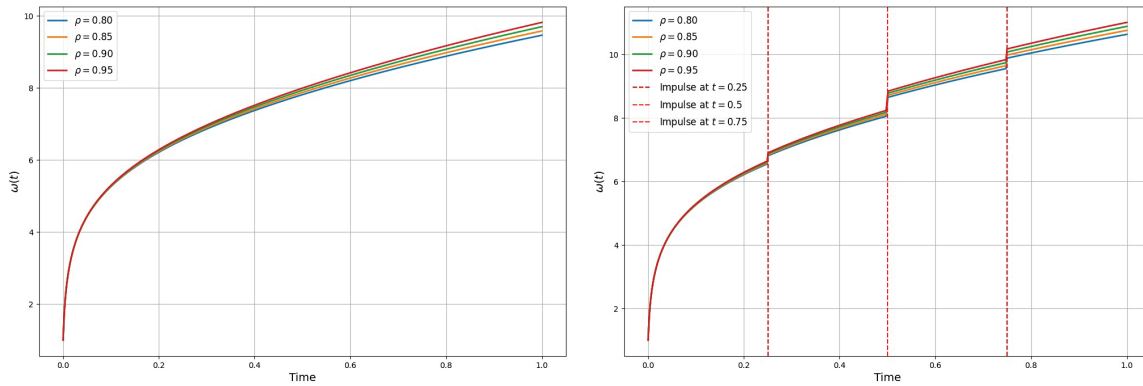


Figure 2: Trajectory of the solution (4.1)–(4.3) with impulses at $L_1 = 0.25$, $L_2 = 0.5$, $L_3 = 0.75$ and different fractional orders $\rho = 0.8, 0.85, 0.90, 0.95$.

($\rho = 0.95$) demonstrate faster growth due to stronger memory effects. Between these impulses, the solution evolves smoothly, driven by the interplay of the conformable fractional derivative and state-dependent delay terms, reflecting the complex dynamics of the system.

5 Conclusion

In conclusion, this paper thoroughly examines the existence and uniqueness of solutions for nonlinear INFDS with SDD and IPs. Using the computational benefits of $\mathcal{CFD}$, we ensured the results are both practical and maintain important mathematical properties. The analysis was based on local Lipschitz continuity of the functions, and the existence and uniqueness were proven using a contractive mapping in Theorem 3.1.

The findings were further substantiated with an illustrative example, which highlight the dynamic interplay between $\mathcal{CFD}$, SDD, and IPs. The example figures reveal key insights: the impact of impulses at specified points, the influence of varying fractional orders (ρ), and the smooth transitions between impulses. Lower fractional orders exhibit slower dynamics, while higher orders demonstrate rapid growth due to stronger memory effects, providing a deeper understanding of fractional-order systems with complex interactions.

This work provides a strong foundation for future research on models with non-instantaneous impulses, controllability, and stability. Using advanced fixed-point theorems in these areas could greatly improve how fractional systems are applied to model real-world problems.

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