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Classical Solutions For A Class Of Primitive Equations*

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Abstract

In this article we investigate an IVP for a class of primitive equations. We propose a new topological approach to prove the existence of at least one classical solution and at least two nonnegative classical solutions. The arguments are based upon recent theoretical results.

1 Introduction

In this paper we investigate the Cauchy problem for the primitive equations of the nemo ocean model

$$\begin{aligned}
&\tfrac{\partial p}{\partial x_3} = pg(t,x,p,U,T,S,\rho),\\
&\nabla\cdot U = 0,\\
&\tfrac{\partial T}{\partial t} = -\nabla\cdot(TU) + D^T(t,x,p,U,T,S,\rho) + F^T(t,x,p,U,T,S,\rho),\\
&\tfrac{\partial S}{\partial t} = -\nabla\cdot(SU) + D^S(t,x,p,U,T,S,\rho) + F^S(t,x,p,U,T,S,\rho),\\
&\rho = TSp, \quad t>0, \quad x=(x_1,x_2,x_3)\in\mathbb{R}^3,\\
&p(0,x) = p_0(x), \quad x\in\mathbb{R}^3,\\
&U(0,x) = U_0(x), \quad x\in\mathbb{R}^3,\\
&T(0,x) = T_0(x), \quad x\in\mathbb{R}^3,\\
&S(0,x) = S_0(x), \quad x\in\mathbb{R}^3,
\end{aligned} \tag{1}$$

where p is the pressure, T is the potential temperature, S is the salinity, ρ is the in sity density, g is the gravitational acceleration, D^T and D^S are the parameterization of small-scale physics for temperature and salinity, respectively, F^T and F^S are surface forcing terms, $\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$, t is the time.

The ocean component of NEMO has been developed from the OPA model. This model has been used for a wide range of applications, both regional or global, as a forced ocean model and as a model coupled with the sea-ice and/or athmosphere.

Efficient numerical solvers for the primitive equations of the ocean are investigated in [8]. The authors derive numerical approximation of this model by the orthogonal sub-scales method via finite elements discretization. Numerical investigations are done in [1]. Classical solutions for a class of Hamilton-Jacobi equations, a class of Broer-Kaup equations and a class of incompressible incompressible Navier-Stokes equations are obtained in [3]. To the best of our knowledge the mathematical studies for existence of classical solutions for the IVP (1) are not known. Our aim in this paper is to ensure conditions for the parameters of (1) so that (1) has at least one and at least two nonnegative classical solutions.

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(H1) $p_0, U_0, T_0, S_0 \in \mathcal{C}^1(\mathbb{R}^3)$, $0 \le p_0, U_0, T_0, S_0 \le B$ on $\mathbb{R}^3$ for some positive constant B.

(H2) $g, D^T, F^T, D^S, F^S \in \mathcal{C}([0,\infty) \times \mathbb{R}^8)$,

$$
\begin{aligned}
0 &\le |g(t,x,u_1,u_2,u_3,u_4,u_5)| \le a_0(t,x) + \sum_{k=1}^{5} a_k(t,x)|u_k|^{p_k},\\
0 &\le |D^T(t,x,u_1,u_2,u_3,u_4,u_5)| \le b_0(t,x) + \sum_{k=1}^{5} b_k(t,x)|u_k|^{q_k},\\
0 &\le |F^T(t,x,u_1,u_2,u_3,u_4,u_5)| \le c_0(t,x) + \sum_{k=1}^{5} c_k(t,x)|u_k|^{r_k},\\
0 &\le |D^S(t,x,u_1,u_2,u_3,u_4,u_5)| \le d_0(t,x) + \sum_{k=1}^{5} d_k(t,x)|u_k|^{s_k},\\
0 &\le |F^S(t,x,u_1,u_2,u_3,u_4,u_5)| \le e_0(t,x) + \sum_{k=1}^{5} e_k(t,x)|u_k|^{p_k},
\end{aligned}
$$

$(t,x,u_1,u_2,u_3,u_4,u_5) \in [0,\infty) \times \mathbb{R}^8$, $a_j, b_j, c_j, d_j, e_j \in \mathcal{C}([0,\infty) \times \mathbb{R}^3)$, $0 \le a_j, b_j, c_j, d_j, e_j \le B$, $j \in \{0,1,2,3,4,5\}$, on $[0,\infty) \times \mathbb{R}^3$.

In addition, we suppose

(H3) there exist a nonnegative function $g \in \mathcal{C}([0,\infty) \times \mathbb{R}^3)$, $g > 0$ on $(0,\infty) \times (\mathbb{R}^3 \backslash (\{(0,x_2,x_3)\} \cup \{(x_1,0,x_3)\} \cup \{(x_1,x_2,0)\}))$, $x_1, x_2, x_3 \in \mathbb{R}$, and a positive constant A such that

$$
2^3(1+t)\prod_{j=1}^{3}(1+|x_j|)\int_0^t \left|\int_0^x g(t_1,s)ds\right| dt_1 \le A, \quad (t,x) \in J \times \mathbb{R}^n,
$$

where $\int_0^x = \int_0^{x_1}\int_0^{x_2}\int_0^{x_3}$, $ds = ds_3 ds_2 ds_1$.

In the last section we will give an example for functions and constants that satisfy $(H1)$–$(H3)$.
Our main results for existence of classical solutions of the IVP (1) are as follows.

Theorem 1 *Suppose (H1)–(H3). Then the IVP (1) has at least one solution* $u \in \mathcal{C}^1([0,\infty) \times \mathbb{R}^3)$.

Theorem 2 *Suppose (H1)–(H3). Then the IVP (1) has at least two nonnegative solutions* $u_1, u_2 \in \mathcal{C}([0,\infty) \times \mathbb{R}^3)$.

The paper is organized as follows. In the next section, we give some auxiliary results. In Section 3 we prove Theorem 1. In Section 4, we prove Theorem 2. In Section 5, we give an example to illustrate our main results. A conclusion is done in Section 6.

2 Preliminary Results

Below, assume that X is a real Banach space. Now, we will recall the definitions of compact and completely continuous mappings in Banach spaces.

Definition 1 *Let $K : M \subset X \to X$ be a map. We say that K is compact if $K(M)$ is contained in a compact subset of X. K is called a completely continuous map if it is continuous and it maps any bounded set into a relatively compact set.*

To prove our existence result we will use the following fixed point theorem. Its proof can be found in [4] or [5].

Theorem 3 *Let E be a Banach space, Y a closed, convex subset of E, U be any open subset of Y with $0 \in U$. Consider two operators T and S, where*

$$Tx = \varepsilon\, x,\ x \in \overline{U},$$

for $\varepsilon > 1$ and $S : \overline{U} \to E$ be such that

(i) $I - S : \overline{U} \to Y$ continuous, compact and

(ii) $\{x \in \overline{U} : x = \lambda(I - S)x, \quad x \in \partial U\} = \emptyset$, for any $\lambda \in \left(0, \frac{1}{\varepsilon}\right)$.

Then there exists $x^* \in \overline{U}$ such that

$$Tx^* + Sx^* = x^*.$$

Definition 2 *Let X and Y be real Banach spaces. A map $K : X \to Y$ is called expansive if there exists a constant $h > 1$ for which one has the following inequality*

$$\|Kx - Ky\|_Y \geq h\|x - y\|_X$$

for any $x, y \in X$.

Now, we will recall the definition for a cone in a Banach space.

Definition 3 *A closed, convex set $\mathcal{P}$ in X is called a cone if*

1. *$\alpha x \in \mathcal{P}$ for any $\alpha \geq 0$ and for any $x \in \mathcal{P}$,*
2. *$x, -x \in \mathcal{P}$ implies $x = 0$.*

Denote $\mathcal{P}^* = \mathcal{P}\backslash\{0\}$. The next result is a fixed point theorem which we will use to prove existence of at least two nonnegative global classical solutions of the IVP (1). For its proof, we refer the reader to [2] and [6].

Theorem 4 *Let $\mathcal{P}$ be a cone of a Banach space E; Ω a subset of $\mathcal{P}$ and U_1, U_2 and U_3 three open bounded subsets of $\mathcal{P}$ such that $\overline{U}_1 \subset \overline{U}_2 \subset U_3$ and $0 \in U_1$. Assume that $T : \Omega \to \mathcal{P}$ is an expansive mapping, $S : \overline{U}_3 \to E$ is a completely continuous map and $S(\overline{U}_3) \subset (I - T)(\Omega)$. Suppose that $(U_2 \setminus \overline{U}_1) \cap \Omega \neq \emptyset$, $(U_3 \setminus \overline{U}_2) \cap \Omega \neq \emptyset$, and there exists $u_0 \in \mathcal{P}^*$ such that the following conditions hold:*

(i) $Sx \neq (I - T)(x - \lambda u_0)$, for all $\lambda > 0$ and $x \in \partial U_1 \cap (\Omega + \lambda u_0)$,

(ii) there exists $\epsilon \geq 0$ such that $Sx \neq (I - T)(\lambda x)$ for all $\lambda \geq 1 + \epsilon$, $x \in \partial U_2$ and $\lambda x \in \Omega$,

(iii) $Sx \neq (I - T)(x - \lambda u_0)$, for all $\lambda > 0$ and $x \in \partial U_3 \cap (\Omega + \lambda u_0)$.

Then $T + S$ has at least two non-zero fixed points $x_1, x_2 \in \mathcal{P}$ such that

$$x_1 \in \partial U_2 \cap \Omega \text{ and } x_2 \in (\overline{U}_3 \setminus \overline{U}_2) \cap \Omega$$

or

$$x_1 \in (U_2 \setminus U_1) \cap \Omega \text{ and } x_2 \in (\overline{U}_3 \setminus \overline{U}_2) \cap \Omega.$$

For convenience, we set $X_1 = \mathcal{C}^1([0, \infty) \times \mathbb{R}^3)$ and suppose that it is endowed with the norm

$$\begin{aligned} \|u\|_1 \quad = \quad \sup\Bigg\{ & \sup_{(t,x)\in[0,\infty)\times\mathbb{R}^3} |u(t, x)|, \quad \sup_{(t,x)\in[0,\infty)\times\mathbb{R}^3} |u_{x_j}(t, x)|, \\ & \sup_{(t,x)\in[0,\infty)\times\mathbb{R}^3} |u_t(t, x)|, \quad j \in \{1, 2, 3\}\Bigg\}, \end{aligned}$$

provided it exists. Let $X = X_1^5$ be endowed with the norm

$$\|(u_1, u_2, u_3, u_4, u_5)\| = \max\{\|u_j\|_1, \quad j \in \{1, 2, 3, 4, 5\}\}.$$

3 Proof of Theorem 1

For $(p, U, T, S, \rho) \in X$, define the operators

$$\begin{aligned} S_1^1(p,U,T,S,\rho)(t,x) &= p(t,x) - p_0(x) + \int_0^t \Big(-\frac{\partial p}{\partial t}(s,x) + \frac{\partial p}{\partial x_3}(s,x) \\ &\quad -p(s,x) g(s,x,p(s,x),U(s,x),T(s,x),S(s,x),\rho(s,x)) \Big) ds, \end{aligned}$$

$$S_1^2(p,U,T,S,\rho)(t,x) = U(t,x) - U_0(x) + \int_0^t \left(-\frac{\partial U}{\partial t}(s,x) + \sum_{j=1}^3 U_{x_j}(s,x) \right) ds,$$

$$\begin{aligned} S_1^3(p,U,T,S,\rho) &= T(t,x) - T_0(x) - \int_0^t \Big(-\sum_{j=1}^3 (TU)_{x_j}(s,x) \\ &\quad + D^T(s,x,p(s,x),U(s,x),T(s,x),S(s,x),\rho(s,x)) \\ &\quad + F^T(s,x,p(s,x),U(s,x),T(s,x),S(s,x),\rho(s,x)) \Big) ds, \end{aligned}$$

$$\begin{aligned} S_1^4(p,U,T,S,\rho) &= S(t,x) - S_0(x) - \int_0^t \Big(-\sum_{j=1}^3 (SU)_{x_j}(s,x) \\ &\quad + D^S(s,x,p(s,x),U(s,x),T(s,x),S(s,x),\rho(s,x)) \\ &\quad + F^S(s,x,p(s,x),U(s,x),T(s,x),S(s,x),\rho(s,x)) \Big) ds, \end{aligned}$$

$$\begin{aligned} S_1(p,U,T,S,\rho)(t,x) &= \Big(S_1^1(p,U,T,S,\rho), S_1^2(p,U,T,S,\rho), \\ &\quad S_1^3(p,U,T,S,\rho), S_1^4(p,U,T,S,\rho), O \Big), \end{aligned}$$

for $(t,x) \in [0,\infty) \times \mathbb{R}^3$. Here O is the zero operator. Let

$$\begin{aligned} B_1 &= \max \Bigg\{ 2 \left(2B + B^2 + \sum_{k=1}^4 B^{2+p_k} + B^{2+3p_5} \right), 4B, \\ &\quad \left(2B + 5B^2 + \sum_{k=1}^4 B^{1+q_k} + \sum_{k=1}^4 B^{1+r_k} + B^{1+3q_5} + B^{1+3r_5} \right), \\ &\quad \left(2B + 5B^2 + \sum_{k=1}^4 B^{1+s_k} + \sum_{k=1}^4 B^{1+t_k} + B^{1+3s_5} + B^{1+3t_5} \right) \Bigg\}. \end{aligned}$$

Lemma 1 *Suppose (H1) and (H2). If $(p, U, T, S, \rho) \in X$ satisfies the equation*

$$S_1(p,U,T,S,\rho)(t,x) = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R}^3,$$

then it is a solution to the IVP (1).

Proof. We have

$$S_1^1(p,U,T,S,\rho)(t,x)=0, \quad S_1^2(p,U,T,S,\rho)(t,x)=0,$$

$$S_1^3(p,U,T,S,\rho)(t,x)=0, \quad S_1^4(p,U,T,S,\rho)(t,x)=0, \quad (t,x)\in[0,\infty)\times\mathbb{R}^3.$$

Consider the first equation, i.e.,

$$\begin{aligned} 0 &= p(t,x)-p_0(x)+\int_0^t\Bigg(-\frac{\partial p}{\partial t}(s,x)+\frac{\partial p}{\partial x_3}(s,x) \\ &\quad -p(s,x)g(s,x,p(s,x),U(s,x),T(s,x),S(s,x),\rho(s,x))\Bigg)ds, \quad (t,x)\in[0,\infty)\times\mathbb{R}^3, \end{aligned}$$

which we differentiate with respect to t and we get

$$\begin{aligned} 0 &= \frac{\partial p}{\partial t}(t,x)-\frac{\partial p}{\partial t}(t,x)+\frac{\partial p}{\partial x_3}(t,x) \\ &\quad -p(t,x)g(t,x,p(t,x),U(t,x),T(t,x),S(t,x),\rho(t,x)), \end{aligned}$$

for $(t,x)\in[0,\infty)\times\mathbb{R}^3$, or

$$0=\frac{\partial p}{\partial x_3}(t,x)-p(t,x)g(t,x,p(t,x),U(t,x),T(t,x),S(t,x),\rho(t,x)),$$

for $(t,x)\in[0,\infty)\times\mathbb{R}^3$. Now, we put $t=0$ and we find

$$p(0,x)=p_0(x), \quad x\in\mathbb{R}^3.$$

As above, one can get the other equations of (1). This completes the proof. ■

Lemma 2 *Suppose (H1)–(H2). If* $(p,U,T,S,\rho)\in X$, $\|(p,U,T,S,\rho)\|\le B$, *then*

$$|g(t,x,p(t,x),U(t,x),S(t,x),\rho(t,x))|\le B+\sum_{k=1}^4 B^{1+p_k}+B^{1+3p_5},$$

$$|D^T(t,x,p(t,x),U(t,x),S(t,x),\rho(t,x))|\le B+\sum_{k=1}^4 B^{1+q_k}+B^{1+3q_5},$$

$$|F^T(t,x,p(t,x),U(t,x),S(t,x),\rho(t,x))|\le B+\sum_{k=1}^4 B^{1+r_k}+B^{1+3r_5},$$

$$|D^S(t,x,p(t,x),U(t,x),S(t,x),\rho(t,x))|\le B+\sum_{k=1}^4 B^{1+s_k}+B^{1+3s_5},$$

$$|F^S(t,x,p(t,x),U(t,x),S(t,x),\rho(t,x))|\le B+\sum_{k=1}^4 B^{1+t_k}+B^{1+3t_5},$$

$$|S_1^j u(t,x)|\le B_1(1+t), \quad (t,x)\in[0,\infty)\times\mathbb{R}^n,$$

for $j\in\{1,2,3,4,5\}$.

Proof. We have, using that $\rho = TSp$,

$$\begin{aligned}
& |g(t,x,p(t,x),U(t,x),S(t,x),\rho(t,x))| \\
\leq\ & a_0(t,x) + a_1(t,x)|p(t,x)|^{p_1} + a_2(t,x)|U(t,x)|^{p_2} \\
& + b_3(t,x)|T(t,x)|^{p_3} + b_4(t,x)|S(t,x)|^{p_4} + b_5(t,x)|\rho(t,x)|^{p_5} \\
\leq\ & B + \sum_{k=1}^{4} B^{1+p_k} + B^{1+3p_5}, \quad (t,x) \in [0,\infty) \times \mathbb{R}^3.
\end{aligned}$$

As in above, we get the estimates for D^T, F^T, D^S, F^S. Next,

$$\begin{aligned}
|S_1^1(p,U,T,S,\rho)(t,x)| &= \bigg| p(t,x) - p_0(x) + \int_0^t \bigg(-\frac{\partial p}{\partial t}(s,x) + \frac{\partial p}{\partial x_3}(s,x) \\
&\quad - p(s,x) g(s,x,p(s,x),U(s,x),T(s,x),S(s,x),\rho(s,x)) \bigg) ds \bigg| \\
&\leq |p(t,x)| + p_0(x) + \int_0^t \bigg(\left|\frac{\partial p}{\partial t}(s,x)\right| + \left|\frac{\partial p}{\partial x_3}(s,x)\right| \\
&\quad + |p(s,x)||g(s,x,p(s,x),U(s,x),T(s,x),S(s,x),\rho(s,x))| \bigg) ds \\
&\leq B + B + \int_0^t \left(B + B + B^2 + \sum_{k=1}^{4} B^{2+p_k} + B^{2+3p_5} \right) ds \\
&= 2B + \left(2B + B^2 + \sum_{k=1}^{4} B^{2+p_k} + B^{2+3p_5} \right) t \\
&\leq 2 \left(2B + B^2 + \sum_{k=1}^{4} B^{2+p_k} + B^{2+3p_5} \right) (1+t) \\
&\leq B_1(1+t), \quad (t,x) \in [0,\infty) \times \mathbb{R}^3.
\end{aligned}$$

As in above, one can get the estimates for S_1^2, S_1^3, S_1^4 and S_1^5. This completes the proof. ∎

For $(p,U,T,S,\rho) \in X$, define the operator

$$S_2(p,U,T,S,\rho)(t,x) = \int_0^t \int_0^x (t-t_1) \prod_{j=1}^{3} (x_j - s_j) g(t_1,s) S_1 u(t_1,s) ds dt_1,$$

for $(t,x) \in [0,\infty) \times \mathbb{R}^3$.

Lemma 3 *Suppose (H1)–(H3). If $(p,U,T,S,\rho) \in X$ and $\|(p,U,T,S,\rho)\| \leq B$, then*

$$\|S_2(p,U,T,S,\rho)\| \leq AB_1.$$

Proof. We have

$$\begin{aligned}
|S_2^1(p,U,T,S,\rho)(t,x)| &= \left|\int_0^t\int_0^x (t-t_1)\prod_{j=1}^3 (x_j-s_j)g(t_1,s)S_1^1 u(t_1,s)dsdt_1\right| \\
&\le \int_0^t\left|\int_0^x (t-t_1)\prod_{j=1}^3 |x_j-s_j|g(t_1,s)|S_1^1(p,U,T,S,\rho)(t_1,s)|ds\right|dt_1 \\
&\le B_1(1+t)^2 2^3\prod_{j=1}^3 |x_j|\int_0^t\left|\int_0^x g(t_1,s)ds\right|dt_1 \\
&\le B_1 2^3(1+t)^2\prod_{j=1}^3(1+|x_j|)\int_0^t\left|\int_0^x g(t_1,s)ds\right|dt_1 \\
&\le AB_1, \quad (t,x)\in[0,\infty)\times\mathbb{R}^3,
\end{aligned}$$

$$\begin{aligned}
\left|\frac{\partial}{\partial t}S_2(p,U,T,S,\rho)(t,x)\right| &= \left|\int_0^t\int_0^x \prod_{j=1}^3 (x_j-s_j)g(t_1,s)S_1^1(p,U,T,S,\rho)(t_1,s)dsdt_1\right| \\
&\le \int_0^t\left|\int_0^x \prod_{j=1}^3 |x_j-s_j|g(t_1,s)|S_1^1(p,U,T,S,\rho)(t_1,s)|ds\right|dt_1 \\
&\le B_1(1+t)2^3\prod_{j=1}^3 |x_j|\int_0^t\left|\int_0^x g(t_1,s)ds\right|dt_1 \\
&\le B_1 2^3(1+t)^2\prod_{j=1}^3(1+|x_j|)\int_0^t\left|\int_0^x g(t_1,s)ds\right|dt_1 \\
&\le AB_1, \quad (t,x)\in[0,\infty)\times\mathbb{R}^3,
\end{aligned}$$

and

$$\begin{aligned}
\left|\frac{\partial}{\partial x_k}S_2^1(p,U,T,S,\rho)(t,x)\right| &= \left|\int_0^t\int_0^x (t-t_1)\prod_{j=1,j\neq k}^3 (x_j-s_j)g(t_1,s)S_1^1(p,U,T,S,\rho)(t_1,s)dsdt_1\right| \\
&\le \int_0^t\left|\int_0^x (t-t_1)\prod_{j=1,j\neq k}^3 |x_j-s_j|g(t_1,s)|S_1^1(p,U,T,S,\rho)(t_1,s)|ds\right|dt_1 \\
&\le B_1(1+t)2^3\prod_{j=1,j\neq k}^3 |x_j|\int_0^t\left|\int_0^x g(t_1,s)ds\right|dt_1 \\
&\le B_1 2^3(1+t)^2\prod_{j=1}^n(1+|x_j|)\int_0^t\left|\int_0^x g(t_1,s)ds\right|dt_1 \\
&\le AB_1, \quad (t,x)\in J\times\mathbb{R}^3, \quad k\in\{1,2,3\}.
\end{aligned}$$

As above,

$$\begin{aligned}
\left|S_2^j(p,U,T,S,\rho)(t,x)\right| &\le AB_1, \quad \left|\frac{\partial}{\partial t}S_2^j(p,U,T,S,\rho)(t,x)\right|\le AB_1, \\
\left|\frac{\partial}{\partial x_k}S_2^j(p,U,T,S,\rho)(t,x)\right| &\le AB_1, \quad (t,x)\in[0,\infty)\times\mathbb{R}^3,
\end{aligned}$$

for $j \in \{2,3,4,5\}$ and $k \in \{1,2,3\}$. Thus,

$$\|S_2(p,U,T,S,\rho)\| \leq AB_1.$$

This completes the proof. ■

Lemma 4 *Suppose (H1), (H2) and let $g \in \mathcal{C}([0,\infty) \times \mathbb{R}^3)$ be a nonnegative function. If $(p,U,T,S,\rho) \in X$ satisfies the equation*

$$S_2(p,U,T,S,\rho)(t,x) = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R}^3, \tag{2}$$

then (p,U,T,S,ρ) is a solution to the IVP (1).

Proof. We differentiate two times with respect to t and with respect to x_k, $k \in \{1,2,3\}$, the equation (2) and we find

$$g(t,x)S_1(p,U,T,S,\rho)(t,x) = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R}^3,$$

whereupon

$$S_1(p,U,T,S,\rho)(t,x) = 0,$$

$(t,x) \in (0,\infty) \times (\mathbb{R}^3 \backslash (\{(0,x_2,x_3)\} \cup \{(x_1,0,x_3)\} \cup \{(x_1,x_2,0)\}))$, $x_1,x_2,x_3 \in \mathbb{R}$. Since $S_1(p,U,T,S,\rho)$ is continuous, we have

$$\begin{aligned}
0 &= \lim_{t \to 0} S_1(p,U,T,S,\rho)(t,x) \\
&= S_1(p,U,T,S,\rho)(0,x) \\
&= \lim_{x_1 \to 0} S_1(p,U,T,S,\rho)(t,x_1,x_2,x_3) \\
&= S_1(p,U,T,S,\rho)(t,0,x_2,x_3) \\
&= \lim_{x_2 \to 0} S_1(p,U,T,S,\rho)(t,x_1,x_2,x_3) \\
&= S_1(p,U,T,S,\rho)(t,x_1,0,x_3) \\
&= \lim_{x_3 \to 0} S_1(p,U,T,S,\rho)(t,x_1,x_2,x_3) \\
&= S_1(p,U,T,S,\rho)(t,x_1,x_2,0).
\end{aligned}$$

Consequently

$$S_1(p,U,T,S,\rho)(t,x) = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R}^3.$$

Hence and Lemma 1, we conclude that (p,U,T,S,ρ) is a solution to the IVP (1). This completes the proof. ■

Below, suppose

(H4) $\epsilon > 1$, A and B satisfy the inequalities $\epsilon B_1(1+A) < 1$ and $AB_1 < 1$.

Let $\widetilde{Y}$ denote the set of all equi-continuous families in X with respect to the norm $\|\cdot\|$. Let also, $Y = \overline{\widetilde{Y}}$ and

$$U = \{u \in Y : \|u\| < B \quad \text{and if} \quad \|u\| \geq \frac{B}{2}, \quad \text{then} \quad u(0,x) > \frac{B}{2}, \quad x \in \mathbb{R}^3\}.$$

For $u \in \overline{U}$ and $\epsilon > 1$, define the operators

$$\begin{aligned} T(u)(t,x) &= \epsilon u(t,x), \\ S(u)(t,x) &= u(t,x) - \epsilon u(t,x) - \epsilon S_2(u)(t,x), \quad t \geq 0, \quad x \in \mathbb{R}^3. \end{aligned}$$

For $u \in \overline{U}$, we have

$$\|(I-S)(u)\| = \|\epsilon u + \epsilon S_2(u)\| \leq \epsilon \|u\| + \epsilon \|S_2(u)\| \leq \epsilon B_1 + \epsilon A B_1.$$

Thus, $S : \overline{U} \to X$ is continuous and $(I-S)(\overline{U})$ resides in a compact subset of Y. Now, suppose that there is a $u \in \partial U$ so that

$$u = \lambda (I-S)(u)$$

or

$$u = \lambda \epsilon \left(u + S_2(u)\right),$$

for some $\lambda \in \left(0, \frac{1}{\epsilon}\right)$. Then, using that $S_2(u)(0,x) = 0$ and $\|u\| = B$, we get $u(0,x) > \frac{B}{2}$, $x \in \mathbb{R}^3$, and

$$u(0,x) = \lambda \epsilon (u(0,x) + S_2(u)(0,x)) = \lambda \epsilon u(0,x), \quad x \in \mathbb{R}^3,$$

whereupon $\lambda \epsilon = 1$. This is a contradiction. Consequently

$$\{u \in \overline{U} : u = \lambda_1 (I-S)(u), \ u \in \partial U\} = \emptyset$$

for any $\lambda_1 \in \left(0, \frac{1}{\epsilon}\right)$. Then, from Theorem 3, it follows that the operator $T + S$ has a fixed point $u^* \in Y$. Therefore

$$\begin{aligned} u^*(t,x) &= T(u^*)(t,x) + S(u^*)(t,x) \\ &= \epsilon u^*(t,x) + u^*(t,x) - \epsilon u^*(t,x) - \epsilon S_2(u^*)(t,x), \quad t \geq 0, \quad x \in \mathbb{R}^3, \end{aligned}$$

whereupon

$$S_2(u^*)(t,x) = 0, \quad t \geq 0, \quad x \in \mathbb{R}^3.$$

From here, u^* is a solution to the problem (1). From here and from Lemma 4, it follows that u is a solution to the IVP (1). This completes the proof.

4 Proof of Theorem 2

Let X be the space used in the previous section. Suppose

(H5) Let $m > 0$ be large enough and A, B, r, L, R_1 be positive constants that satisfy the following conditions

$$r < L < R_1, \quad \epsilon > 1, \quad R_1 > \left(\frac{2}{5m} + 1\right) L,$$

$$AB_1 < \frac{L}{5}.$$

For $u = (u_1, u_2, u_3, u_4, u_5) \in X$, we will write $u \geq 0$ if $u_j \geq 0$, $j \in \{1,2,3,4,5\}$. Let

$$\widetilde{P} = \{u = (p, U, T, S, \rho) \in X : u \geq 0 \quad \text{on} \quad [0,\infty) \times \mathbb{R}^3\}.$$

With $\mathcal{P}$ we will denote the set of all equi-continuous families in $\widetilde{P}$. For $v = (p_1, U_1, T_1, S_1, \rho_1) \in X$, define the operators

$$T_1 v(t) = (1 + m\epsilon) v(t) - \left(\epsilon \frac{L}{10}, \epsilon \frac{L}{10}, \epsilon \frac{L}{10}, \epsilon \frac{L}{10}, \epsilon \frac{L}{10}\right),$$

$$S_3v(t) = -\epsilon S_2v(t) - m\epsilon v(t) - \left(\epsilon\frac{L}{10}, \epsilon\frac{L}{10}, \epsilon\frac{L}{10}, \epsilon\frac{L}{10}, \epsilon\frac{L}{10}\right),$$

for $t \in [0, \infty)$. Note that any fixed point $v = (p, U, T, S, \rho) \in X$ of the operator $T_1 + S_3$ is a solution to the IVP (1). Define

$$\begin{aligned}
U_1 &= \mathcal{P}_r = \{v = (p, U, T, S, \rho) \in \mathcal{P} : \|v\| < r\}, \\
U_2 &= \mathcal{P}_L = \{v = (p, U, T, S, \rho) \in \mathcal{P} : \|v\| < L\}, \\
U_3 &= \mathcal{P}_{R_1} = \{v = (p, U, T, S, \rho) \in \mathcal{P} : \|v\| < R_1\}, \\
R_2 &= R_1 + \frac{A}{m}B_1 + \frac{L}{5m}, \\
\Omega &= \overline{\mathcal{P}_{R_2}} = \{v = (p, U, T, S, \rho) \in \mathcal{P} : \|v\| \le R_2\}.
\end{aligned}$$

1. For $v_1 = (p_1, U_1, T_1, S_1, \rho_1), v_2 = (p_2, U_2, T_2, S_2, \rho_2) \in \Omega$, we have

$$\|T_1v_1 - T_1v_2\| = (1 + m\varepsilon)\|v_1 - v_2\|,$$

whereupon $T_1 : \Omega \to X$ is an expansive operator with a constant $h = 1 + m\varepsilon > 1$.

2. For $v = (p, U, T, S, \rho) \in \overline{\mathcal{P}}_{R_1}$, we get

$$\|S_3v\| \le \varepsilon\|S_2v\| + m\varepsilon\|v\| + \varepsilon\frac{L}{10} \le \varepsilon\left(AB_1 + mR_1 + \frac{L}{10}\right).$$

Therefore $S_3(\overline{\mathcal{P}}_{R_1})$ is uniformly bounded. Since $S_3 : \overline{\mathcal{P}}_{R_1} \to X$ is continuous, we have that $S_3(\overline{\mathcal{P}}_{R_1})$ is equi-continuous. Consequently $S_3 : \overline{\mathcal{P}}_{R_1} \to X$ is a 0-set contraction.

3. Let $v_1 = (p, U, T, S, \rho) \in \overline{\mathcal{P}}_{R_1}$. Set

$$v_2 = v_1 + \frac{1}{m}S_2v_1 + \left(\frac{L}{5m}, \frac{L}{5m}, \frac{L}{5m}, \frac{L}{5m}, \frac{L}{5m}\right).$$

Note that $S_2v_1 + \left(\frac{L}{5m}, \frac{L}{5m}, \frac{L}{5m}, \frac{L}{5m}, \frac{L}{5m}\right) \ge 0$ on $[0, \infty) \times \mathbb{R}^3$. We have $v_2 \ge 0$ on $[0, \infty) \times \mathbb{R}^3$ and

$$\|v_2\| \le \|v_1\| + \frac{1}{m}\|S_2v_1\| + \frac{L}{5m} \le R_1 + \frac{A}{m}B_1 + \frac{L}{5m} = R_2.$$

Therefore $v_2 \in \Omega$ and

$$\begin{aligned}
-\varepsilon mv_2 &= -\varepsilon mv_1 - \varepsilon S_2v_1 - \left(\varepsilon\frac{L}{10}, \varepsilon\frac{L}{10}, \varepsilon\frac{L}{10}, \varepsilon\frac{L}{10}, \varepsilon\frac{L}{10}\right) \\
&\quad - \left(\varepsilon\frac{L}{10}, \varepsilon\frac{L}{10}, \varepsilon\frac{L}{10}, \varepsilon\frac{L}{10}, \varepsilon\frac{L}{10}\right)
\end{aligned}$$

or

$$\begin{aligned}
(I - T_1)v_2 &= -\varepsilon mv_2 + \left(\varepsilon\frac{L}{10}, \varepsilon\frac{L}{10}, \varepsilon\frac{L}{10}, \varepsilon\frac{L}{10}, \varepsilon\frac{L}{10}\right) \\
&= S_3v_1.
\end{aligned}$$

Consequently $S_3(\overline{\mathcal{P}}_{R_1}) \subset (I - T_1)(\Omega)$.

4. Assume that for any $u_0 = (p_0, U_0, T_0, S_0, \rho_0) \in \mathcal{P}^*$ there exist $\lambda \ge 0$ and $x = (p, U, T, S, \rho) \in \partial\mathcal{P}_r \cap (\Omega + \lambda u_0)$ or $x \in \partial\mathcal{P}_{R_1} \cap (\Omega + \lambda u_0)$ such that

$$S_3x = (I - T_1)(x - \lambda u_0).$$

Then

$$\begin{aligned} & -\epsilon S_2 x - m\epsilon x - \left(\varepsilon\frac{L}{10}, \varepsilon\frac{L}{10}, \varepsilon\frac{L}{10}, \varepsilon\frac{L}{10}, \varepsilon\frac{L}{10}\right) \\ = \quad & -m\epsilon(x - \lambda u_0) + \left(\varepsilon\frac{L}{10}, \varepsilon\frac{L}{10}, \varepsilon\frac{L}{10}, \varepsilon\frac{L}{10}, \varepsilon\frac{L}{10}\right) \end{aligned}$$

or

$$-S_2 x = \lambda m u_0 + \left(\frac{L}{5}, \frac{L}{5}, \frac{L}{5}, \frac{L}{5}, \frac{L}{5}\right).$$

Hence,

$$\|S_2 x\| = \left\|\lambda m u_0 + \frac{L}{5}\right\| > \frac{L}{5}.$$

This is a contradiction.

5. Suppose that for any $\epsilon_1 \geq 0$ small enough there exist a $x_1 = (p, U, T, S, \rho) \in \partial\mathcal{P}_L$ and $\lambda_1 \geq 1 + \epsilon_1$ such that $\lambda_1 x_1 \in \overline{\mathcal{P}}_{R_1}$ and

$$S_3 x_1 = (I - T_1)(\lambda_1 x_1). \tag{3}$$

In particular, for $\epsilon_1 > \frac{2}{5m}$, we have $x_1 \in \partial\mathcal{P}_L$, $\lambda_1 x_1 \in \overline{\mathcal{P}}_{R_1}$, $\lambda_1 \geq 1 + \epsilon_1$ and (3) holds. Since $x_1 \in \partial\mathcal{P}_L$ and $\lambda_1 x_1 \in \overline{\mathcal{P}}_{R_1}$, it follows that

$$\left(\frac{2}{5m} + 1\right) L < \lambda_1 L = \lambda_1 \|x_1\| \leq R_1.$$

Moreover,

$$\begin{aligned} & -\epsilon S_2 x_1 - m\epsilon x_1 - \left(\varepsilon\frac{L}{10}, \varepsilon\frac{L}{10}, \varepsilon\frac{L}{10}, \varepsilon\frac{L}{10}, \varepsilon\frac{L}{10}\right) \\ = \quad & -\lambda_1 m\epsilon x_1 + \left(\varepsilon\frac{L}{10}, \varepsilon\frac{L}{10}, \varepsilon\frac{L}{10}, \varepsilon\frac{L}{10}, \varepsilon\frac{L}{10}\right), \end{aligned}$$

or

$$S_2 x_1 + \left(\frac{L}{5}, \frac{L}{5}, \frac{L}{5}, \frac{L}{5}, \frac{L}{5}\right) = (\lambda_1 - 1) m x_1.$$

From here,

$$2\frac{L}{5} \geq \left\|S_2 x_1 + \left(\frac{L}{5}, \frac{L}{5}, \frac{L}{5}, \frac{L}{5}, \frac{L}{5}\right)\right\| = (\lambda_1 - 1) m \|x_1\| = (\lambda_1 - 1) m L,$$

and

$$\frac{2}{5m} + 1 \geq \lambda_1,$$

which is a contradiction.

Therefore all conditions of Theorem 2 hold. Hence, the IVP (1) has at least two solutions $u_1 = (p_1, U_1, T_1, S_1, \rho_1)$ and $u_2 = (p_2, U_2, T_2, S_2, \rho_2)$ so that

$$\|u_1\| = L < \|u_2\| < R_1 \quad \text{or} \quad r < \|u_1\| < L < \|u_2\| < R_1.$$

5 An Example

Below, we will illustrate our main results. Let

$$B = 1, \quad p_1 = q_2 = r_3 = s_4 = t_5 = 2,$$

$$p_2 = p_3 = p_4 = p_5 = q_1 = q_3 = q_4 = q_5 = 0,$$
$$r_1 = r_2 = r_4 = r_5 = s_1 = s_2 = s_3 = s_5 = 0,$$
$$t_1 = t_2 = t_3 = t_4 = 0,$$

and

$$R_1 = 10, \quad L = 5, \quad r = 4, \quad m = 10^{50}, \quad A = \frac{1}{5B_1}, \quad \epsilon = 10.$$

Then

$$B_1 = \max\{4, 16, 17\} = 17.$$

Note that (H4) holds. Next,

$$r < L < R_1, \quad \epsilon > 0, \quad R_1 > \left(\frac{2}{5m} + 1\right) L, \quad AB_1 < \frac{L}{5},$$

i.e., (H5) holds. Take

$$h(s) = \log \frac{1 + s^{11}\sqrt{2} + s^{22}}{1 - s^{11}\sqrt{2} + s^{22}}, \quad l(s) = \arctan \frac{s^{11}\sqrt{2}}{1 - s^{22}}, \quad s \in \mathbb{R}, \quad s \neq \pm 1.$$

Then

$$h'(s) = \frac{22\sqrt{2}s^{10}(1 - s^{22})}{(1 - s^{11}\sqrt{2} + s^{22})(1 + s^{11}\sqrt{2} + s^{22})},$$
$$l'(s) = \frac{11\sqrt{2}s^{10}(1 + s^{20})}{1 + s^{40}}, \quad s \in \mathbb{R}, \quad s \neq \pm 1.$$

Therefore

$$-\infty < \lim_{s \to \pm\infty} (1 + s + s^2)h(s) < \infty \quad \text{and} \quad -\infty < \lim_{s \to \pm\infty} (1 + s + s^2)l(s) < \infty.$$

Hence, there exists a positive constant C_1 so that

$$(1 + s + s^2)^3 \left(\frac{1}{44\sqrt{2}} \log \frac{1 + s^{11}\sqrt{2} + s^{22}}{1 - s^{11}\sqrt{2} + s^{22}} + \frac{1}{22\sqrt{2}} \arctan \frac{s^{11}\sqrt{2}}{1 - s^{22}} \right) \leq C_1,$$

for $s \in \mathbb{R}$. Note that $\lim_{s \to \pm 1} l(s) = \frac{\pi}{2}$ and by [7, pp. 707, Integral 79], we have

$$\int \frac{dz}{1 + z^4} = \frac{1}{4\sqrt{2}} \log \frac{1 + z\sqrt{2} + z^2}{1 - z\sqrt{2} + z^2} + \frac{1}{2\sqrt{2}} \arctan \frac{z\sqrt{2}}{1 - z^2}.$$

Let

$$Q(s) = \frac{s^{10}}{(1 + s^{44})(1 + s + s^2)^2}, \quad s \in \mathbb{R},$$

and

$$g_1(t, x) = Q(t)Q(x_1)Q(x_2)Q(x_3), \quad t \in [0, \infty), \quad x \in \mathbb{R}^3.$$

Then there exists a constant $C > 0$ such that

$$2^3(1 + t)^2 \prod_{j=1}^{3} (1 + |x_j|) \int_0^t \left| \int_0^x g_1(t_1, x_1)dx_1 \right| dt_1 \leq C, \quad (t, x) \in [0, \infty) \times \mathbb{R}^3.$$

Let

$$g(t,x) = \frac{A}{C} g_1(t,x), \quad (t,x) \in [0,\infty) \times \mathbb{R}^3.$$

Then

$$2^3(1+t)^2 \prod_{j=1}^{3} (1+|x_j|) \int_0^t \left| \int_0^x g(t_1,y_1)dy_1 \right| dt_1 \le A, \quad (t,x) \in [0,\infty) \times \mathbb{R}^3,$$

i.e., (H3) holds. Therefore for the IVP

$$\begin{aligned}
\frac{\partial p}{\partial x_3} &= \frac{p^3}{1+x_1^2}\\
\nabla \cdot U &= 0\\
\frac{\partial T}{\partial t} &= -\nabla \cdot (TU) + \frac{U^2}{1+x_2^8} + \frac{T^2}{1+x_3^6}\\
\frac{\partial S}{\partial t} &= -\nabla \cdot (SU) + \frac{S^2}{1+x_1^2+x_2^4} + \frac{\rho^2}{1+x_1^2}\\
\rho &= TSp, \quad t>0, \quad x=(x_1,x_2,x_3) \in \mathbb{R}^3,\\
p(0,x) &= \frac{1}{1+x_1^8}, \quad x \in \mathbb{R}^n,\\
U(0,x) &= \frac{1}{1+x_2^8}, \quad x \in \mathbb{R}^3,\\
T(0,x) &= \frac{1}{1+x_3^8}, \quad x \in \mathbb{R}^3,\\
S(0,x) &= \frac{1}{1+2x_1^4}, \quad x \in \mathbb{R}^3 \quad x \in \mathbb{R},
\end{aligned}$$

are fulfilled all conditions of Theorem 1 and Theorem 2.

6 Conclusions

In this paper, a class of primitive equations is investigated for existence of at least one classical solution and for existence of at least two nonnegative classical solutions. The main idea for the proof of the main results is as follows. Firstly, we give an integral representation of the solutions of the considered equations. Then we define suitable integral operators so that any fixed point of the sum of these operators is a solution of the considered problem. Using these operators and suitable fixed point theorems, we prove existence and nonuniqueness of the solutions. The proposed technique in this paper can be used for investigations for existence of solutions of initial value problems and boundary value problems for different classes of ordinary and partial differential equations.

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