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# On Milne Type Inequalities For $h$-Convex Functions Via Conformable Fractional Integral Operators*

Bouharket Benaissa†, Mehmet Zeki Sarikaya‡

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**Abstract**

In this study, Milne-type inequalities for $h$-convex functions involving conformable operators are established. In addition, new results are presented that generalize various inequalities known in the literature.

## 1 Introduction

Convexity theory offers powerful processes and notions for dealing with a wide range of pure and applied mathematics problems. Convex functions have been employed in a range of mathematical disciplines, resulting in the discovery of many inequalities in the literature. In [1], the author presents a new class of functions, $h$-convex functions.

**Definition 1** *Let $h : J \subseteq \mathbb{R} \to \mathbb{R}$, where $(0,1) \subseteq J$, be a non-negative function, $h \neq 0$. We say that $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is an $h$-convex function, if $f$ is non-negative and for all $x, y \in I$ and $\lambda \in (0,1)$, then we have*

$$f(\lambda x + (1-\lambda)y) \leq h(\lambda)f(x) + h(1-\lambda)f(y). \tag{1}$$

*If inequality (1) is reversed. Then $f$ is said to be $h$-concave.*

By setting

- $h(\lambda) = \lambda$, Definition 1 reduces to convex function [2].
- $h(\lambda) = 1$, Definition 1 reduces to $P$-functions [3, 4].
- $h(\lambda) = \lambda^s$, Definition 1 reduces to $s$-convex functions [5].

Let $f \in L[a,b]$. The left and right-sided conformable fractional integral operators of order $\alpha > 0$ and $\rho \in (0,1]$ are expressed as follows [6]:

$${}^{\rho}\mathfrak{I}^{\alpha}_{a^+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left( \frac{(x-a)^{\rho} - (t-a)^{\rho}}{\rho} \right)^{\alpha-1} (t-a)^{\rho-1} f(t)dt, \quad x > a,$$

$${}^{\rho}\mathfrak{I}^{\alpha}_{b^-} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \left( \frac{(b-x)^{\rho} - (b-t)^{\rho}}{\rho} \right)^{\alpha-1} (b-t)^{\rho-1} f(t)dt, \quad x < b.$$

For $\rho = 1$, the previous operators are reduced to Riemann-Liouville fractional operators with order $\alpha > 0$ as follows:

$$\mathfrak{I}^{\alpha}_{a^+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a,$$

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†Faculty of Material Sciences, University of Tiaret. Laboratory of Informatics and Mathematics, University of Tiaret-Algeria

‡Department of Mathematics, Faculty of Science and Arts, Düzce University, Konuralp Campus, Düzce, Turkey

$$\mathfrak{J}_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b.$$

The Beta-Euler function $\beta(\cdot,\cdot)$ is defined for any $x,\, y > 0$ as follows:

$$\beta(x,y) = \int_0^1 (1-t)^{x-1}\, t^{y-1}\, dt.$$

In 2022, Djenaoui and Meftah presented a Milne inequality for convex functions with Riemann integral as follows [7, Corollary 2.4.]

$$\left| \frac{1}{3}\left[2f(a) - f\left(\frac{a+b}{2}\right) + 2f(b)\right] - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \frac{5(b-a)}{24}\left(|f'(a)| + |f'(b)|\right). \tag{2}$$

In 2023, Budak et al. presented new Milne-type inequalities for fractional integrals for convex functions using Riemann-Liouville fractional operators [8, Theorem 2]

$$\begin{aligned} &\left| \frac{1}{3}\left[2f(a) - f\left(\frac{a+b}{2}\right) + 2f(b)\right] - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\mathfrak{J}_{b^-}^{\alpha} f\left(\frac{a+b}{2}\right) + \mathfrak{J}_{a^+}^{\alpha} f\left(\frac{a+b}{2}\right)\right] \right| \\ \leq\ & \frac{b-a}{12}\left(\frac{\alpha+4}{\alpha+1}\right)\left(|f'(a)| + |f'(b)|\right). \end{aligned} \tag{3}$$

In [9], Çelik et al. presented the following interesting Lemma 2.1 involving conformable fractional integral operators as follows:

**Lemma 1** *If $\alpha > 0$ and $f : [a,b] \to \mathbb{R}$ is a differentiable mapping such that $f' \in L_1([a,b])$. Then the following identity holds.*

$$\begin{aligned} &\frac{1}{3\rho^{\alpha}}\left[2f(a) - f\left(\frac{a+b}{2}\right) + 2f(b)\right] - \frac{2^{\alpha\rho-1}\Gamma(\alpha+1)}{(b-a)^{\alpha\rho}}\left[{}^{\rho}\mathfrak{J}_{a^+}^{\alpha} f\left(\frac{a+b}{2}\right) + {}^{\rho}\mathfrak{J}_{b^-}^{\alpha} f\left(\frac{a+b}{2}\right)\right] \\ =\ & \frac{(b-a)}{4}\int_0^1 \left(\left(\frac{1-(1-t)^{\rho}}{\rho}\right)^{\alpha} + \frac{1}{3\rho^{\alpha}}\right) \times \left[f'\left(\left(\frac{1-t}{2}\right)a + \left(\frac{1+t}{2}\right)b\right) - f'\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)b\right)\right] dt. \end{aligned}$$

Based on previous research, we developed an additional version of Milne inequality for $h$-convex functions using conformable fractional integral operators.

## 2 Milne Type Inequalities

Taking $\tau = 1-t$, we can rewrite the above Lemma 1 as follows:

**Lemma 2** *If $\alpha > 0$ and $f : [a,b] \to \mathbb{R}$ is a differentiable mapping such that $f' \in L_1([a,b])$. Then the following identity holds:*

$$\begin{aligned} &\frac{1}{3}\left[2f(a) - f\left(\frac{a+b}{2}\right) + 2f(b)\right] - \frac{\rho^{\alpha}\, 2^{\alpha\rho-1}\Gamma(\alpha+1)}{(b-a)^{\alpha\rho}}\left[{}^{\rho}\mathfrak{J}_{a^+}^{\alpha} f\left(\frac{a+b}{2}\right) + {}^{\rho}\mathfrak{J}_{b^-}^{\alpha} f\left(\frac{a+b}{2}\right)\right] \\ =\ & \frac{(b-a)}{4}\int_0^1 \left((1-t^{\rho})^{\alpha} + \frac{1}{3}\right)\left[f'\left(\left(\frac{t}{2}\right)a + \left(1-\frac{t}{2}\right)b\right) - f'\left(\left(1-\frac{t}{2}\right)a + \left(\frac{t}{2}\right)b\right)\right] dt. \end{aligned} \tag{4}$$

We present the first result for Milne inequality with conformable fractional integral operators.

**Theorem 3** *Assume that the assumptions of Lemma 2 hold. If $|f'|$ is a $h$-convex mapping on $[a,b]$. Then the following Milne-type inequality for conformable fractional integral operators holds:*

$$\begin{aligned}&\left|\frac{1}{3}\left[2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right]-\frac{\rho^{\alpha}\,2^{\alpha\rho-1}\Gamma(\alpha+1)}{(b-a)^{\alpha\rho}}\left[{}^{\rho}\mathfrak{J}^{\alpha}_{a+}f\left(\frac{a+b}{2}\right)+{}^{\rho}\mathfrak{J}^{\alpha}_{b-}f\left(\frac{a+b}{2}\right)\right]\right|\\ \leq\;& \frac{b-a}{4}\left[|f'(a)|+|f'(b)|\right]\int_0^1\left((1-t^{\rho})^{\alpha}+\frac{1}{3}\right)\left[h\left(\frac{t}{2}\right)+h\left(1-\frac{t}{2}\right)\right]dt. \qquad (5)\end{aligned}$$

**Proof.** By using the absolute value of identity 4 and the $h$-convexity of the function $|f'|$, we get

$$\begin{aligned}&\left|\frac{1}{3}\left[2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right]-\frac{\rho^{\alpha}\,2^{\alpha\rho-1}\Gamma(\alpha+1)}{(b-a)^{\alpha\rho}}\left[{}^{\rho}\mathfrak{J}^{\alpha}_{a+}f\left(\frac{a+b}{2}\right)+{}^{\rho}\mathfrak{J}^{\alpha}_{b-}f\left(\frac{a+b}{2}\right)\right]\right|\\ \leq\;& \frac{b-a}{4}\int_0^1\left((1-t^{\rho})^{\alpha}+\frac{1}{3}\right)\left[\left|f'\left(\left(\frac{t}{2}\right)a+\left(1-\frac{t}{2}\right)b\right)\right|+\left|f'\left(\left(1-\frac{t}{2}\right)a+\left(\frac{t}{2}\right)b\right)\right|\right]dt\\ \leq\;& \frac{b-a}{4}\int_0^1\left((1-t^{\rho})^{\alpha}+\frac{1}{3}\right)\\ &\times\left[h\left(\frac{t}{2}\right)|f'(a)|+h\left(1-\frac{t}{2}\right)|f'(b)|+h\left(1-\frac{t}{2}\right)|f'(a)|+h\left(\frac{t}{2}\right)|f'(b)|\right]dt\\ =\;& \frac{b-a}{4}\left[|f'(a)|+|f'(b)|\right]\int_0^1\left((1-t^{\rho})^{\alpha}+\frac{1}{3}\right)\left[h\left(\frac{t}{2}\right)+h\left(1-\frac{t}{2}\right)\right]dt.\end{aligned}$$

This completes the proof. ■

The following Lemma is required to prove the main results.

**Lemma 4** *Let $t\in(0,1)$. Then*

$$\text{for all } s\in(0,1]:\ (1+s)^{\frac{1}{s}}\geq 2, \qquad (6)$$

*and*

$$\text{for all } s\in[0,1]:\ \left(\frac{t}{2}\right)^{s}+\left(1-\frac{t}{2}\right)^{s}\leq\left(\frac{1}{2}\right)^{s-1}. \qquad (7)$$

**Proof.** Let $\Psi(s)=\ln(s+1)$ and $A(1,\ln 2)$. On the interval $[0,1]$, the graph of the function $\Psi$ appears over the line $(OA)$. This gives us for all $s\in(0,1]$

$$\ln(s+1)\geq s\ln 2\Leftrightarrow(1+s)^{\frac{1}{s}}\geq 2.$$

Taking $s=0$ in (7), we get equality. To demonstrate inequality (7) for $s\in(0,1]$, we use absurdity. Suppose that exist $s\in(0,1]$ verified

$$\left(\frac{t}{2}\right)^{s}+\left(1-\frac{t}{2}\right)^{s}>\left(\frac{1}{2}\right)^{s-1}.$$

Hence,

$$\int_0^1\left[\left(\frac{t}{2}\right)^{s}+\left(1-\frac{t}{2}\right)^{s}\right]dt>\left(\frac{1}{2}\right)^{s-1}.$$

This gives

$$2\left(\frac{1}{s+1}\right)>\left(\frac{1}{2}\right)^{s-1}\Leftrightarrow s+1<2^{s}.$$

Therefore,

$$(1+s)^{\frac{1}{s}}<2$$

which is contrary to (6). ■

Next, consider some particular cases of Theorem 3 with $h$-convexity involving conformable fractional integral operators.

1. Given $h(\lambda) = \lambda^s$ with $s \in [0, 1]$ in Theorem 3 and using (7), we obtain

$$\begin{aligned} I_1 &= \int_0^1 \left((1-t^\rho)^\alpha + \frac{1}{3}\right)\left[h\left(\frac{t}{2}\right) + h\left(1-\frac{t}{2}\right)\right] dt \\ &= \int_0^1 \left((1-t^\rho)^\alpha + \frac{1}{3}\right)\left[\left(\frac{t}{2}\right)^s + \left(1-\frac{t}{2}\right)^s\right] dt \\ &\leq \left(\frac{1}{2}\right)^{s-1} \int_0^1 \left((1-t^\rho)^\alpha + \frac{1}{3}\right) dt \\ &\leq \left(\frac{1}{2}\right)^{s-1} \left(\frac{1}{3} + \frac{1}{\rho}\beta\left(\alpha+1, \frac{1}{\rho}\right)\right), \end{aligned}$$

where we used

$$\int_0^1 (1-t^\rho)^\alpha \, dt = \frac{1}{\rho}\int_0^1 (1-t)^\alpha t^{\frac{1}{\rho}-1} dt = \frac{1}{\rho}\beta\left(\alpha+1, \frac{1}{\rho}\right).$$

**Corollary 5** *Assume $\alpha$ and $f$ are defined according to Theorem 3. If $|f'|$ is a $s$-convex function on $[a,b]$. Then*

$$\begin{aligned} &\left|\frac{1}{3}\left[2f(a) - f\left(\frac{a+b}{2}\right) + 2f(b)\right] - \frac{\rho^\alpha\, 2^{\alpha\rho-1}\Gamma(\alpha+1)}{(b-a)^{\alpha\rho}}\left[{}^\rho\mathfrak{I}^\alpha_{a+} f\left(\frac{a+b}{2}\right) + {}^\rho\mathfrak{I}^\alpha_{b-} f\left(\frac{a+b}{2}\right)\right]\right| \\ \leq\;& \frac{b-a}{4}\left(\frac{1}{2}\right)^{s-1}\left(\frac{1}{3} + \frac{1}{\rho}\beta\left(\alpha+1, \frac{1}{\rho}\right)\right)\left[|f'(a)| + |f'(b)|\right]. \end{aligned} \tag{8}$$

**Remark 1** *1.* • *Corollary 5 is a generalization of Theorem 2.2 in [9], simply by setting $s = 1$.*

• *Taking $\rho = 1$ in inequality (8), we get Milne inequality via Riemann-Liouville operators*

$$\begin{aligned} &\left|\frac{1}{3}\left[2f(a) - f\left(\frac{a+b}{2}\right) + 2f(b)\right] - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\mathfrak{I}^\alpha_{a+} f\left(\frac{a+b}{2}\right) + \mathfrak{I}^\alpha_{b-} f\left(\frac{a+b}{2}\right)\right]\right| \\ \leq\;& \frac{b-a}{4}\left(\frac{1}{2}\right)^{s-1}\left(\frac{\alpha+4}{3(\alpha+1)}\right)\left[|f'(a)| + |f'(b)|\right]. \end{aligned} \tag{9}$$

*The inequality (9) is a new generalization of the inequality (3).*

• *Putting $\rho = 1$ and $\alpha = 1$ in inequality (8), we get Milne inequality via Riemann integral*

$$\left|\frac{1}{3}\left[2f(a) - f\left(\frac{a+b}{2}\right) + 2f(b)\right] - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{5(b-a)}{24}\left(\frac{1}{2}\right)^{s-1}\left[|f'(a)| + |f'(b)|\right]. \tag{10}$$

*The inequality (10) is a new generalization of the inequality (2).*

*2. Setting $h(\lambda) = 1$ in Theorem 3 gives the following new result about the class $P$-function: Take $s = 0$ in the above inequalities (8), (9) and (10).*

**Corollary 6** *Assume $\alpha$ and $f$ are defined according to Theorem 3. If $|f'|$ is a $P$-function on $[a,b]$. Then*

$$\begin{aligned} &\left|\frac{1}{3}\left[2f(a) - f\left(\frac{a+b}{2}\right) + 2f(b)\right] - \frac{\rho^\alpha\, 2^{\alpha\rho-1}\Gamma(\alpha+1)}{(b-a)^{\alpha\rho}}\left[{}^\rho\mathfrak{I}^\alpha_{a+} f\left(\frac{a+b}{2}\right) + {}^\rho\mathfrak{I}^\alpha_{b-} f\left(\frac{a+b}{2}\right)\right]\right| \\ \leq\;& \frac{b-a}{2}\left(\frac{1}{3} + \frac{1}{\rho}\beta\left(\alpha+1, \frac{1}{\rho}\right)\right)\left[|f'(a)| + |f'(b)|\right]. \end{aligned}$$

**Remark 2** *Taking $\rho = 1$, we derive the following Milne inequality via Riemann-Liouville operators, where $|f'|$ is a $P$-function:*

$$\begin{aligned}&\left|\frac{1}{3}\left[2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\mathfrak{I}_{a^+}^{\alpha}f\left(\frac{a+b}{2}\right)+\mathfrak{I}_{b^-}^{\alpha}f\left(\frac{a+b}{2}\right)\right]\right|\\ \leq\ &\frac{(b-a)(\alpha+4)}{6(\alpha+1)}\left[|f'(a)|+|f'(b)|\right].\end{aligned}$$

**Remark 3** *Setting $\rho = 1$ and $\alpha = 1$ yields the Milne inequality for the Riemann integral.*

$$\left|\frac{1}{3}\left[2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\frac{5(b-a)}{12}\left(|f'(a)|+|f'(b)|\right).$$

**Theorem 7** *Let $p > 1$, $\frac{1}{p'}+\frac{1}{p}=1$ and assume that $\alpha, f$ are defined as in Lemma 2. If $|f'|^p$ is a $h$-convex mapping on $[a, b]$, we get the following Milne type inequality:*

$$\begin{aligned}&\left|\frac{1}{3}\left[2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right]-\frac{\rho^{\alpha}\,2^{\alpha\rho-1}\Gamma(\alpha+1)}{(b-a)^{\alpha\rho}}\left[{}^{\rho}\mathfrak{I}_{a^+}^{\alpha}f\left(\frac{a+b}{2}\right)+{}^{\rho}\mathfrak{I}_{b^-}^{\alpha}f\left(\frac{a+b}{2}\right)\right]\right|\\ \leq\ &\frac{b-a}{4}\left(2\int_0^1\left((1-t^{\rho})^{\alpha}+\frac{1}{3}\right)^{p'}dt\right)^{\frac{1}{p'}}\left(\int_0^1\left[h\left(\frac{t}{2}\right)+h\left(1-\frac{t}{2}\right)\right]dt\right)^{\frac{1}{p}}\left[|f'(a)|^p+|f'(b)|^p\right]^{\frac{1}{p}}\\ \leq\ &\frac{b-a}{4}\left(2\int_0^1\left((1-t^{\rho})^{\alpha}+\frac{1}{3}\right)^{p'}dt\right)^{\frac{1}{p'}}\left(\int_0^1\left[h\left(\frac{t}{2}\right)+h\left(1-\frac{t}{2}\right)\right]dt\right)^{\frac{1}{p}}\left[|f'(a)|+|f'(b)|\right].\end{aligned}\tag{11}$$

**Proof.** By applying the absolute value of identity 4, we obtain that

$$\begin{aligned}&\left|\frac{1}{3}\left[2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right]-\frac{\rho^{\alpha}\,2^{\alpha\rho-1}\Gamma(\alpha+1)}{(b-a)^{\alpha\rho}}\left[{}^{\rho}\mathfrak{I}_{a^+}^{\alpha}f\left(\frac{a+b}{2}\right)+{}^{\rho}\mathfrak{I}_{b^-}^{\alpha}f\left(\frac{a+b}{2}\right)\right]\right|\\ \leq\ &\frac{b-a}{4}\int_0^1\left((1-t^{\rho})^{\alpha}+\frac{1}{3}\right)\left|f'\left(\left(\frac{t}{2}\right)a+\left(1-\frac{t}{2}\right)b\right)\right|dt\\ &+\frac{b-a}{4}\int_0^1\left((1-t^{\rho})^{\alpha}+\frac{1}{3}\right)\left|f'\left(\left(1-\frac{t}{2}\right)a+\left(\frac{t}{2}\right)b\right)\right|dt,\end{aligned}$$

using the Hölder inequality and $A^{\frac{1}{p}}+B^{\frac{1}{p}}=2^{1-\frac{1}{p}}(A+B)^{\frac{1}{p}}$. Hence, we conclude

$$\begin{aligned}&\left|\frac{1}{3}\left[2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right]-\frac{\rho^{\alpha}\,2^{\alpha\rho-1}\Gamma(\alpha+1)}{(b-a)^{\alpha\rho}}\left[{}^{\rho}\mathfrak{I}_{a^+}^{\alpha}f\left(\frac{a+b}{2}\right)+{}^{\rho}\mathfrak{I}_{b^-}^{\alpha}f\left(\frac{a+b}{2}\right)\right]\right|\\ \leq\ &\frac{b-a}{4}\left(\int_0^1\left((1-t^{\rho})^{\alpha}+\frac{1}{3}\right)^{p'}dt\right)^{\frac{1}{p'}}\left(\int_0^1\left|f'\left(\left(\frac{t}{2}\right)a+\left(1-\frac{t}{2}\right)b\right)\right|^p dt\right)^{\frac{1}{p}}\\ &+\frac{b-a}{4}\left(\int_0^1\left((1-t^{\rho})^{\alpha}+\frac{1}{3}\right)^{p'}dt\right)^{\frac{1}{p'}}\left(\int_0^1\left|f'\left(\left(1-\frac{t}{2}\right)a+\left(\frac{t}{2}\right)b\right)\right|^p dt\right)^{\frac{1}{p}}\\ \leq\ &\frac{b-a}{4}\left(\int_0^1\left((1-t^{\rho})^{\alpha}+\frac{1}{3}\right)^{p'}dt\right)^{\frac{1}{p'}}2^{1-\frac{1}{p}}\\ &\times\left[\int_0^1\left|f'\left(\left(\frac{t}{2}\right)a+\left(1-\frac{t}{2}\right)b\right)\right|^p dt+\int_0^1\left|f'\left(\left(1-\frac{t}{2}\right)a+\left(\frac{t}{2}\right)b\right)\right|^p dt\right]^{\frac{1}{p}}.\end{aligned}$$

Assuming $|f'|^p$ is a $h$-convex function, we get

$$\begin{aligned}
&\left|\frac{1}{3}\left[2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right]-\frac{\rho^{\alpha}\,2^{\alpha\rho-1}\Gamma(\alpha+1)}{(b-a)^{\alpha\rho}}\left[{}^{\rho}\mathfrak{I}^{\alpha}_{a+}f\left(\frac{a+b}{2}\right)+{}^{\rho}\mathfrak{I}^{\alpha}_{b-}f\left(\frac{a+b}{2}\right)\right]\right|\\
\leq\ & \frac{b-a}{4}\left(2\int_0^1\left((1-t^{\rho})^{\alpha}+\frac{1}{3}\right)^{p'}dt\right)^{\frac{1}{p'}}\left[\int_0^1\left(h\left(\frac{t}{2}\right)|f'(a)|^p+h\left(1-\frac{t}{2}\right)|f'(b)|^p\right)dt\right.\\
&\left.+\int_0^1\left(h\left(1-\frac{t}{2}\right)|f'(a)|^p+h\left(\frac{t}{2}\right)|f'(b)|^p\right)dt\right]^{\frac{1}{p}}\\
\leq\ & \frac{b-a}{4}\left(2\int_0^1\left((1-t^{\rho})^{\alpha}+\frac{1}{3}\right)^{p'}dt\right)^{\frac{1}{p'}}\left(\int_0^1\left[h\left(\frac{t}{2}\right)+h\left(1-\frac{t}{2}\right)\right]dt\right)^{\frac{1}{p}}\left[|f'(a)|^p+|f'(b)|^p\right]^{\frac{1}{p}}.
\end{aligned}$$

This results in the first inequality noticed in (11).

For $p>1$ and $A,B\geq 0$, we deduce $A^p+B^p\leq(A+B)^p$, which yields the second inequality in (11). ■

Next, let's study some particular cases of Theorem 7. Then that involve conformable fractional integral operators and $h$-convexity.

1. Given $h(\lambda)=\lambda^s$ with $s\in(0,1]$ in Theorem 7 and using (7), we obtain that

$$\begin{aligned}
I_2 &= \int_0^1\left[h\left(\frac{t}{2}\right)+h\left(1-\frac{t}{2}\right)\right]dt\\
&= \int_0^1\left[\left(\frac{t}{2}\right)^s+\left(1-\frac{t}{2}\right)^s\right]dt\\
&\leq \left(\frac{1}{2}\right)^{s-1}.
\end{aligned}$$

**Corollary 8** *Assume that $\alpha,\rho$ and $f$ are defined according to Theorem 7. If $|f'|^p$ is a $s$-convex function on $[a,b]$, then*

$$\begin{aligned}
&\left|\frac{1}{3}\left[2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right]-\frac{\rho^{\alpha}\,2^{\alpha\rho-1}\Gamma(\alpha+1)}{(b-a)^{\alpha\rho}}\left[{}^{\rho}\mathfrak{I}^{\alpha}_{a+}f\left(\frac{a+b}{2}\right)+{}^{\rho}\mathfrak{I}^{\alpha}_{b-}f\left(\frac{a+b}{2}\right)\right]\right|\\
\leq\ & \frac{b-a}{2}\left(\int_0^1\left((1-t^{\rho})^{\alpha}+\frac{1}{3}\right)^{p'}dt\right)^{\frac{1}{p'}}\left(\frac{1}{2}\right)^{\frac{s}{p}}\left[|f'(a)|^p+|f'(b)|^p\right]^{\frac{1}{p}}\\
\leq\ & \frac{b-a}{2}\left(\int_0^1\left((1-t^{\rho})^{\alpha}+\frac{1}{3}\right)^{p'}dt\right)^{\frac{1}{p'}}\left(\frac{1}{2}\right)^{\frac{s}{p}}\left[|f'(a)|+|f'(b)|\right].
\end{aligned}\tag{12}$$

1. 
   - With $s=1$, Corollary 8 improves Theorem 2.4 from [9].
   - Taking $\rho=1$ in the inequality (12), we get Milne inequality via Riemann-Liouville operators.

$$\begin{aligned}
&\left|\frac{1}{3}\left[2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\mathfrak{I}^{\alpha}_{a+}f\left(\frac{a+b}{2}\right)+\mathfrak{I}^{\alpha}_{b-}f\left(\frac{a+b}{2}\right)\right]\right|\\
\leq\ & \frac{b-a}{2}\left(\int_0^1\left((1-t)^{\alpha}+\frac{1}{3}\right)^{p'}dt\right)^{\frac{1}{p'}}\left(\frac{1}{2}\right)^{\frac{s}{p}}\left[|f'(a)|^p+|f'(b)|^p\right]^{\frac{1}{p}}\\
\leq\ & \frac{b-a}{2}\left(\int_0^1\left((1-t)^{\alpha}+\frac{1}{3}\right)^{p'}dt\right)^{\frac{1}{p'}}\left(\frac{1}{2}\right)^{\frac{s}{p}}\left[|f'(a)|+|f'(b)|\right].
\end{aligned}\tag{13}$$

- By putting $\rho = 1$ and $\alpha = 1$ in inequality (12), we get Milne inequality via Riemann integral.

$$\begin{aligned}
&\left| \frac{1}{3}\left[ 2f(a) - f\left(\frac{a+b}{2}\right) + 2f(b) \right] - \frac{1}{b-a}\int_a^b f(t)dt \right| \\
\leq\ & \frac{b-a}{2}\left[ \frac{1}{1+p'}\left( \left(\frac{4}{3}\right)^{1+p'} - \left(\frac{1}{3}\right)^{1+p'} \right) \right]^{\frac{1}{p'}} \left(\frac{1}{2}\right)^{\frac{s}{p}} \left[ |f'(a)|^p + |f'(b)|^p \right]^{\frac{1}{p}} \\
\leq\ & \frac{b-a}{2}\left[ \frac{1}{1+p'}\left( \left(\frac{4}{3}\right)^{1+p'} - \left(\frac{1}{3}\right)^{1+p'} \right) \right]^{\frac{1}{p'}} \left(\frac{1}{2}\right)^{\frac{s}{p}} \left[ |f'(a)| + |f'(b)| \right]. \qquad (14)
\end{aligned}$$

2. The following new result regarding the class $P$-function is obtained by setting $h(\lambda) = 1$ in the Theorem 7. Consider $s = 0$ in the inequalities (12), (13) and (14).

   Assume $\alpha$, $\rho$ and $f$ are defined according to Theorem 7. If $|f'|^p$ is a $P$-function on $[a, b]$. Then

$$\begin{aligned}
&\left| \frac{1}{3}\left[ 2f(a) - f\left(\frac{a+b}{2}\right) + 2f(b) \right] - \frac{\rho^\alpha\, 2^{\alpha\rho-1}\Gamma(\alpha+1)}{(b-a)^{\alpha\rho}} \left[ {}^\rho\mathfrak{I}^\alpha_{a+} f\left(\frac{a+b}{2}\right) + {}^\rho\mathfrak{I}^\alpha_{b-} f\left(\frac{a+b}{2}\right) \right] \right| \\
\leq\ & \frac{b-a}{2}\left( \int_0^1 \left( (1-t^\rho)^\alpha + \frac{1}{3} \right)^{p'} dt \right)^{\frac{1}{p'}} \left[ |f'(a)|^p + |f'(b)|^p \right]^{\frac{1}{p}} \\
\leq\ & \frac{b-a}{2}\left( \int_0^1 \left( (1-t^\rho)^\alpha + \frac{1}{3} \right)^{p'} dt \right)^{\frac{1}{p'}} \left[ |f'(a)| + |f'(b)| \right].
\end{aligned}$$

   Taking $\rho = 1$, we derive the following Milne inequality via Riemann-Liouville operators, where $|f'|^p$ is a $P$-function.

$$\begin{aligned}
&\left| \frac{1}{3}\left[ 2f(a) - f\left(\frac{a+b}{2}\right) + 2f(b) \right] - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}} \left[ \mathfrak{I}^\alpha_{a+} f\left(\frac{a+b}{2}\right) + \mathfrak{I}^\alpha_{b-} f\left(\frac{a+b}{2}\right) \right] \right| \\
\leq\ & \frac{b-a}{2}\left( \int_0^1 \left( (1-t)^\alpha + \frac{1}{3} \right)^{p'} dt \right)^{\frac{1}{p'}} \left[ |f'(a)|^p + |f'(b)|^p \right]^{\frac{1}{p}} \\
\leq\ & \frac{b-a}{2}\left( \int_0^1 \left( (1-t)^\alpha + \frac{1}{3} \right)^{p'} dt \right)^{\frac{1}{p'}} \left[ |f'(a)| + |f'(b)| \right].
\end{aligned}$$

   By setting $\rho = 1$ and $\alpha = 1$, we use the Riemann integral to derive the following Milne inequality for the class $P$-function.

$$\begin{aligned}
&\left| \frac{1}{3}\left[ 2f(a) - f\left(\frac{a+b}{2}\right) + 2f(b) \right] - \frac{1}{b-a}\int_a^b f(t)dt \right| \\
\leq\ & \frac{b-a}{2}\left[ \frac{1}{1+p'}\left( \left(\frac{4}{3}\right)^{1+p'} - \left(\frac{1}{3}\right)^{1+p'} \right) \right]^{\frac{1}{p'}} \left[ |f'(a)|^p + |f'(b)|^p \right]^{\frac{1}{p}} \\
\leq\ & \frac{b-a}{2}\left[ \frac{1}{1+p'}\left( \left(\frac{4}{3}\right)^{1+p'} - \left(\frac{1}{3}\right)^{1+p'} \right) \right]^{\frac{1}{p'}} \left[ |f'(a)| + |f'(b)| \right].
\end{aligned}$$

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