Applied Mathematics E-Notes, 25(2025), 207-212 © ISSN 1607-2510
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A Note On Stability Of A Neutral Type Single Neuron System*

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Received 3 March 2024

Abstract

This note is concerned with the exposition of sufficient conditions ensuring the asymptotic stability of a single neuron system of neutral type. Delay independent criteria are given by means of a linear matrix inequality (LMI). A given numerical example illustrates the result.

1 Introduction

In numerous fields of modern science and technology, systems incorporating time delays are frequently encountered. These systems exhibit intricate dynamics, which are often modeled by neutral differential equations [3, 5, 6]. Neutral differential equations (NDEs) constitute a class of differential equations that incorporate a time delay in the derivative of the highest order [8, 9, 10]. Such delays can arise due to various factors, such as signal propagation time or feedback mechanisms within the system. As a result, the analysis and understanding of NDEs play a pivotal role in unraveling the behavior of diverse real-world systems, ranging from biological networks to control systems in engineering applications.

Recent years have witnessed a surge of interest in exploring the dynamic characteristics of neural networks. Researchers have delved deeply into understanding phenomena such as bifurcation and chaos, which are inherent in these networks. Consequently, various neural network architectures have been extensively studied, with a particular emphasis on their dynamic behavior. Furthermore, neutral delay differential equations have emerged as valuable tools for modeling the dynamics of neural networks. These equations offer a flexible framework for capturing the intricate interplay of time delays and nonlinear dynamics observed in neural systems. As a result, researchers have proposed and investigated multiple types of neutral delay differential equations to elucidate the complex dynamics exhibited by neural networks [5, 11, 15].

In this note, our aim is to investigate the stability of the null solution for neutral-type delay differential equations modeling a single neuron system, which take the form

$$\frac{d}{dt}[x(t)+px(t-\sigma)] = -ax(t) + b\tanh[x(t)+cx(t-\sigma)], \tag{1}$$

for all $t \geq t_1 = t_0 + \sigma$, where $\sigma > 0$ represents the delay, and $a > 0$, b, c, and p are real constants. In this equation, b denotes the range of the continuous variable x, while c measures the inhibitory influence from the past history. By analyzing the parameters listed above, we aim to provide sufficient conditions that ensure stability for this type of neutral-neuron systems.

For each solution of (1), we assume the following initial condition:

$$x(t) = \phi(t), \quad t \in [-\sigma, t_0], \quad \phi \in C([-\sigma, 0], \mathbb{R}).$$

Remark 1 *Readers are referred to [8, 9, 10, 16] for general references related to delay and neutral differential equations. Additionally, it can be observed that if $c = 0$ and $\sigma = \tau$, then Eq. (1) reduces to the case discussed in [2], implying that the equation treated here generalizes the previous one.*

We define the operator $D(x_t)$ by: $D(x_t) = x(t) + px(t-\sigma)$.

*Mathematics Subject Classifications: 34K20, 34D20.

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Lemma 1 ([8]) *The operator D is said to be stable if the zero solution of the homogeneous difference equation $D(x_t) = 0$, $t \geq 0$ is uniformly asymptotically stable.*

Remark 2 *The operator D is stable if $|p| < 1$. Refer to [8] for further information.*

The remainder of this note is divided into two sections. The first section presents the main result obtained, along with its proof, while the second section provides numerical affirmations of the obtained result.

2 Main Result

Instead of the actual form of Eq. (1) , with the change $y(t) = x(t) + cx(t-\sigma)$, we have

$$\begin{aligned} & \frac{d}{dt}[x(t) + px(t-\sigma)] + c\frac{d}{dt}[x(t-\sigma) + px(t-2\sigma)] \\ = \; & -ax(t) + b\tanh[x(t) + cx(t-\sigma)] \\ & -acx(t-\sigma) + bc\tanh[x(t-\sigma) + cx(t-2\sigma)] \\ = \; & -ay(t) + b\tanh[y(t)] + bc\tanh[y(t-\sigma)], \end{aligned}$$

hence, we get

$$\frac{d}{dt}[y(t) + py(t-\sigma)] = -ay(t) + b\tanh[y(t)] + bc\tanh[y(t-\sigma)],$$

and in a descriptor fashion

$$\left\{ \begin{array}{l} \frac{d}{dt}z(t) = -ay(t) + b\tanh[y(t)] + bc\tanh[y(t-\sigma)], \\ 0 = -z(t) + y(t) + py(t-\sigma). \end{array} \right. \tag{2}$$

Now we give the main result of this work.

Theorem 1 *For given positive constants $\alpha, \beta, \epsilon_1$ and real constants ϵ_2, ϵ_3, if $|p| < 1$, then Eq. (1) is asymptotically stable provided the following linear matrix inequality holds:*

$$\Psi = \begin{bmatrix} -2\epsilon_2 & b\epsilon_1\sqrt{1+c^2} & -(a\epsilon_1 + \epsilon_3 - \epsilon_2) & p\epsilon_2 \\ b\epsilon_1\sqrt{1+c^2} & -\beta & 0 & 0 \\ -(a\epsilon_1 + \epsilon_3 - \epsilon_2) & 0 & 2\epsilon_3 + \alpha + 2\beta & p\epsilon_3 \\ p\epsilon_2 & 0 & p\epsilon_3 & -\alpha \end{bmatrix} < 0. \tag{3}$$

Proof. Define a Lyapunov function by

$$V(t) = V_1(t) + V_2(t), \tag{4}$$

where

$$V_1(t) = \begin{bmatrix} z(t) & y(t) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \epsilon_1 & 0 \\ \epsilon_2 & \epsilon_3 \end{bmatrix} \begin{bmatrix} z(t) \\ y(t) \end{bmatrix},$$

with $\epsilon_1 > 0, \epsilon_2$ and ϵ_3 real,

$$V_2(t) = \alpha\int_{t-\sigma}^{t} y^2(s)ds + \beta\int_{t-\sigma}^{t} \tanh^2[y(s)]ds,$$

with $\alpha > 0, \beta > 0$. The derivative of (4) along trajectories of (2) is given by $V'(t) = V_1'(t) + V_2'(t)$, and we have

$$\begin{aligned}
V_1'(t) &= 2\begin{bmatrix} z(t) & y(t) \end{bmatrix}\begin{bmatrix} \epsilon_1 & \epsilon_2 \\ 0 & \epsilon_3 \end{bmatrix}\begin{bmatrix} z'(t) \\ 0 \end{bmatrix} \\
&= 2\begin{bmatrix} z(t) & y(t) \end{bmatrix}\begin{bmatrix} \epsilon_1 & \epsilon_2 \\ 0 & \epsilon_3 \end{bmatrix}\begin{bmatrix} -ay(t) + b\tanh[y(t)] + bc\tanh[y(t-\sigma)] \\ -z(t) + y(t) + py(t-\sigma) \end{bmatrix} \\
&= 2\epsilon_1 z(t)\left[-ay(t) + b\tanh[y(t)] + bc\tanh[y(t-\sigma)]\right] \\
&\quad + \left[2\epsilon_2 z(t) + 2\epsilon_3 y(t)\right]\left[-z(t) + y(t) + py(t-\sigma)\right] \\
&= -2a\epsilon_1 y(t)z(t) + 2b\epsilon_1 \tanh[y(t)]z(t) + 2bc\epsilon_1 \tanh[y(t-\sigma)]z(t) \\
&\quad -2\epsilon_2 z^2(t) + 2\epsilon_2 y(t)z(t) + 2p\epsilon_2 y(t-\sigma)z(t) \\
&\quad -2\epsilon_3 y(t)z(t) + 2\epsilon_3 y^2(t) + 2p\epsilon_3 y(t-\sigma)y(t).
\end{aligned}$$

Using inequality $2uv \leq \frac{1}{\beta}u^2 + \beta v^2$, for $\beta > 0$, we get

$$\begin{aligned}
V_1'(t) &\leq -2a\epsilon_1 y(t)z(t) + \frac{b^2\epsilon_1^2}{\beta}z^2(t) + \beta\tanh^2[y(t)] + \frac{b^2c^2\epsilon_1^2}{\beta}z^2(t) + \beta\tanh^2[y(t-\sigma)] \\
&\quad -2\epsilon_2 z^2(t) + 2\epsilon_2 y(t)z(t) + 2p\epsilon_2 y(t-\sigma)z(t) \\
&\quad -2\epsilon_3 y(t)z(t) + 2\epsilon_3 y^2(t) + 2p\epsilon_3 y(t-\sigma)y(t) \\
&\leq 2\epsilon_3 y^2(t) - \left(2\epsilon_2 - \frac{\left(1+c^2\right)b^2\epsilon_1^2}{\beta}\right)z^2(t) + \beta\tanh^2[y(t)] + \beta\tanh^2[y(t-\sigma)] \\
&\quad -2\left(a\epsilon_1 + \epsilon_3 - \epsilon_2\right)y(t)z(t) + 2p\epsilon_2 y(t-\sigma)z(t) + 2p\epsilon_3 y(t-\sigma)y(t).
\end{aligned}$$

Now, $V_2'(t)$ is equal to

$$V_2'(t) = \alpha y^2(t) - \alpha y^2(t-\sigma) + \beta\tanh^2[y(t)] - \beta\tanh^2[y(t-\sigma)].$$

Therefore

$$\begin{aligned}
V'(t) &= V_1'(t) + V_2'(t) \\
&\leq 2\epsilon_3 y^2(t) - \left(2\epsilon_2 - \frac{\left(1+c^2\right)b^2\epsilon_1^2}{\beta}\right)z^2(t) + \beta\tanh^2[y(t)] + \beta\tanh^2[y(t-\sigma)] \\
&\quad -2\left(a\epsilon_1 + \epsilon_3 - \epsilon_2\right)y(t)z(t) + 2p\epsilon_2 y(t-\sigma)z(t) + 2p\epsilon_3 y(t-\sigma)y(t) \\
&\quad +\alpha y^2(t) - \alpha y^2(t-\sigma) + \beta\tanh^2[y(t)] - \beta\tanh^2[y(t-\sigma)] \\
&\leq \left(2\epsilon_3 + \alpha\right)y^2(t) - \left(2\epsilon_2 - \frac{\left(1+c^2\right)b^2\epsilon_1^2}{\beta}\right)z^2(t) - \alpha y^2(t-\sigma) + 2\beta\tanh^2[y(t)] \\
&\quad -2\left(a\epsilon_1 + \epsilon_3 - \epsilon_2\right)y(t)z(t) + 2p\epsilon_2 y(t-\sigma)z(t) + 2p\epsilon_3 y(t-\sigma)y(t).
\end{aligned}$$

Using the relation $\tanh^2[x] \leq x^2$, we have

$$\begin{aligned}
V'(t) &\leq \left(2\epsilon_3 + \alpha + 2\beta\right)y^2(t) - \left(2\epsilon_2 - \frac{\left(1+c^2\right)b^2\epsilon_1^2}{\beta}\right)z^2(t) - \alpha y^2(t-\sigma) \\
&\quad -2\left(a\epsilon_1 + \epsilon_3 - \epsilon_2\right)y(t)z(t) + 2p\epsilon_2 y(t-\sigma)z(t) + 2p\epsilon_3 y(t-\sigma)y(t) \\
&\leq A^T(t)\Omega A(t),
\end{aligned}$$

where

$$A(t) = [z(t), y(t), y(t-\sigma)]^T,$$

and

$$\Omega = \begin{pmatrix} -2\epsilon_2 + \frac{(1+c^2)b^2\epsilon_1^2}{\beta} & -(a\epsilon_1 + \epsilon_3 - \epsilon_2) & p\epsilon_2 \\ -(a\epsilon_1 + \epsilon_3 - \epsilon_2) & 2\epsilon_3 + \alpha + 2\beta & p\epsilon_3 \\ p\epsilon_2 & p\epsilon_3 & -\alpha \end{pmatrix}.$$

Thus negative definiteness of V follows negative definiteness of the matrix Ω. Using Schur complement [17], the inequality $\Omega < 0$ is equivalent to $\Psi < 0$. Noting that $|p| < 1$, the operator $x(t) + px(t-\sigma)$ is stable, hence Eq. (1) is asymptotically stable according to [8]. This completes the proof. ■

3 Example

To verify the feasibility of inequality (3), various convex optimization algorithms can be employed. In this study, we utilize the YALMIP (Yet Another Linear Matrix Inequality Parser) toolbox in MATLAB [18], which offers a user-friendly interface and efficient problem-solving capabilities for such problems. Consider the following equation:

$$\frac{d}{dt}[x(t) + 0.6x(t-0.3)] = -0.6x(t) + 0.2\tanh[x(t) + 0.2x(t-0.3)]. \tag{5}$$

Observe that $|p| = a = 0.6$, $b = c = 0.2$, and $\sigma = 0.3$. By solving the linear matrix inequality (3), we get its

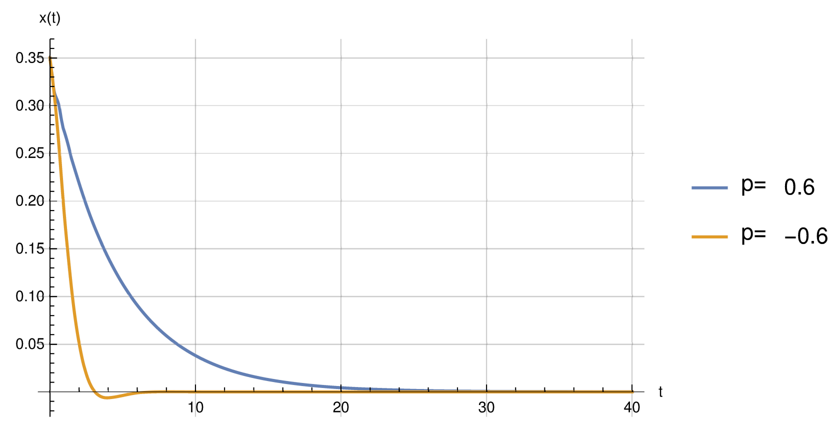

Figure 1: Numerical simulation of Eq. (5) for $p = \pm 0.6 < 1$.

solution

$$\alpha = 0.6182, \quad \beta = 0.3019, \quad \epsilon_1 = 1.6585, \quad \epsilon_2 = 0.4183, \quad \epsilon_3 = -0.9859.$$

We have $|p| = 0.6 < 1$ which ensure stability of the D operator in this case. Moreover, substitution of the previous values on Ω gives

$$\Omega = \begin{pmatrix} -0.45758 & 0.40910 & 0.25098 \\ 0.40910 & -0.74980 & -0.59154 \\ 0.25098 & -0.59154 & -0.61820 \end{pmatrix}.$$

This matrix is negative definite, by Theorem (1), Eq. (5) is asymptotically stable.

4 Conclusion

To summarize, this research concentrated on identifying the sufficient conditions to guarantee the long-term stability of a single neuron system of neutral type. By employing delay-independent criteria, expressed as linear matrix inequalities (LMIs), the stability analysis of such systems was simplified. The applicability and effectiveness of the proposed method were demonstrated through a numerical example, highlighting its practical significance.

Acknowledgment. The author would like to thank the reviewer for comments and suggestions that led to the improvement of the paper. This research was supported by DGRSDT, Algeria.

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