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Approximating Fixed Points For Class Of Generalized Nonexpansive Operators Using Efficient Iteration Process With Application*

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Abstract

In this paper, $\mathcal{SRJ}$ iteration process has been defined for class of generalized nonexpansive operators. By means of an numerical example we show that our iteration process converges at a rate faster then some of the leading multi-step iterative algorithm in the existing literature which have been used recently to find the solutions of a nonlinear integral equation.

1 Introduction

We know that analytical methods may fail to find exact solution of mathematical problems. Therefore, fixed point theory recommends some alternate techniques for solving these problems. First, we expressed the solution of the problem as the fixed point of a certain map (the map may be contraction, nonexpansive, or generalized nonexpansive [29]). In this situation, an existence of a solution and the existence of a fixed point have the same meanings. We used some suitable iterative methods to find approximate unique fixed point (see, e.g., [3] and others). The Banach Contraction Principle indicates, among other things, that if the operator is a contraction and the subset is a closed subset of a Banach space, there may be a single fixed point. As an operator $\mathcal{P}$ on a subset $\mathcal{G}$ of a Banach space $\mathfrak{B}$ is called a contraction, if for all $a, b \in \mathcal{G}$, we have the following equation:

$$||\mathcal{P}a - \mathcal{P}b|| \leq \theta||a - b|| \tag{1}$$

where $\theta \in [0, 1)$.

A nonexpansive mapping $\mathcal{P}$ is a mapping that satisfies equation (1) for $\theta = 1$. Actually, the Picard iteration method ($\mathcal{A}_{n+1} = \mathcal{P}\mathcal{A}_n$) was suggested by the Banach contraction principle proof to determine the estimated value of the unique fixed point of the contraction $\mathcal{P}$. If $\mathcal{G}$ is closed bounded and convex and $\mathfrak{B}$ is a uniformly convex Banach space (uniformly convex Banach space). Then $\mathcal{P}$ has a fixed point (see, e.g., Kirk [17], Browder [8], and Gohde [14]). In applied sciences, nonexpansive mappings have a key role for solving fixed points problems. Therefore, we tried to use some extensions of these mappings.

Suppose $\mathcal{P}$ is a self-map, that is, $\mathcal{P}\colon \mathcal{G} \to \mathcal{G}$ and $a, b \in \mathcal{G}$, where $\mathcal{G}$ is any subset of a Banach space, then $\mathcal{P}$ is called as follows:

(a) Suzuki generalized nonexpansive [26] if

$$\frac{1}{2}||a - \mathcal{P}a|| \leq ||a - b|| \implies ||\mathcal{P}a - \mathcal{P}b|| \leq ||a - b||.$$

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(b) Generalized α-nonexpansive [22] if

$$\frac{1}{2}||a-\mathcal{P}a|| \leq ||a-b|| \implies ||\mathcal{P}a-\mathcal{P}b|| \leq ||a-b|| \leq \alpha||a-\mathcal{P}a|| + \alpha||a-\mathcal{P}b|| + (1-2\alpha)||a-b||.$$

(c) Generalized (α,β)-nonexpansive [29] if

$$\frac{1}{2}||a-\mathcal{P}a|| \leq ||a-b||,$$

then

$$||\mathcal{P}a-\mathcal{P}b|| \leq ||a-b|| \leq \alpha||a-\mathcal{P}b|| + \alpha||b-\mathcal{P}a|| + \beta||a-\mathcal{P}a|| + \beta||b-\mathcal{P}b|| + (1-2\alpha-2\beta)||a-b||.$$

The classes of Generalized α-nonexpansive and Generalized (α,β)-nonexpansive self-maps properly include the class of Suzuki nonexpansive self-maps.

Ullah et al. [30] presented a new class of nonlinear mapping.

Definition 1 *A self-map $\mathcal{P}$ on a subset $\mathcal{G}$ of a Banach space is said to be (α,β,γ)-nonexpansive if for all $a,b\in\mathcal{G}$,*

$$||\mathcal{P}a-\mathcal{P}b|| \leq \alpha||a-b|| + \beta||a-\mathcal{P}a|| + \gamma||a-\mathcal{P}b||,$$

where $\alpha,\beta,\gamma\in\mathbb{R}^+$ such that $\alpha+\gamma\leq 1$.

The class of (α,β,γ)-nonexpansive mappings includes all these mappings, and thus, the concept of (α,β,γ)-nonexpansive mappings is more difficult but more important than the other mappings mentioned above. In this work we present a numerical example to illustrate our main result and then display the efficiency of the proposed algorithm compared to different iterative algorithms in the literature. Our results obtained in this paper improve, extend and unify some related result Ullah et al. [30].

2 Preliminaries

We need some of the known results. Suppose a Banach space $\mathfrak{B}$ is equipped with $||.||$. The space $\mathfrak{B}$ will be called a uniformly convex Banach space [10] provided that for each choice of $0\leq\epsilon<1$, a real number $0<\delta<\infty$ can be found satisfying $\left\|\frac{a+b}{2}\right\|\leq 1-\delta$, for all two elements $a,b\in\mathfrak{B}$ with $||a||\leq 1$, $||b||\leq 1$ and $||a+b||\geq\epsilon$. On the other side, if $\mathfrak{B}$ satisfies the property that $||a+b||<2$ for all two different $a,b\in\mathfrak{B}$ with $||a||=||b||=1$. Then $\mathfrak{B}$ is called strictly convex.

The space $\mathfrak{B}$ is said to be equipped with the Opial's property [21], if and only if for any given weakly convergent sequence, namely, $\{\mathcal{A}_n\}$ in $\mathfrak{B}$ having limit $a_0\in\mathfrak{B}$. Then for all $b_0\in\mathfrak{B}-a_0$, one has

$$\limsup_{n\to\infty}||\mathcal{A}_n-a_0|| < \limsup_{n\to\infty}||\mathcal{A}_n-b_0||.$$

Definition 2 ([2, 27]) *Suppose $\{\mathcal{A}_n\}$ denotes any bounded sequence in a closed convex subset $\mathcal{G}$ of a uniformly convex Banach space $\mathfrak{B}$. In this case, one denotes and defines the asymptotic radius of $\{\mathcal{A}_n\}$ on the set $\mathcal{G}$ by*

$$u(\mathcal{G},\{\mathcal{A}_n\}) = \inf\{\limsup_{n\to\infty}||\mathcal{A}_n-a||\colon a\in\mathcal{G}\},$$

while the asymptotic center of $\{\mathcal{A}_n\}$ on the set $\mathcal{G}$ is denoted and defined as

$$\mathcal{K}(\mathcal{G},\{\mathcal{A}_n\}) = \{a\in\mathcal{G}\colon \limsup_{n\to\infty}||\mathcal{A}_n-a|| = u(\mathcal{G},\{\mathcal{A}_n\})\}.$$

Furthermore, in a uniformly convex Banach space, the asymptotic center $\mathcal{K}(\mathcal{G},\{\mathcal{A}_n\})$ contains exactly one element.

Remark 1 *The set $\mathcal{K}(\mathcal{G}, \{\mathcal{A}_n\})$ contains only one point provided that $\mathfrak{B}$ is a uniformly convex Banach space. The property that $\mathcal{K}(\mathcal{G}, \{\mathcal{A}_n\})$ is convex is also known in the setting of weakly compact convex sets (e.g., [23, 27] and others).*

Every uniformly convex Banach space has the following important property [24].

Lemma 1 *Consider two sequences $\{\mathcal{A}_n\}$ and $\{\mathcal{B}_n\}$ in a uniformly convex Banach space $\mathfrak{B}$ with*

$$\limsup_{n\to\infty} ||\mathcal{A}_n|| \leq \kappa \quad \textit{and} \quad \limsup_{n\to\infty} ||\mathcal{B}_n|| \leq \kappa.$$

In addition, if $0 < \mu \leq \mu_n \leq \nu < 1$ and $\lim_{n\to\infty} ||\mu_n \mathcal{A}_n + (1-\mu_n)\mathcal{B}_n|| = \kappa$ for some $\kappa \geq 0$. Then $\lim_{n\to\infty} ||\mathcal{A}_n - \mathcal{B}_n|| = 0$.

Proposition 1 *Suppose $\mathcal{P}$ is nonexpansive self-map whose domain of definition is possibly a subset $\mathcal{G}$ of $\mathfrak{B}$. Then $\mathcal{P}$ is (α, β, γ)-nonexpansive.*

Example 1 *We now suggest a self-map $\mathcal{P}\colon [0,2] \to [0,2]$ by the formula*

$$\mathcal{P}a = \begin{cases} 0 & \textit{if } a \neq 0, \\ 1 & \textit{if } a = 0. \end{cases}$$

Here $\mathcal{P}$ is discontinuous and so not nonexpansive but $\mathcal{P}$ is (α, β, γ)-nonexpansive. Accordingly, the class of (α, β, γ)-nonexpansive self-maps properly contains as a subset the class of all nonexpansive self-maps.

Lemma 2 *Suppose $\mathcal{P}$ is (α, β, γ)-nonexpansive self-map whose domain of definition is possibly a subset $\mathcal{G}$ of $\mathfrak{B}$ with a fixed point, namely, ρ. In such a case, the estimate $||\mathcal{P}a - \mathcal{P}\rho|| \leq ||a - \rho||$ holds for all $a \in \mathcal{G}$ and $\rho \in \mathcal{F}(\mathcal{P})$.*

Now Lemma 2 suggests the following result.

Lemma 3 *Suppose $\mathcal{P}$ is (α, β, γ)-nonexpansive self-map whose domain of definition is possibly a subset $\mathcal{G}$ of a Banach space $\mathfrak{B}$. Consequently, the set $\mathcal{F}(\mathcal{P})$ is closed. Also, the set $\mathcal{F}(\mathcal{P})$ is convex provided that $\mathcal{G}$ is convex and the space $\mathfrak{B}$ is strictly convex.*

The next lemma shows a very basic property of the (α, β, γ)-nonexpansive mappings.

Lemma 4 ([30]) *Suppose $\mathcal{P}$ is (α, β, γ)-nonexpansive self-map whose domain of definition is possibly a subset $\mathcal{G}$ of $\mathfrak{B}$. Then for all $a, b \in \}$,*

$$||a - \mathcal{P}b|| \leq \frac{(1+\beta)}{1-\gamma} ||a - \mathcal{P}a|| + \frac{\alpha}{1-\gamma} ||a - b||.$$

This is what we need. Now we prove a demiclosedness principle.

Lemma 5 ([30]) *Suppose $\mathcal{P}$ is (α, β, γ)-nonexpansive self-map whose domain of definition is possibly a subset $\mathcal{G}$ of $\mathfrak{B}$. If the given $\mathfrak{B}$ satisfies the Opial's property. Then the following implication holds:*

$$\mathcal{A}_n \in \mathcal{G},\ ||\mathcal{P}\mathcal{A}_n - \mathcal{A}_n|| \to 0 \ \textit{ and } \mathcal{A}_n \to \rho \implies \rho \in \mathcal{F}(\mathcal{P}).$$

Proof. From Lemma 4, we have

$$||\mathcal{A}_n - \mathcal{P}\rho|| \leq \frac{(1+\beta)}{1-\gamma} ||\mathcal{A}_n - \mathcal{P}\mathcal{A}_n|| + \frac{\alpha}{1-\gamma} ||\mathcal{A}_n - \rho||.$$

Since $\alpha + \gamma \leq 1$, we see that $\alpha \leq 1 - \gamma$. It follows that

$$\limsup_{n\to\infty} ||\mathcal{A}_n - \mathcal{P}\rho|| < \limsup_{n\to\infty} ||\mathcal{A}_n - \rho||.$$

Since the underlying space has the Opial's property, one get $\mathcal{P}\rho = \rho$. This finishes the proof. ■

3 $\mathcal{SRJ}$ Iteration Process and its Convergence Analysis

The study of iterative scheme is an important area of research on its own [31, 32]. As we know Picard iteration is not necessarily convergent in the case of nonexpansive operators. This example suggests that we use other iterative methods. In the literature of fixed-point iterations, one can search for many iterative methods that converge in the case of nonexpansive operators and also suggest better accuracy as compared to the Picard iteration method. If $\mathcal{G}$ is a closed and convex subset of a Banach space, $n \in \mathbb{N}$ and $a_n, b_n, c_n \in (0,1)$. Then for $\mathcal{A}_1 = \mathcal{A} \in \mathcal{G}$, Mann [19], Ishikawa [16], Noor [20], Agarwal [2], Abbas [1], Thakur [28] and Ullah [15] iterative methods respectively read as follows:

$$\{\mathcal{A}_{n+1} = a_n \mathcal{P}\mathcal{A}_n + (1-a_n)\mathcal{A}_n, \tag{2}$$

$$\begin{cases} \mathcal{A}_{n+1} = a_n \mathcal{P}\mathcal{B}_n + (1-a_n)\mathcal{A}_n, \\ \mathcal{B}_{n+1} = b_n \mathcal{P}\mathcal{A}_n + (1-b_n)\mathcal{A}_n, \end{cases} \tag{3}$$

$$\begin{cases} \mathcal{A}_{n+1} = a_n \mathcal{P}\mathcal{B}_n + (1-a_n)\mathcal{A}_n, \\ \mathcal{B}_n = b_n \mathcal{P}\mathcal{C}_n + (1-b_n)\mathcal{A}_n, \\ \mathcal{C}_n = c_n \mathcal{P}\mathcal{A}_n + (1-c_n)\mathcal{A}_n, \end{cases} \tag{4}$$

$$\begin{cases} \mathcal{A}_{n+1} = a_n \mathcal{P}\mathcal{B}_n + (1-a_n)\mathcal{A}_n, \\ \mathcal{B}_n = b_n \mathcal{P}\mathcal{A}_n + (1-b_n)\mathcal{P}\mathcal{A}_n, \end{cases} \tag{5}$$

$$\begin{cases} \mathcal{A}_{n+1} = a_n \mathcal{P}\mathcal{C}_n + (1-a_n)\mathcal{P}\mathcal{B}_n, \\ \mathcal{B}_n = b_n \mathcal{P}\mathcal{C}_n + (1-b_n)\mathcal{P}\mathcal{A}_n, \\ \mathcal{C}_n = c_n \mathcal{P}\mathcal{A}_n + (1-c_n)\mathcal{A}_n, \end{cases} \tag{6}$$

$$\begin{cases} \mathcal{A}_{n+1} = \mathcal{P}\mathcal{B}_n, \\ \mathcal{B}_n = \mathcal{P}(a_n \mathcal{C}_n + (1-a_n)\mathcal{A}_n), \\ \mathcal{C}_n = b_n \mathcal{P}\mathcal{A}_n + (1-b_n)\mathcal{A}_n, \end{cases} \tag{7}$$

$$\begin{cases} \mathcal{A}_{n+1} = \mathcal{P}\mathcal{B}_n, \\ \mathcal{B}_n = \mathcal{P}(a_n \mathcal{P}\mathcal{C}_n + (1-a_n)\mathcal{P}\mathcal{A}_n), \\ \mathcal{C}_n = b_n \mathcal{P}\mathcal{A}_n + (1-b_n)\mathcal{A}_n. \end{cases} \tag{8}$$

It is known from [15] that the Ullah iterative method (8) converges faster than the iterative methods (2)-(7) under certain assumptions.

A natural question arises: does there exist an iterative method that is essentially better than all of the above iterative methods, including the Ullah iterative method (8)? To answer this question, Dashputre et al. [12] introduced and studied the following $\mathcal{SRJ}$-iterative method:

Let $\mathcal{G}$ be a nonempty, closed and convex subset of a uniformly convex Banach space $\mathfrak{B}$ and $\mathcal{P}\colon \mathcal{G} \to \mathcal{G}$ be a mapping and the sequence $\{\mathcal{A}_n\}$ generated iteratively by

$$\begin{cases} \mathcal{A}_{n+1} = \mathcal{P}(c_n\mathcal{P}\mathcal{B}_n + (1-c_n)\mathcal{B}_n), \\ \mathcal{B}_n = \mathcal{P}(b_n\mathcal{P}\mathcal{C}_n + (1-b_n)\mathcal{C}_n), \\ \mathcal{C}_n = \mathcal{P}(a_n\mathcal{P}\mathcal{A}_n + (1-a_n)\mathcal{A}_n). \end{cases} \tag{9}$$

Now, we apply the previously established properties of (α, β, γ)-nonexpansive mappings in this paper and prove the convergence of the $\mathcal{SRJ}$-iterative method (9) to the fixed point of these mappings. After this, we then provide another example of these mappings, which essentially exceed nonexpansive mappings, to compare the high accuracy of the $\mathcal{SRJ}$ iterative method in this new setting.

4 Main Results

Theorem 1 *Let $\mathcal{P}\colon \mathcal{G} \to \mathcal{G}$ be a (α, β, γ)-nonexpansive mapping defined on a nonempty closed convex subset $\mathcal{G}$ of a uniformly convex Banach space $\mathfrak{B}$ such that $\mathcal{F}(\mathcal{P}) \neq \emptyset$. If $\{\mathcal{A}_n\}$ is a sequence generated by (9). Then $\lim_{n\to\infty} ||\mathcal{A}_n - \rho||$ exists for all $\rho \in \mathcal{F}(\mathcal{P})$.*

Proof. Let $\rho \in \mathcal{F}(\mathcal{P})$. By Lemma 2, we have

$$\begin{aligned} ||\mathcal{C}_n - \rho|| &= ||\mathcal{P}((1-a_n)\mathcal{A}_n + a_n\mathcal{P}\mathcal{A}_n) - \rho|| \\ &\leq ((1-a_n))||\mathcal{A}_n - \rho|| + a_n||\mathcal{P}\mathcal{C}_n - \rho|| \\ &\leq (1-a_n)||\mathcal{A}_n - \rho|| + a_n||\mathcal{A}_n - \rho|| \\ &\leq ||\mathcal{A}_n - \rho||. \end{aligned} \tag{10}$$

Using Lemma 2 and (10), we get

$$\begin{aligned} ||\mathcal{B}_n - \rho|| &= ||\mathcal{P}((1-b_n)\mathcal{C}_n + b_n\mathcal{P}\mathcal{C}_n) - \rho|| \\ &\leq (1-b_n)||\mathcal{C}_n - \rho|| + b_n||\mathcal{P}\mathcal{C}_n - \rho|| \\ &\leq (1-b_n)||\mathcal{C}_n - \rho|| + b_n||\mathcal{C}_n - \rho|| \\ &\leq (1-b_n)||\mathcal{A}_n - \rho|| + b_n||\mathcal{A}_n - \rho|| \\ &\leq ||\mathcal{A}_n - \rho||. \end{aligned} \tag{11}$$

Using Lemma 2, (10) and (11), we get

$$\begin{aligned} ||\mathcal{A}_{n+1} - \rho|| &= ||\mathcal{P}((1-c_n)y_n + c_n\mathcal{P}y_n), \rho|| \\ &\leq (1-c_n)||\mathcal{B}_n - \rho|| + c_n||\mathcal{P}\mathcal{B}_n - \rho|| \\ &\leq (1-c_n)||\mathcal{B}_n - \rho|| + c_n||\mathcal{B}_n - \rho|| \\ &\leq (1-c_n)||\mathcal{A}_n - \rho|| + c_n||\mathcal{A}_n - \rho|| \\ &\leq ||\mathcal{A}_n - \rho||. \end{aligned} \tag{12}$$

Thus, $\{||\mathcal{A}_n - \rho||\}$ is a non-increasing sequence of reals which is bounded below by zero and hence convergent. Therefore, $\lim_{n\to\infty} ||\mathcal{A}_n - \rho||$ exists $\forall \rho \in \mathcal{F}(\mathcal{P})$. ■

Theorem 2 *Let $\mathcal{P}\colon \mathcal{G} \to \mathcal{G}$ be a (α, β, γ)-nonexpansive mapping defined on a nonempty closed convex subset $\mathcal{G}$ of a uniformly convex Banach space $\mathfrak{B}$ and $\{\mathcal{A}_n\}$ is generated by the algorithm (9). Then $\mathcal{F}(\mathcal{P}) \neq \emptyset$ if and only if $\{\mathcal{A}_n\}$ is bounded and $\lim_{n\to\infty} ||\mathcal{P}\mathcal{A}_n - \mathcal{A}_n|| = 0$.*

Proof. Suppose that $\mathcal{F}(\mathcal{P}) \neq \emptyset$ and $\rho \in \mathcal{F}(\mathcal{P})$.

Then by theorem 1, we see that $\lim_{n\to\infty} ||\mathcal{A}_n - \rho||$ exists and $\{\mathcal{A}_n\}$ is bounded. Put

$$\lim_{n\to\infty} ||\mathcal{A}_n - \rho|| = r. \tag{13}$$

By the proof of Theorem 1 and (13), we have

$$\lim_{n\to\infty} \sup ||\mathcal{P}\mathcal{A}_n - \rho|| \leq \lim_{n\to\infty} \sup ||\mathcal{A}_n - \rho|| = r. \tag{14}$$

Again by the proof of Theorem 1 from (11), we have $||\mathcal{B}_n - \rho|| \leq ||\mathcal{A}_n - \rho||$.

Therefore,

$$\begin{aligned} ||\mathcal{A}_{n+1} - \rho|| &= ||\mathcal{P}((1-c_n)\mathcal{B}_n + c_n\mathcal{P}\mathcal{B}_n) - \rho|| \\ &\leq (1-c_n)||\mathcal{B}_n - \rho|| + c_n d(\mathcal{P}y_n, p) \\ &\leq (1-c_n)||\mathcal{A}_n - \rho|| + c_n||\mathcal{B}_n - \rho||. \end{aligned}$$

It follows that

$$\begin{aligned} ||\mathcal{A}_{n+1} - \rho|| - ||\mathcal{A}_n - \rho|| &\leq \frac{||\mathcal{A}_{n+1} - \rho|| - ||\mathcal{A}_n - \rho||}{c_n} \\ &\leq ||\mathcal{B}_n - \rho|| - ||\mathcal{A}_n - \rho|| \\ &\leq (1-b_n)||\mathcal{C}_n - \rho|| + b_n||\mathcal{C}_n - \rho|| - ||\mathcal{A}_n - \rho|| \\ &\leq ||\mathcal{C}_n - \rho|| - ||\mathcal{A}_n - \rho||. \end{aligned}$$

So, we can get $||\mathcal{A}_{n+1} - \rho|| \leq ||\mathcal{C}_n - \rho||$ and from (13), we have

$$r \leq \lim_{n\to\infty} \inf ||\mathcal{C}_n - \rho||. \tag{15}$$

Hence, from (14) and (15), we obtain

$$r = \lim_{n\to\infty} ||\mathcal{C}_n - \rho||. \tag{16}$$

Therefore, from (16), we have

$$\begin{aligned} r &= \lim_{n\to\infty} ||\mathcal{C}_n - \rho|| = \lim_{n\to\infty} ||\mathcal{P}((1-a_n)\mathcal{A}_n + a_n\mathcal{P}\mathcal{A}_n) - \rho|| \\ &\leq \lim_{n\to\infty} (1-a_n)||\mathcal{A}_n - \rho|| + a_n||\mathcal{P}\mathcal{A}_n - \rho|| \\ &\leq \lim_{n\to\infty} (1-a_n)||\mathcal{A}_n - \rho|| + \lim_{n\to\infty} a_n||\mathcal{P}\mathcal{A}_n - \rho|| \\ &\leq r. \end{aligned} \tag{17}$$

Hence,

$$\lim_{n\to\infty} (1-a_n)||\mathcal{A}_n - \rho|| + a_n||\mathcal{P}\mathcal{A}_n - \rho|| = c. \tag{18}$$

Now, from (15),(18) and Lemma 1, we conclude that

$$\lim_{n\to\infty} ||\mathcal{P}\mathcal{A}_n - \mathcal{A}_n|| = 0.$$

Conversely, suppose that $\{\mathcal{A}_n\}$ is bounded and $\lim_{n\to\infty} ||\mathcal{P}\mathcal{A}_n - \mathcal{A}_n|| = 0$ and $\rho \in \mathcal{K}(\{\mathcal{A}_n\})$. By Lemma 4 and Definition 2, we have

$$\begin{aligned} u(\mathcal{P}\rho, \{\mathcal{A}_n\}) &= \lim_{n\to\infty} \sup \|\mathcal{A}_n - \mathcal{P}\rho\| \\ &\leq \lim_{n\to\infty} \sup \left[\frac{(1+\beta)}{1-\gamma} ||\mathcal{A}_n - \mathcal{P}\mathcal{A}_n|| + \frac{\alpha}{1-\gamma} ||\mathcal{A}_n - \rho|| \right] \end{aligned}$$

$$\begin{aligned}
&\leq \left(\frac{1+\beta}{1-\gamma}\right)\lim_{n\to\infty}\sup||\mathcal{A}_n-\mathcal{P}\mathcal{A}_n||+\left(\frac{\alpha}{1-\gamma}\right)\lim_{n\to\infty}\sup||\mathcal{A}_n-\rho|| \\
&\leq \lim_{n\to\infty}\sup||\mathcal{A}_n-\rho|| \\
&= u(\rho,\{\mathcal{A}_n\}).
\end{aligned}$$

So, $\mathcal{P}\rho\in\mathcal{K}(\mathcal{G},\{\mathcal{A}_n\})$. As $\mathfrak{B}$ is uniformly convex Banach space. Therefore, $\mathcal{K}(\mathcal{G},\{\mathcal{A}_n\})$ consists of a single point. Hence, $\mathcal{P}\rho=\rho$, that is the fixed point of $\mathcal{P}$ is nonempty. ■

Sometimes the strong convergence for a certain operator is not possible in general; therefore, we need the weak convergence in such a case. Under the following conditions, we establish the weak convergence result for (α,β,γ)-nonexpansive self-maps.

Theorem 3 *Let $\mathcal{P}\colon\mathcal{G}\to\mathcal{G}$ be a (α,β,γ)-nonexpansive mapping defined on a nonempty closed convex subset $\mathcal{G}$ of a uniformly convex Banach space $\mathfrak{B}$ such that $\mathcal{F}(\mathcal{P})\neq\emptyset$. If $\{\mathcal{A}_n\}$ is a sequence generated by (9) and $\mathfrak{B}$ satisfies Opial's property, then it converges weakly to some fixed point of $\mathcal{P}$.*

Proof. It has been established in Theorem 1 that $\lim_{n\to\infty}||\mathcal{A}_n-\rho||$ exists and that $\{\mathcal{A}_n\}$ is bounded. Now, since $\mathfrak{B}$ is uniformly convex, we can find a subsequence say $\{\mathcal{A}_{n_m}\}$ of $\{\mathcal{A}_n\}$ that converges weakly in $\mathcal{G}$. We establish that $\{\mathcal{A}_n\}$ has a unique weak subsequential limit in $\mathcal{F}(\mathcal{P})$. Let ρ and ρ_0 be weak limits of the subsequences $\{\mathcal{A}_{n_m}\}$ and $\{\mathcal{A}_{n_r}\}$ of $\{\mathcal{A}_n\}$ respectively. By Theorem 2, we have that $\lim_{n\to\infty}||\mathcal{A}_n-\mathcal{P}\mathcal{A}_n||=0$ and $I-\mathcal{P}$ is demiclosed with respect to zero by Lemma 5. Therefore, we have that $\mathcal{P}\rho=\rho$. Using a similar approach, we can show that $\rho_0=\mathcal{P}\rho_0$. It follows from Theorem 1 that $\lim_{n\to\infty}||\mathcal{A}_n-\rho_0||$ exists. Now, suppose that $\rho\neq\rho_0$. Then by the Opial condition, we have

$$\begin{aligned}
\limsup_{n\to\infty}||\mathcal{A}_n-\rho|| &= \limsup_{m\to\infty}||\mathcal{A}_{n_m}-\rho||<\limsup_{m\to\infty}||\mathcal{A}_{n_m}-\rho_0|| \\
&= \limsup_{n\to\infty}||\mathcal{A}_n-\rho_0||<\limsup_{r\to\infty}||\mathcal{A}_{n_r}-\rho_0|| \\
&< \limsup_{r\to\infty}||\mathcal{A}_{n_r}-\rho||<\limsup_{n\to\infty}||\mathcal{A}_n-\rho||.
\end{aligned}$$

This is a contradiction. So $\rho\neq\rho_0$. Hence, $\{\mathcal{A}_n\}$ converges weakly to a fixed point of $\mathcal{F}(\mathcal{P})$ and this completes the proof. ■

The next result is related to the strong convergence, which is based on the assumption that the domain of $\mathcal{P}$ is a compact set.

Theorem 4 *Let $\mathcal{P}\colon\mathcal{G}\to\mathcal{G}$ be a (α,β,γ)-nonexpansive mapping defined on a nonempty closed convex subset $\mathcal{G}$ of a uniformly convex Banach space $\mathfrak{B}$ such that $\mathcal{F}(\mathcal{P})\neq\emptyset$. If $\{\mathcal{A}_n\}$ is a sequence generated by (9), then the sequence $\{\mathcal{A}_n\}$ converges strongly to some fixed point of $\mathcal{P}$.*

Proof. Since the set $\mathcal{G}$ is convex and compact, $\{\mathcal{A}_n\}$ contained in $\mathcal{G}$ and has a convergent subsequence. We denote this subsequence by $\{\mathcal{A}_{n_m}\}$ with a strong limit $\rho\in\mathcal{G}$, that is, $\lim_{n_m\to\infty}||\mathcal{A}_{n_m}-\rho||$. Suppose $a=\mathcal{A}_{n_m}$ and $b=\rho$, then applying Lemma 4, one has

$$||\mathcal{A}_{n_m}-\mathcal{F}\rho||\leq\frac{(1+\beta)}{1-\gamma}||\mathcal{A}_{n_m}-\mathcal{P}\mathcal{A}_{n_m}||+\frac{\alpha}{1-\gamma}||\mathcal{A}_{n_m}-\rho||. \tag{19}$$

By Theorem 2, $\lim_{n_m\to\infty}||\mathcal{A}_{n_m}-\mathcal{P}\mathcal{A}_{n_m}||=0$ and also from the above $\lim_{n_m\to\infty}||\mathcal{A}_{n_m}-\rho||=0$. Accordingly, (19) provides that $\mathcal{A}_{n_m}\to\mathcal{P}\rho$. It follows that $\mathcal{P}\rho=\rho$. By Theorem 1, $\lim_{n\to\infty}||\mathcal{A}_n-\rho||$ exists. Consequently, we have proved that $\rho\in\mathcal{F}(\mathcal{P})$ and $\mathcal{A}_n\to\rho$. This finishes proof. ■

In the following result, we drop the assumption that the domain of $\mathcal{P}$ is a compact set.

Theorem 5 *Let $\mathcal{P}\colon\mathcal{G}\to\mathcal{G}$ be a (α,β,γ)-nonexpansive mapping defined on a nonempty closed convex subset $\mathcal{G}$ of a uniformly convex Banach space $\mathfrak{B}$ such that $\mathcal{F}(\mathcal{P})\neq\emptyset$. Assume that $\{\mathcal{A}_n\}$ is a sequence generated by (9). If $\liminf_{n\to\infty}d_s(\mathcal{A}_n,\mathcal{F}(\mathcal{P}))$ holds. Then the sequence $\{\mathcal{A}_n\}$ converges strongly to some fixed point of $\mathcal{P}$.*

Proof. If the sequence $\{\mathcal{A}_n\}$ converges to a point $\rho \in \mathcal{F}(\mathcal{P})$, then

$$\lim_{n\to\infty} inf ||\mathcal{A}_n - \rho|| = 0.$$

So

$$\lim_{n\to\infty} ||\mathcal{A}_n - \mathcal{F}(\mathcal{P})|| = 0.$$

For converse part, assume that $\lim_{n\to\infty} \inf ||\mathcal{A}_n - \mathcal{F}(\mathcal{P})|| = 0$. From Theorem 1, we have

$$||\mathcal{A}_{n+1} - \rho|| \leq ||\mathcal{A}_n - \rho|| \text{ for any } \rho \in \mathcal{F}(\mathcal{P}).$$

So we have,

$$||\mathcal{A}_{n+1} - \mathcal{F}(\mathcal{P})|| \leq ||\mathcal{A}_n - \mathcal{F}(\mathcal{P})||. \tag{20}$$

Thus, $||\mathcal{A}_n - \mathcal{F}(\mathcal{P})||$ forms a decreasing sequence which is bounded below by zero as well, thus $\lim_{n\to\infty} ||\mathcal{A}_n - \mathcal{F}(\mathcal{P})||$ exists. Since $\lim_{n\to\infty} \inf ||\mathcal{A}_n - \mathcal{F}(\mathcal{P})|| = 0$, we see that $\lim_{n\to\infty} ||\mathcal{A}_n - \mathcal{F}(\mathcal{P})|| = 0$.

Now, there exists a subsequence $\{\mathcal{A}_{n_j}\}$ of $\{\mathcal{A}_n\}$ and a sequence $\{\mathcal{A}_j\}$ in $\mathcal{F}(\mathcal{P})$ such that $||\mathcal{A}_{n_j} - \mathcal{A}_j|| \leq \frac{1}{2^j}$ for all $j \in \mathbb{N}$. From the proof of Theorem 1, we have

$$||\mathcal{A}_{n_{j+1}} - \mathcal{A}_j|| \leq ||\mathcal{A}_{n_j} - \mathcal{A}_j|| \leq \frac{1}{2^j}.$$

Using triangle inequality, we get

$$\begin{aligned} ||\mathcal{A}_{n_{j+1}} - \mathcal{A}_j|| &\leq ||\mathcal{A}_{j+1} - \mathcal{A}_{n_{j+1}}|| + ||\mathcal{A}_{n_{j+1}} - \mathcal{A}_j|| \\ &\leq \frac{1}{2^{j+1}} + \frac{1}{2^j} \\ &\leq \frac{1}{2^{j-1}} \\ &\to 0 \text{ as } j \to \infty. \end{aligned}$$

So, $\{\mathcal{A}_j\}$ is a Cauchy sequence in $\mathcal{F}(\mathcal{P})$. From Lemma 3 $\mathcal{F}(\mathcal{P})$ is closed, so $\{\mathcal{A}_j\}$ converges to some $\rho \in \mathcal{F}(\mathcal{P})$. Again, owing to triangle inequality, we have

$$||\mathcal{A}_{n_j} - \rho|| \leq ||\mathcal{A}_{n_j} - \mathcal{A}_j|| + ||\mathcal{A}_j - \rho||.$$

Letting $j \to \infty$, we have $\{\mathcal{A}_{n_j}\}$ converges strongly to $x \in \mathcal{F}(\mathcal{P})$. Since $\lim_{n\to\infty} inf||\mathcal{A}_n, \rho)$ exists by Theorem 1, we observe that $\{\mathcal{A}_n\}$ converges to $\rho \in \mathcal{F}(\mathcal{P})$. ■

Now, we establish the main result of this section, which is related to the work of Senter and Dotson [25].

Theorem 6 *Let $\mathcal{P}: \mathcal{G} \to \mathcal{G}$ be a (α, β, γ)-nonexpansive mapping defined on a nonempty closed convex subset $\mathcal{G}$ of a uniformly convex Banach space $\mathfrak{B}$ such that $\mathcal{F}(\mathcal{P}) \neq \emptyset$. If $\{\mathcal{A}_n\}$ is a sequence generated by (9) and $\mathfrak{B}$ satisfies condition (1). Then the sequence $\{\mathcal{A}_n\}$ converges strongly to same fixed point of $\mathcal{P}$.*

Proof. It follows from Theorem 2 the $\liminf_{n\to\infty} ||\mathcal{A}_n - \mathcal{P}\mathcal{A}_n|| = 0$. By condition (1) of $\mathcal{P}$, we have $\liminf_{n\to\infty} d(\mathcal{A}_n, \mathcal{F}(\mathcal{P}))$. Theorem 4 leads to the conclusions. ■

5 Numerical Example

This section presents a novel example of a (α, β, γ)-nonexpansive mapping and proves that it is not nonexpansive. In this example, we compare our strategy to various iterative approaches (see tables and graphs below). According to the data, the $\mathcal{SRJ}$-iterative approach represents a novel class of mappings that converges quicker than the comparable iterative methods.

Example 2 *Let $\mathcal{G} = [0,1]$ and set $\mathcal{P}$ on $\mathcal{G}$ as:*

$$\mathcal{P}a = \begin{cases} \frac{a}{3} & \text{if } a \in [0, \frac{1}{2}), \\ \frac{a}{4} & \text{if } a \in [\frac{1}{2}, 1]. \end{cases}$$

Case (I). If $a, b \in [0, \frac{1}{2})$, then

$$\begin{aligned} ||\mathcal{P}a - \mathcal{P}b|| &= \left|\left|\frac{a-b}{3}\right|\right| \leq \left|\left|\frac{a-b}{2}\right|\right| \\ &\leq \left|\left|\frac{a-b}{2}\right|\right| + \left|\left|\frac{2(a-\frac{a}{3})}{3}\right|\right| \\ &\leq \left|\left|\frac{a-b}{2}\right|\right| + \left|\left|\frac{2(a-\frac{a}{3})}{3}\right|\right| + \left|\left|\frac{(a-\frac{b}{3})}{2}\right|\right| \\ &= \alpha||a-b|| + \beta||a-\mathcal{P}a|| + \gamma||a = \mathcal{P}b||. \end{aligned}$$

Case (II). If $a, b \in [\frac{1}{2}, 1]$, then

$$\begin{aligned} ||\mathcal{P}a - \mathcal{P}b|| &= \left|\left|\frac{a-b}{4}\right|\right| \leq \left|\left|\frac{a-b}{2}\right|\right| \\ &\leq \left|\left|\frac{a-b}{2}\right|\right| + \left|\left|\frac{2(a-\frac{a}{4})}{3}\right|\right| \\ &\leq \left|\left|\frac{a-b}{2}\right|\right| + \left|\left|\frac{2(a-\frac{a}{4})}{3}\right|\right| + \left|\left|\frac{(a-\frac{b}{4})}{2}\right|\right| \\ &= \alpha||a-b|| + \beta||a-\mathcal{P}a|| + \gamma||a = \mathcal{P}b||. \end{aligned}$$

Case (III). If $a \in [0, \frac{1}{2})$ and $b \in [\frac{1}{2}, 1]$, then

$$\begin{aligned} ||\mathcal{P}a - \mathcal{P}b|| &= \left|\left|\frac{a}{3} - \frac{b}{4}\right|\right| \leq \left|\left|\frac{b}{4}\right|\right| + \left|\left|\frac{a}{3}\right|\right| \\ &\leq \left|\left|\frac{3b}{8}\right|\right| + \left|\left|\frac{4a}{9}\right|\right| \\ &= \left|\left|\frac{(b-\frac{b}{4})}{2}\right|\right| + \left|\left|\frac{4a}{9}\right|\right| \\ &= \left|\left|\frac{((a-b)-(a-\frac{b}{4}))}{2}\right|\right| + \left|\left|\frac{2(a-\frac{a}{3})}{3}\right|\right| \\ &\leq \left|\left|\frac{a-b}{2}\right|\right| + \left|\left|\frac{2(a-\frac{a}{3})}{3}\right|\right| + \left|\left|\frac{(a-\frac{b}{4})}{2}\right|\right| \\ &= \alpha||a-b|| + \beta||a-\mathcal{P}a|| + \gamma||a = \mathcal{P}b||. \end{aligned}$$

Case (IV). If $b \in [0, \frac{1}{2})$ and $a \in [\frac{1}{2}, 1]$, then

$$\begin{aligned} ||\mathcal{P}a - \mathcal{P}b|| &= \left|\left|\frac{a}{4} - \frac{b}{3}\right|\right| \leq \left|\left|\frac{b}{3}\right|\right| + \left|\left|\frac{a}{4}\right|\right| \\ &\leq \left|\left|\frac{2b}{6}\right|\right| + \left|\left|\frac{a}{2}\right|\right| \\ &= \left|\left|\frac{(b-\frac{b}{3})}{2}\right|\right| + \left|\left|\frac{6a}{12}\right|\right| \end{aligned}$$

$$\begin{aligned} &= \left|\left|\frac{((a-b)-(a-\frac{b}{3}))}{2}\right|\right| + \left|\left|\frac{2(a-\frac{a}{4})}{3}\right|\right| \\ &\leq \left|\left|\frac{a-b}{2}\right|\right| + \left|\left|\frac{2(a-\frac{a}{4})}{3}\right|\right| + \left|\left|\frac{(a-\frac{b}{3})}{2}\right|\right| \\ &= \alpha||a-b|| + \beta||a-\mathcal{P}a|| + \gamma||a=\mathcal{P}b||. \end{aligned}$$

As a result, all of the preceding cases indicate that the operator $\mathcal{P}$ is (α, β, γ)-nonexpansive.

By using example (2), we tried to show that the rate of convergence of the $\mathcal{SRJ}$ iteration is better then some known iteration processes for (α, β, γ)-nonexpansive mapping. Parameters are $a_n = 1 - \frac{1}{(2n+8)}$, $b_n = \frac{n}{16n+1}$ and $c_n = \frac{n}{(n+5)}$, $\forall n \in \mathbb{N}$.

Table 1: Convergence of SRJ iteration (9) for fixed point 0.

No. of iteration	**Agrawal**	**Thakur**	**K**	**SRJ**
1	0.5	0.5	0.5	0.5
2	0.158481	0.052984	0.018082	0.001869
3	0.050654	0.005645	0.000657	0.000009
4	0.01619	0.000601	0.000023	0
5	0.005174	0.000064	0.000008	0
6	0.001653	0.000006	0	0
7	0.000528	0	0	0
8	0.000168	0	0	0
9	0.000054	0	0	0
10	0.000017	0	0	0

Clearly $\rho = 0$ is a fixed point of (α, β, γ)-nonexpansive mapping. Table 1 shows that behaviour of some iteration processes to fixed point of $\mathcal{P}$ for initial value 0.50. Furthermore, we have examined the influence of parameters a_n, b_n and c_n. For this we have considered various sets of parameters and present a study regarding the number of iterations required. Each iteration starts with a particular initial value and the respective number of iterations, average of the number of iterations for different initial points are given in Figure 5. We have examined the fastness and stability of different iterations relative to above mentioned set of parameters.

In what follows, we numerically compare our new iteration process (9) with some existing iteration processes.

Case I: Taking, $a_n = \frac{1}{\sqrt{n+9}}$, $b_n = \frac{1}{n+1}$ and $c_n = \frac{n}{n+5}$.

Case II: Taking, $a_n = \frac{3n}{8n+4}$, $b_n = \frac{1}{n+4}$ and $c_n = \frac{n}{(2n+6)^2}$.

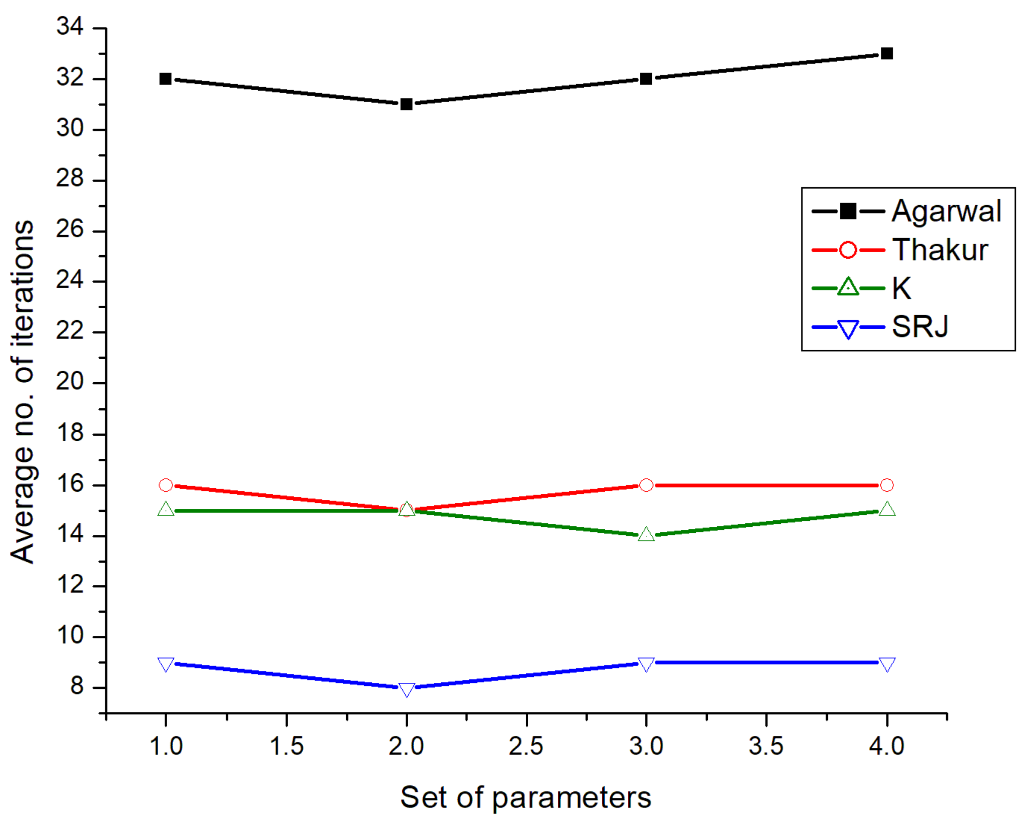
Case III: Taking, $a_n = \frac{n}{n+1}$, $b_n = \frac{1}{(n+7)^2}$ and $c_n = \frac{2n}{(5n+2)}$.

Case IV: Taking, $a_n = 1 - \frac{1}{(2n+8)}$, $b_n = \frac{n}{16n+1}$ and $c_n = \frac{n}{(n+5)}$.

Comparison of various iteration processes for example 2.

Iterations																				
Init. Value	0.25				0.50				0.75				1				Iteration Average			
Case	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
Agarwal	31	31	31	30	32	32	32	31	32	32	32	31	32	32	32	31	30.75	31.75	31.75	31.75
Thakur	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16
K	15	14	11	11	15	14	11	11	16	14	11	11	16	15	11	11	12.75	12.75	13	13.75
SRJ	8	10	8	7	9	10	8	7	9	9	8	7	9	9	8	7	8.25	11	8.25	8.25

Table depicting comparison of various iteration process under distinct parameters for Example 2.



Average no. of iterations under distinct parameters for Example 2.

The observations are given in Figures 5 and 5. We have concluded that the new iteration process (9) not only converges faster than the known iterations but also is stable with respect to the parameters a_n, b_n and c_n. From Figure 5, we also observe that the average number of iterations of the new iteration process (9) is the smallest with respect to other processes.

6 An Application to Volterra-Fredholm Integral Equation

In this section, we use iteration (9) to solve the following Volterra-Fredholm integral equation given by Lungu and Rus [18]:

$$f(x,y) = g(x,y,h(f(x,y))) + \int_0^x \int_0^y \mathbf{K}(x,y,u,v,f(u,v))dudv, \tag{21}$$

where $x, y \in \mathbb{R}_+$. Let $(\mathcal{W}, ||.||)$ be a Banach space. Let $\tau > 0$ and

$$\mathcal{X}_\tau = \{f \in \mathcal{G}(\mathbb{R}^2, \mathcal{W}) |\ \exists \mathcal{M}(f) > 0 \colon |f(x,y)|e^{\tau(x+y)} \leq \mathcal{M}(f)\}.$$

Now, we consider BieleckiáÂŹ norm on $\mathcal{X}_\tau$ as follows:

$$||f||_\tau = \sup_{x,y\in\mathbb{R}_+} (|f(x,y)|e^{\tau(x+y)}).$$

Obviously, $(\mathcal{X}_\tau, ||.||_\tau)$ is a Banach space (see [7]).

The following result will play a major role in proving the main result.

Theorem 7 ([18]) *Suppose the following conditions are fulfilled*

(A) $g \in \mathcal{G}(\mathbb{R}^2_+ \times \mathcal{W}, \mathcal{W})$, $\mathbf{K} \in \mathcal{G}(\mathbb{R}^4_+ \times \mathcal{W}, \mathcal{W})$;

(B) $h \colon \mathbf{X}_\tau \to \mathbf{X}_\tau$ *such that*

$$\exists\ l_h > 0 \colon |h(f_1(x,y)) - h(f_1(x,y))| \leq ||f_1 - f_2||e^{\tau(x+y)},$$

for all $x, y \in \mathbb{R}_+$ *and* $f_1, f_2 \in \mathcal{X}_\tau$;

(C) $\exists\ l_g > 0 \colon |g(x,y,u_1) - g(x,y,u_2)| \leq l_g||u_1 - u_2||$, *for all* $x, y \in \mathbb{R}_+$ *and* $u_1, u_2 \in \mathcal{W}$;

(D) $\exists\ l_{\mathbf{K}}(x,y,u,v) \colon |\mathbf{K}(x,y,u,v,u_1) - \mathbf{K}(x,y,u,v,u_1)| \leq l_{\mathbf{K}}(x,y,u,v)|u_1 - u_2|$, *for all* $x, y \in \mathbb{R}_+$ *and* $u_1, u_2 \in \mathcal{W}$;

(E) $l_{\mathbf{K}} \in \mathcal{G}(\mathbb{R}^4_+, \mathbb{R}_+)$ *and*

$$\int_0^x \int_0^y l_{\mathbf{K}}(x,y,u,v)e^{\tau(x+y)}dudv \leq le^{\tau(x+y)},$$

for all $x, y \in \mathbb{R}_+$.

(F) $(l_g l_h + l) < 1$.

Then the equation (21) has a unique solution $z \in \mathcal{Z}_\tau$ *and the sequence of successive approximations*

$$f_{n+1}(x,y) = g(x,y,h(f_n(x,y))) + \int_0^x \int_0^y \mathbf{K}(x,y,u,v,f_n(u,v))dudv, \tag{22}$$

for all $n \in \mathbb{N}$ *converges uniformly to* z.

Our main result is as follows:

Theorem 8 *If all the conditions from (A) to (F) in Theorem 7 are satisfied, then the equation (22) has a unique fixed point* $\rho \in \mathcal{X}_\tau$ *and the iteration (9) with sequence* $\{a_n\}$, $\{b_n\}$ *and* $\{c_n\} \in (0,1)$ *such that* $\sum_{n=0}^{\infty} a_n = \infty$ *converges strongly to* ρ.

Proof. Let $\{\mathcal{A}_n\}$ be the sequence defined by the SRJ iteration process (9) for the operator $\mathcal{P}\colon \mathcal{X}_\tau \to \mathcal{X}_\tau$ defined by

$$\mathcal{P}(f(x,y)) = g(x,y,h(f(x,y))) + \int_0^x \int_0^y \mathbf{K}(x,y,u,v,f_n(u,v))dudv.$$

We have to show that $\{\mathcal{A}_n\} \to 0$ as $n \to \infty$. By (9)

$$||\mathcal{A}_{n+1} - \rho||_\tau = \sup_{x,y\in\mathbb{R}_+}(|\mathcal{P}(c_n\mathcal{P}\mathcal{B}_n + (1-c_n)\mathcal{B}_n)(x,y) - \mathcal{P}(\rho(x,y))|e^{\tau(x+y)}).$$

Now

$$\begin{aligned}
& |\mathcal{P}(c_n\mathcal{P}\mathcal{B}_n + (1-c_n)\mathcal{B}_n)(x,y) - \mathcal{P}(\rho(x,y))| \\
\leq\ & |g(x,y,h(c_n\mathcal{P}\mathcal{B}_n + (1-c_n)\mathcal{B}_n)(x,y)) - g(x,y,h(\rho(x,y)))| \\
& +|\int_0^x \int_0^y \mathbf{K}(x,y,u,v,(c_n\mathcal{P}\mathcal{B}_n + ((1-c_n)\mathcal{B}_n)(u,v))du\ dv - \int_0^x \int_0^y \mathbf{K}(x,y(\rho(u,v))dudv| \\
\leq\ & l_g|h((c_n\mathcal{P}\mathcal{B}_n + (1-c_n)\mathcal{B}_n)(x,y) - h(\rho(x,y))| \\
& + \int_0^x \int_0^y |\mathbf{K}(x,y,u,v,(c_n\mathcal{P}\mathcal{B}_n + ((1-c_n)\mathcal{B}_n)(u,v)) - \mathbf{K}(x,y(\rho(u,v))|dudv \\
\leq\ & l_g l_h||((c_n\mathcal{P}\mathcal{B}_n + ((1-c_n)\mathcal{B}_n) - \rho||_\tau e^{\tau(x+y)} \\
& + \int_0^x \int_0^y l_{\mathbf{K}}(x,y,u,v,|(c_n\mathcal{P}\mathcal{B}_n + ((1-c_n)\mathcal{B}_n)(u,v) - \rho(u,v)|dudv \\
\leq\ & l_g l_h||(c_n\mathcal{P}\mathcal{B}_n + ((1-c_n)\mathcal{B}_n) - \rho||_\tau e^{\tau(x+y)} \\
& +l||(c_n\mathcal{P}\mathcal{B}_n + ((1-c_n)\mathcal{B}_n) - \rho||_\tau e^{\tau(x+y)} \\
=\ & (l_g l_h + l)||(c_n\mathcal{P}\mathcal{B}_n + ((1-c_n)\mathcal{B}_n) - \rho||_\tau e^{\tau(x+y)} \\
& (l_g l_h + l)||(c_n\mathcal{P}\mathcal{B}_n + ((1-c_n)\mathcal{B}_n) - \rho||_\tau,
\end{aligned} \tag{23}$$

and

$$\begin{aligned}
||(c_n\mathcal{P}\mathcal{B}_n + ((1-c_n)\mathcal{B}_n) - \rho||_\tau &= ||c_n(\mathcal{P}\mathcal{B}_n - \rho) + (1-c_n)(\mathcal{B}_n - \rho)||_\tau \\
&\leq c_n||\mathcal{P}\mathcal{B}_n - \rho||_\tau + (1-c_n)||\mathcal{B}_n - \rho||_\tau.
\end{aligned} \tag{24}$$

So

$$||\mathcal{P}\mathcal{B}_n - \rho|| \leq \sup_{x,y\in\mathbb{R}_+}(|\mathcal{P}\mathcal{B}_n(x,y) - \mathcal{P}\rho(x,y)|e^{\tau(x+y)}).$$

Now

$$\begin{aligned}
|\mathcal{P}\mathcal{B}_n(x,y) - \mathcal{P}\rho(x,y)| &\leq |g(x,y,h(\mathcal{B}_n(x,y))) - g(x,y,h(\rho(x,y)))| \\
&\quad +|\int_0^x \int_0^y \mathcal{K}(x,y,h(\mathcal{B}_n(u,v)))du\ dv - \mathcal{K}(x,y,h(\rho_n(u,v)))dudv| \\
&\leq l_g|h(\mathcal{B}_n(x,y)) - \rho(x,y)| \\
&\quad + \int_0^x \int_0^y |\mathcal{K}(x,y,u,v,\mathcal{B}_n(u,v) - \mathcal{K}(x,y,u,v,\rho(u,v)|dudv \\
&\leq l_g l_h||\mathcal{B}_n - \rho||_\tau e^{\tau(x+y)} + \int_0^x \int_0^y l_{\mathcal{K}}(x,y,u,v)|\mathcal{B}_n - \rho|dudv \\
&\leq l_g l_h||\mathcal{B}_n - \rho||_\tau e^{\tau(x+y)} + l||\mathcal{B}_n - \rho||_\tau e^{\tau(x+y)} \\
&\leq (l_g l_h + l)||\mathcal{B}_n - \rho||_\tau e^{\tau(x+y)} \\
&\leq (l_g l_h + l)||\mathcal{B}_n - \rho||_\tau.
\end{aligned} \tag{25}$$

From equation (24) and (25), we get

$$\begin{aligned}||(c_n\mathcal{P}\mathcal{B}_n+(1-c_n)\mathcal{B}_n)-\rho||_\tau &\le c_n(l_gl_h+l)||\mathcal{B}_n-\rho||_\tau+(1-c_n)||\mathcal{B}_n-\rho||_\tau\\ &= [1-c_n\{1-(l_gl_h+l)\}]||\mathcal{B}_n-\rho||_\tau. \qquad (26)\end{aligned}$$

Thus, from equation (23) and (26), we obtain

$$|\mathcal{P}(c_n\mathcal{P}\mathcal{B}_n+(1-c_n)\mathcal{B}_n)(x,y)-\mathcal{P}(\rho(x,y))|\le[1-c_n\{1-(l_gl_h+l)\}]||\mathcal{B}_n-\rho||_\tau.$$

Therefore,

$$||\mathcal{A}_{n+1}-\rho||_\tau\le(l_gl_h+l)[1-c_n\{1-(l_gl_h+l)\}]||\mathcal{B}_n-\rho||_\tau. \qquad (27)$$

Similarly,

$$\begin{aligned}||\mathcal{B}_n-\rho||_\tau &= \sup_{x,y\in\mathbb{R}_+}(|\mathcal{P}(b_n\mathcal{P}\mathcal{C}_n+(1-b_n)\mathcal{C}_n)(x,y)-\mathcal{P}(\rho(x,y))|e^{\tau(x+y)})\\ &= (l_gl_h+l)[1-b_n\{1-(l_gl_h+l)\}]||\mathcal{C}_n-\rho||_\tau. \qquad (28)\end{aligned}$$

From equation (27) and (28), we obtain

$$||\mathcal{A}_{n+1}-\rho||_\tau\le(l_gl_h+l)^2[1-c_n\{1-(l_gl_h+l)\}][1-b_n\{1-(l_gl_h+l)\}]||\mathcal{C}_n-\rho||_\tau. \qquad (29)$$

Again,

$$\begin{aligned}||\mathcal{C}_n-\rho||_\tau &= \sup_{x,y\in\mathbb{R}_+}(|\mathcal{P}(a_n\mathcal{P}\mathcal{A}_n+(1-a_n)\mathcal{A}_n)(x,y)-\mathcal{P}(\rho(x,y))|e^{\tau(x+y)})\\ &= (l_gl_h+l)[1-a_n\{1-(l_gl_h+l)\}]||\mathcal{A}_n-\rho||_\tau. \qquad (30)\end{aligned}$$

Putting equation (29) and (30)

$$||\mathcal{A}_{n+1}-\rho||_\tau\le(l_gl_h+l)^3[1-c_n\{1-(l_gl_h+l)\}][1-b_n\{1-(l_gl_h+l)\}][1-a_n\{1-(l_gl_h+l)\}]||\mathcal{A}_n-\rho||_\tau. \qquad (31)$$

Recalling assumption $(l_gl_h+l)\le 1$ and $a_n,b_n\in(0,1]$, it follows that

$$[1-b_n\{1-(l_gl_h+l)\}][1-a_n\{1-(l_gl_h+l)\}]\le 1.$$

Thus, equation (31) becomes.

$$||\mathcal{A}_{n+1}-\rho||_\tau\le[1-c_n\{1-(l_gl_h+l)\}]||\mathcal{A}_n-\rho||_\tau. \qquad (32)$$

From equation (32), Inductively we obtain

$$||\mathcal{A}_{n+1}-\rho||_\tau\le||\mathcal{A}_0-\rho||_\tau\prod_{i=0}^{i=n}[1-c_i\{1-(l_gl_h+l)\}]. \qquad (33)$$

Since $c_i\in(0,1]$ for all $i\in\mathbb{N}$ and condition (F) we have $(l_gl_h+l<1)$. Thus, $[1-c_i\{1-(l_gl_h+l)\}]\le 1$. We also know that $1-\theta\le e^{-\theta}$ for all $\theta\in[0,1]$. From (33) we have

$$||\mathcal{A}_{n+1}-\rho||_\tau\le||\mathcal{A}_0-\rho||_\tau e^{-[c_i\{1-(l_gl_h+l)\}]\sum_{i=0}^{i=n}c_i}. \qquad (34)$$

Therefore, $\lim_{n\to\infty}||\mathcal{A}_n-\rho||_\tau=0$ as $\sum_{i=0}^{i=n}c_i=\infty$ whenever $n\to\infty$ and $e^{-\infty}=0$. This completes the proof. ■

7 Conclusion

In this paper, we use $\mathcal{SRJ}$ iteration for (α,β,γ)-nonexpansive mapping in uniformly convex Banach space $\mathfrak{B}$. Our result improves results of Ullah et al. [30] in the sense of faster iteration process. Finally, we used the $\mathcal{SRJ}$ iterative approach to analyze and solve the Volterra-Fredholm integral equation.

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